CHAPTER 7 TECHNIQUES OF INTEGRATION, L'HÔPITAL'S RULE, AND IMPROPER INTEGRALS

7.1 BASIC INTEGRATION FORMULAS

1. $\int \frac{16x\,dx}{\sqrt{8x^2+1}}$; $\begin{bmatrix} u = 8x^2+1 \\ du = 16x\,dx \end{bmatrix} \to \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{8x^2+1} + C$

2. $\int \frac{3\cos x\,dx}{\sqrt{1+3\sin x}}$; $\begin{bmatrix} u = 1+3\sin x \\ du = 3\cos x\,dx \end{bmatrix} \to \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{1+3\sin x} + C$

3. $\int 3\sqrt{\sin v}\cos v\,dv$; $\begin{bmatrix} u = \sin v \\ du = \cos v\,dv \end{bmatrix} \to \int 3\sqrt{u}\,du = 3 \cdot \frac{2}{3} u^{3/2} + C = 2(\sin v)^{3/2} + C$

4. $\int \cot^3 y \csc^2 y\,dy$; $\begin{bmatrix} u = \cot y \\ du = -\csc^2 y\,dy \end{bmatrix} \to \int u^3(-du) = -\frac{u^4}{4} + C = \frac{-\cot^4 y}{4} + C$

5. $\int_0^1 \frac{16x\,dx}{8x^2+2}$; $\begin{bmatrix} u = 8x^2+2 \\ du = 16x\,dx \\ x = 0 \Rightarrow u = 2, \quad x = 1 \Rightarrow u = 10 \end{bmatrix} \to \int_2^{10} \frac{du}{u} = [\ln|u|]_2^{10} = \ln 10 - \ln 2 = \ln 5$

6. $\int_{\pi/4}^{\pi/3} \frac{\sec^2 z\,dz}{\tan z}$; $\begin{bmatrix} u = \tan z \\ du = \sec^2 z\,dz \\ z = \frac{\pi}{4} \Rightarrow u = 1,\ z = \frac{\pi}{3} \Rightarrow u = \sqrt{3} \end{bmatrix} \to \int_1^{\sqrt{3}} \frac{1}{u}\,du = [\ln|u|]_1^{\sqrt{3}} = \ln\sqrt{3} - \ln 1 = \ln\sqrt{3}$

7. $\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$; $\begin{bmatrix} u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\,dx \\ 2\,du = \frac{dx}{\sqrt{x}} \end{bmatrix} \to \int \frac{2\,du}{u} = 2\ln|u| + C = 2\ln(\sqrt{x}+1) + C$

8. $\int \frac{dx}{x-\sqrt{x}} = \int \frac{dx}{\sqrt{x}(\sqrt{x}-1)}$; $\begin{bmatrix} u = \sqrt{x}-1 \\ du = \frac{1}{2\sqrt{x}}\,dx \\ 2\,du = \frac{dx}{\sqrt{x}} \end{bmatrix} \to \int \frac{2\,du}{u} = 2\ln|u| + C = 2\ln\left|\sqrt{x}-1\right| + C$

9. $\int \cot(3-7x)\,dx;\ \begin{bmatrix} u = 3-7x \\ du = -7\,dx \end{bmatrix} \to -\frac{1}{7}\int \cot u\,du = -\frac{1}{7}\ln|\sin u| + C = -\frac{1}{7}\ln|\sin(3-7x)| + C$

10. $\int \csc(\pi x - 1)\,dx;\ \begin{bmatrix} u = \pi x - 1 \\ du = \pi\,dx \end{bmatrix} \to \int \csc u \cdot \frac{du}{\pi} = \frac{-1}{\pi}\ln|\csc u + \cot u| + C$

$= -\frac{1}{\pi}\ln|\csc(\pi x - 1) + \cot(\pi x - 1)| + C$

11. $\int e^{\theta}\csc(e^{\theta}+1)\,d\theta;\ \begin{bmatrix} u = e^{\theta}+1 \\ du = e^{\theta}\,d\theta \end{bmatrix} \to \int \csc u\,du = -\ln|\csc u + \cot u| + C = -\ln|\csc(e^{\theta}+1) + \cot(e^{\theta}+1)| + C$

12. $\int \frac{\cot(3+\ln x)}{x}\,dx;\ \begin{bmatrix} u = 3+\ln x \\ du = \frac{dx}{x} \end{bmatrix} \to \int \cot u\,du = \ln|\sin u| + C = \ln|\sin(3+\ln x)| + C$

13. $\int \sec\frac{t}{3}\,dt;\ \begin{bmatrix} u = \frac{t}{3} \\ du = \frac{dt}{3} \end{bmatrix} \to \int 3\sec u\,du = 3\ln|\sec u + \tan u| + C = 3\ln\left|\sec\frac{t}{3} + \tan\frac{t}{3}\right| + C$

14. $\int x\sec(x^2-5)\,dx;\ \begin{bmatrix} u = x^2-5 \\ du = 2x\,dx \end{bmatrix} \to \int \frac{1}{2}\sec u\,du = \frac{1}{2}\ln|\sec u + \tan u| + C$

$= \frac{1}{2}\ln|\sec(x^2-5) + \tan(x^2-5)| + C$

15. $\int \csc(s-\pi)\,ds;\ \begin{bmatrix} u = s-\pi \\ du = ds \end{bmatrix} \to \int \csc u\,du = -\ln|\csc u + \cot u| + C = -\ln|\csc(s-\pi) + \cot(s-\pi)| + C$

16. $\int \frac{1}{\theta^2}\csc\frac{1}{\theta}\,d\theta;\ \begin{bmatrix} u = \frac{1}{\theta} \\ du = \frac{-d\theta}{\theta^2} \end{bmatrix} \to \int -\csc u\,du = \ln|\csc u + \cot u| + C = \ln\left|\csc\frac{1}{\theta} + \cot\frac{1}{\theta}\right| + C$

17. $\int_0^{\sqrt{\ln 2}} 2xe^{x^2}\,dx;\ \begin{bmatrix} u = x^2 \\ du = 2x\,dx \\ x = 0 \Rightarrow u = 0,\ x = \sqrt{\ln 2} \Rightarrow u = \ln 2 \end{bmatrix} \to \int_0^{\ln 2} e^u\,du = [e^u]_0^{\ln 2} = e^{\ln 2} - e^0 = 2 - 1 = 1$

18. $\int_{\pi/2}^{\pi} \sin(y)\,e^{\cos y}\,dy;\ \begin{bmatrix} u = \cos y \\ du = -\sin y\,dy \\ x = \frac{\pi}{2} \Rightarrow u = 0,\ x = \pi \Rightarrow u = -1 \end{bmatrix} \to \int_0^{-1} -e^u\,du = \int_{-1}^{0} e^u\,du = [e^u]_{-1}^{0} = 1 - e^{-1} = \frac{e-1}{e}$

19. $\int e^{\tan v}\sec^2 v\,dv;\ \begin{bmatrix} u = \tan v \\ du = \sec^2 v\,dv \end{bmatrix} \to \int e^u\,du = e^u + C = e^{\tan v} + C$

20. $\int \frac{e^{\sqrt{t}}\,dt}{\sqrt{t}};\ \begin{bmatrix} u=\sqrt{t} \\ du=\frac{dt}{2\sqrt{t}} \end{bmatrix} \to \int 2e^u\,du = 2e^u + C = 2e^{\sqrt{t}} + C$

21. $\int 3^{x+1}\,dx;\ \begin{bmatrix} u=x+1 \\ du=dx \end{bmatrix} \to \int 3^u\,du = \left(\frac{1}{\ln 3}\right)3^u + C = \frac{3^{(x+1)}}{\ln 3} + C$

22. $\int \frac{2^{\ln x}}{x}\,dx;\ \begin{bmatrix} u=\ln x \\ du=\frac{dx}{x} \end{bmatrix} \to \int 2^u\,du = \frac{2^u}{\ln 2} + C = \frac{2^{\ln x}}{\ln 2} + C$

23. $\int \frac{2^{\sqrt{w}}\,dw}{2\sqrt{w}};\ \begin{bmatrix} u=\sqrt{w} \\ du=\frac{dw}{2\sqrt{w}} \end{bmatrix} \to \int 2^u\,du = \frac{2^u}{\ln 2} + C = \frac{2^{\sqrt{w}}}{\ln 2} + C$

24. $\int 10^{2\theta}\,d\theta;\ \begin{bmatrix} u=2\theta \\ du=2\,d\theta \end{bmatrix} \to \int \frac{1}{2}10^u\,du = \frac{10^u}{2\ln 10} + C = \frac{1}{2}\left(\frac{10^{2\theta}}{\ln 10}\right) + C$

25. $\int \frac{9\,du}{1+9u^2};\ \begin{bmatrix} x=3u \\ dx=3\,du \end{bmatrix} \to \int \frac{3\,dx}{1+x^2} = 3\tan^{-1}x + C = 3\tan^{-1}3u + C$

26. $\int \frac{4\,dx}{1+(2x+1)^2};\ \begin{bmatrix} u=2x+1 \\ du=2\,dx \end{bmatrix} \to \int \frac{2\,du}{1+u^2} = 2\tan^{-1}u + C = 2\tan^{-1}(2x+1) + C$

27. $\int_0^{1/6} \frac{dx}{\sqrt{1-9x^2}};\ \begin{bmatrix} u=3x \\ du=3\,dx \\ x=0 \Rightarrow u=0,\ x=\frac{1}{6} \Rightarrow u=\frac{1}{2} \end{bmatrix} \to \int_0^{1/2} \frac{1}{3}\frac{du}{\sqrt{1-u^2}} = \left[\frac{1}{3}\sin^{-1}u\right]_0^{1/2} = \frac{1}{3}\left(\frac{\pi}{6}-0\right) = \frac{\pi}{18}$

28. $\int_0^1 \frac{dt}{\sqrt{4-t^2}} = \left[\sin^{-1}\frac{t}{2}\right]_0^1 = \sin^{-1}\left(\frac{1}{2}\right) - 0 = \frac{\pi}{6}$

29. $\int \frac{2s\,ds}{\sqrt{1-s^4}};\ \begin{bmatrix} u=s^2 \\ du=2s\,ds \end{bmatrix} \to \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}s^2 + C$

30. $\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}};\ \begin{bmatrix} u=2\ln x \\ du=\frac{2\,dx}{x} \end{bmatrix} \to \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}(2\ln x) + C$

31. $\int \frac{6\,dx}{x\sqrt{25x^2-1}} = \int \frac{6\,dx}{5x\sqrt{x^2-\frac{1}{25}}} = \frac{6}{5}\cdot 5\sec^{-1}|5x| + C = 6\sec^{-1}|5x| + C$

32. $\int \frac{dr}{r\sqrt{r^2-9}} = \frac{1}{3}\sec^{-1}\left|\frac{r}{3}\right| + C$

33. $\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x\,dx}{e^{2x}+1};\ \begin{bmatrix} u = e^x \\ du = e^x\,dx \end{bmatrix} \to \int \frac{du}{u^2+1} = \tan^{-1}u + C = \tan^{-1}e^x + C$

34. $\int \frac{dy}{\sqrt{e^{2y}-1}} = \int \frac{e^y\,dy}{e^y\sqrt{(e^y)^2-1}};\ \begin{bmatrix} u = e^y \\ du = e^y\,dy \end{bmatrix} \to \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \sec^{-1}e^y + C$

35. $\int_1^{e^{\pi/3}} \frac{dx}{x\cos(\ln x)};\ \begin{bmatrix} u = \ln x \\ du = \frac{dx}{x} \\ x = 1 \Rightarrow u = 0,\ x = e^{\pi/3} \Rightarrow u = \frac{\pi}{3} \end{bmatrix} \to \int_0^{\pi/3} \frac{du}{\cos u} = \int_0^{\pi/3} \sec u\,du = \left[\ln|\sec u + \tan u|\right]_0^{\pi/3}$

$= \ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \ln|\sec 0 + \tan 0| = \ln\left(2+\sqrt{3}\right) - \ln(1) = \ln\left(2+\sqrt{3}\right)$

36. $\int \frac{\ln x\,dx}{x + 4x\ln^2 x} = \int \frac{\ln x\,dx}{x\left(1+4\ln^2 x\right)};\ \begin{bmatrix} u = \ln^2 x \\ du = \frac{2}{x}\ln x\,dx \end{bmatrix} \to \int \frac{1}{2}\frac{du}{1+4u} = \frac{1}{8}\ln|1+4u| + C = \frac{1}{8}\ln\left(1+4\ln^2 x\right) + C$

37. $\int_1^2 \frac{8\,dx}{x^2-2x+2} = 8\int_1^2 \frac{dx}{1+(x-1)^2};\ \begin{bmatrix} u = x-1 \\ du = dx \\ x = 1 \Rightarrow u = 0,\ x = 2 \Rightarrow u = 1 \end{bmatrix} \to 8\int_0^1 \frac{du}{1+u^2} = 8\left[\tan^{-1}u\right]_0^1$

$= 8\left(\tan^{-1}1 - \tan^{-1}0\right) = 8\left(\frac{\pi}{4} - 0\right) = 2\pi$

38. $\int_2^4 \frac{2\,dx}{x^2-6x+10} = 2\int_2^4 \frac{dx}{(x-3)^2+1};\ \begin{bmatrix} u = x-3 \\ du = dx \\ x = 2 \Rightarrow u = -1,\ x = 4 \Rightarrow u = 1 \end{bmatrix} \to 2\int_{-1}^1 \frac{du}{u^2+1} = 2\left[\tan^{-1}u\right]_{-1}^1$

$= 2\left[\tan^{-1}1 - \tan^{-1}(-1)\right] = 2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi$

39. $\int \frac{dt}{\sqrt{-t^2+4t-3}} = \int \frac{dt}{\sqrt{1-(t-2)^2}};\ \begin{bmatrix} u = t-2 \\ du = dt \end{bmatrix} \to \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}(t-2) + C$

40. $\int \frac{d\theta}{\sqrt{2\theta-\theta^2}} = \int \frac{d\theta}{\sqrt{1-(\theta-1)^2}};\ \begin{bmatrix} u = \theta-1 \\ du = d\theta \end{bmatrix} \to \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}(\theta-1) + C$

41. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}}; \left[\begin{array}{l} u = x+1 \\ du = dx \end{array}\right] \to \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \sec^{-1}|x+1| + C,$
$|u| = |x+1| > 1$

42. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}}; \left[\begin{array}{l} u = x-2 \\ du = dx \end{array}\right] \to \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C$
$= \sec^{-1}|x+2| + C, |u| = |x+2| > 1$

43. $\int (\sec x + \cot x)^2\, dx = \int (\sec^2 x + 2 \sec x \cot x + \cot^2 x)\, dx = \int \sec^2 x\, dx + \int 2 \csc x\, dx + \int (\csc^2 x - 1)\, dx$
$= \tan x - 2 \ln|\csc x + \cot x| - \cot x - x + C$

44. $\int (\csc x - \tan x)^2\, dx = \int (\csc^2 x - 2 \csc x \tan x + \tan^2 x)\, dx = \int \csc^2 x\, dx - \int 2 \sec x\, dx + \int (\sec^2 x - 1)\, dx$
$= -\cot x - 2 \ln|\sec x + \tan x| + \tan x - x + C$

45. $\int \csc x \sin 3x\, dx = \int (\csc x)(\sin 2x \cos x + \sin x \cos 2x)\, dx = \int (\csc x)(2 \sin x \cos^2 x + \sin x \cos 2x)\, dx$
$= \int (2 \cos^2 x + \cos 2x)\, dx = \int [(1 + \cos 2x) + \cos 2x]\, dx = \int (1 + 2 \cos 2x)\, dx = x + \sin 2x + C$

46. $\int (\sin 3x \cos 2x - \cos 3x \sin 2x)\, dx = \int \sin(3x - 2x)\, dx = \int \sin x\, dx = -\cos x + C$

47. $\int \frac{x}{x+1}\, dx = \int \left(1 - \frac{1}{x+1}\right) dx = x - \ln|x+1| + C$

48. $\int \frac{x^2}{x^2+1}\, dx = \int \left(1 - \frac{1}{x^2+1}\right) dx = x - \tan^{-1} x + C$

49. $\int_{\sqrt{2}}^{3} \frac{2x^3}{x^2-1}\, dx = \int_{\sqrt{2}}^{3} \left(2x + \frac{2x}{x^2-1}\right) dx = \left[x^2 + \ln|x^2-1|\right]_{\sqrt{2}}^{3} = (9 + \ln 8) - (2 + \ln 1) = 7 + \ln 8$

50. $\int_{-1}^{3} \frac{4x^2-7}{2x+3}\, dx = \int_{-1}^{3} \left[(2x-3) + \frac{2}{2x+3}\right] dx = \left[x^2 - 3x + \ln|2x+3|\right]_{-1}^{3} = (9 - 9 + \ln 9) - (1 + 3 + \ln 1) = \ln 9 - 4$

51. $\int \frac{4t^3 - t^2 + 16t}{t^2+4}\, dt = \int \left[(4t-1) + \frac{4}{t^2+4}\right] dt = 2t^2 - t + 2 \tan^{-1}\left(\frac{t}{2}\right) + C$

52. $\int \frac{2\theta^3 - 7\theta^2 + 7\theta}{2\theta - 5}\, d\theta = \int \left[(\theta^2 - \theta + 1) + \frac{5}{2\theta - 5}\right] d\theta = \frac{\theta^3}{3} - \frac{\theta^2}{2} + \theta + \frac{5}{2} \ln|2\theta - 5| + C$

53. $\int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1} x + \sqrt{1-x^2} + C$

54. $\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx = \int \frac{dx}{2\sqrt{x-1}} + \int \frac{dx}{x} = (x-1)^{1/2} + \ln|x| + C$

55. $\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\,dx = \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\,dx = [\tan x + \sec x]_0^{\pi/4} = (1+\sqrt{2}) - (0+1) = \sqrt{2}$

56. $\int_0^{1/2} \frac{2-8x}{1+4x^2}\,dx = \int_0^{1/2} \left(\frac{2}{1+4x^2} - \frac{8x}{1+4x^2}\right) dx = \left[\tan^{-1}(2x) - \ln\left|1+4x^2\right|\right]_0^{1/2}$
$= (\tan^{-1} 1 - \ln 2) - (\tan^{-1} 0 - \ln 1) = \frac{\pi}{4} - \ln 2$

57. $\int \frac{dx}{1+\sin x} = \int \frac{(1-\sin x)}{(1-\sin^2 x)}\,dx = \int \frac{(1-\sin x)}{\cos^2 x}\,dx = \int (\sec^2 x - \sec x \tan x)\,dx = \tan x - \sec x + C$

58. $1+\cos x = 1+\cos\left(2\cdot\frac{x}{2}\right) = 2\cos^2 \frac{x}{2} \Rightarrow \int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2\left(\frac{x}{2}\right)} = \frac{1}{2}\int \sec^2\left(\frac{x}{2}\right) dx = \tan\frac{x}{2} + C$

59. $\int \frac{1}{\sec\theta + \tan\theta}\,d\theta = \int \frac{\cos\theta}{1+\sin\theta}\,d\theta; \left[\begin{array}{l} u = 1+\sin\theta \\ du = \cos\theta\,d\theta \end{array}\right] \to \int \frac{du}{u} = \ln|u| + C = \ln|1+\sin\theta| + C$

60. $\int \frac{1}{\csc\theta + \cot\theta}\,d\theta = \int \frac{\sin\theta}{1+\cos\theta}\,d\theta; \left[\begin{array}{c} u = 1+\cos\theta \\ du = -\sin\theta\,d\theta \end{array}\right] \to \int \frac{-du}{u} = -\ln|u| + C = -\ln|1+\cos\theta| + C$

61. $\int \frac{1}{1-\sec x}\,dx = \int \frac{\cos x}{\cos x - 1}\,dx = \int \left(1 + \frac{1}{\cos x - 1}\right) dx = \int \left(1 - \frac{1+\cos x}{\sin^2 x}\right) dx = \int \left(1 - \csc^2 x - \frac{\cos x}{\sin^2 x}\right) dx$
$= \int (1 - \csc^2 x - \csc x \cot x)\,dx = x + \cot x + \csc x + C$

62. $\int \frac{1}{1-\csc x}\,dx = \int \frac{\sin x}{\sin x - 1}\,dx = \int \left(1 + \frac{1}{\sin x - 1}\right) dx = \int \left(1 + \frac{\sin x + 1}{(\sin x - 1)(\sin x + 1)}\right) dx$
$= \int \left(1 - \frac{1+\sin x}{\cos^2 x}\right) dx = \int \left(1 - \sec^2 x - \frac{\sin x}{\cos^2 x}\right) dx = \int (1 - \sec^2 x - \sec x \tan x)\,dx = x - \tan x - \sec x + C$

63. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx = \int_0^{2\pi} \left|\sin\frac{x}{2}\right| dx; \left[\begin{array}{c} \sin\frac{x}{2} \geq 0 \\ \text{for } 0 \leq \frac{x}{2} \leq \pi \end{array}\right] \to \int_0^{2\pi} \sin\left(\frac{x}{2}\right) dx = \left[-2\cos\frac{x}{2}\right]_0^{2\pi} = -2(\cos\pi - \cos 0)$
$= (-2)(-2) = 4$

64. $\int_0^{\pi} \sqrt{1-\cos 2x}\, dx = \int_0^{\pi} \sqrt{2}\,|\sin x|\, dx; \begin{bmatrix} \sin x \ge 0 \\ \text{for } 0 \le x \le \pi \end{bmatrix} \rightarrow \sqrt{2}\int_0^{\pi} \sin x\, dx = \left[-\sqrt{2}\cos x\right]_0^{\pi}$
$= -\sqrt{2}(\cos \pi - \cos 0) = 2\sqrt{2}$

65. $\int_{\pi/2}^{\pi} \sqrt{1+\cos 2t}\, dt = \int_{\pi/2}^{\pi} \sqrt{2}\,|\cos t|\, dt; \begin{bmatrix} \cos t \le 0 \\ \text{for } \frac{\pi}{2} \le t \le \pi \end{bmatrix} \rightarrow \int_{\pi/2}^{\pi} -\sqrt{2}\cos t\, dt = \left[-\sqrt{2}\sin t\right]_{\pi/2}^{\pi}$
$= -\sqrt{2}\left(\sin \pi - \sin \frac{\pi}{2}\right) = \sqrt{2}$

66. $\int_{-\pi}^{0} \sqrt{1+\cos t}\, dt = \int_{-\pi}^{0} \sqrt{2}\left|\cos \frac{t}{2}\right| dt; \begin{bmatrix} \cot \frac{t}{2} \ge 0 \\ \text{for } -\pi \le t \le 0 \end{bmatrix} \rightarrow \int_{-\pi}^{0} \sqrt{2}\cos \frac{t}{2}\, dt = \left[2\sqrt{2}\sin \frac{t}{2}\right]_{-\pi}^{0}$
$= 2\sqrt{2}\left[\sin 0 - \sin\left(-\frac{\pi}{2}\right)\right] = 2\sqrt{2}$

67. $\int_{-\pi}^{0} \sqrt{1-\cos^2\theta}\, d\theta = \int_{-\pi}^{0} |\sin \theta|\, d\theta; \begin{bmatrix} \sin \theta \le 0 \\ \text{for } -\pi \le \theta \le 0 \end{bmatrix} \rightarrow \int_{-\pi}^{0} -\sin \theta\, d\theta = [\cos \theta]_{-\pi}^{0} = \cos 0 - \cos(-\pi)$
$= 1 - (-1) = 2$

68. $\int_{\pi/2}^{\pi} \sqrt{1-\sin^2\theta}\, d\theta = \int_{\pi/2}^{\pi} |\cos \theta|\, d\theta; \begin{bmatrix} \cos \theta \le 0 \\ \text{for } \frac{\pi}{2} \le \theta \le \pi \end{bmatrix} \rightarrow \int_{\pi/2}^{\pi} -\cos \theta\, d\theta = [-\sin \theta]_{\pi/2}^{\pi} = -\sin \pi + \sin \frac{\pi}{2} = 1$

69. $\int_{-\pi/4}^{\pi/4} \sqrt{\tan^2 y + 1}\, dy = \int_{-\pi/4}^{\pi/4} |\sec y|\, dy; \begin{bmatrix} \sec y \ge 0 \\ \text{for } -\frac{\pi}{4} \le y \le \frac{\pi}{4} \end{bmatrix} \rightarrow \int_{-\pi/4}^{\pi/4} \sec y\, dy = \left[\ln|\sec y + \tan y|\right]_{-\pi/4}^{\pi/4}$
$= \ln\left|\sqrt{2}+1\right| - \ln\left|\sqrt{2}-1\right|$

70. $\int_{-\pi/4}^{0} \sqrt{\sec^2 y - 1}\, dy = \int_{-\pi/4}^{0} |\tan y|\, dy; \begin{bmatrix} \tan y \le 0 \\ \text{for } -\frac{\pi}{4} \le y \le 0 \end{bmatrix} \rightarrow \int_{-\pi/4}^{0} -\tan y\, dy = \left[\ln|\cos y|\right]_{-\pi/4}^{0} = -\ln\left(\frac{1}{\sqrt{2}}\right)$
$= \ln \sqrt{2}$

71. $\int_{\pi/4}^{3\pi/4} (\csc x - \cot x)^2\, dx = \int_{\pi/4}^{3\pi/4} \left(\csc^2 x - 2\csc x \cot x + \cot^2 x\right) dx = \int_{\pi/4}^{3\pi/4} \left(2\csc^2 x - 1 - 2\csc x \cot x\right) dx$
$= \left[-2\cot x - x + 2\csc x\right]_{\pi/4}^{3\pi/4} = \left(-2\cot \frac{3\pi}{4} - \frac{3\pi}{4} + 2\csc \frac{3\pi}{4}\right) - \left(-2\cot \frac{\pi}{4} - \frac{\pi}{4} + 2\csc \frac{\pi}{4}\right)$
$= \left[-2(-1) - \frac{3\pi}{4} + 2\left(\sqrt{2}\right)\right] - \left[-2(1) - \frac{\pi}{4} + 2\left(\sqrt{2}\right)\right] = 4 - \frac{\pi}{2}$

72. $\int_0^{\pi/4} (\sec x + 4 \cos x)^2\, dx = \int_0^{\pi/4} \left[\sec^2 x + 8 + 16\left(\frac{1+\sin 2x}{2}\right)\right] dx = \left[\tan x + 16x - 4 \cos 2x\right]_0^{\pi/4}$
$= \left(\tan \frac{\pi}{4} + 4\pi - 4 \cos \frac{\pi}{2}\right) - (\tan 0 + 0 - 4 \cos 0) = 5 + 4\pi$

73. $\int \cos \theta \csc (\sin \theta)\, d\theta; \begin{bmatrix} u = \sin \theta \\ du = \cos \theta\, d\theta \end{bmatrix} \to \int \csc u\, du = -\ln |\csc u + \cot u| + C$
$= -\ln |\csc (\sin \theta) + \cot (\sin \theta)| + C$

74. $\int \left(1 + \frac{1}{x}\right) \cot (x + \ln x)\, dx; \begin{bmatrix} u = x + \ln x \\ du = \left(1 + \frac{1}{x}\right) dx \end{bmatrix} \to \int \cot u\, du = \ln |\sin u| + C = \ln |\sin (x + \ln x)| + C$

75. $\int (\csc x - \sec x)(\sin x + \cos x)\, dx = \int (1 + \cot x - \tan x - 1)\, dx = \int \cot x\, dx - \int \tan x\, dx$
$= \ln |\sin x| + \ln |\cos x| + C$

76. $\int 3 \sinh\left(\frac{x}{2} + \ln 5\right) dx = \begin{bmatrix} u = \frac{x}{2} + \ln 5 \\ 2\, du = dx \end{bmatrix} = 6 \int \sinh u\, du = 6 \cosh u + C = 6 \cosh\left(\frac{x}{2} + \ln 5\right) + C$

77. $\int \frac{6\, dy}{\sqrt{y}(1+y)}; \begin{bmatrix} u = \sqrt{y} \\ du = \frac{1}{2\sqrt{y}}\, dy \end{bmatrix} \to \int \frac{12\, du}{1+u^2} = 12 \tan^{-1} u + C = 12 \tan^{-1} \sqrt{y} + C$

78. $\int \frac{dx}{x\sqrt{4x^2 - 1}} = \int \frac{2\, dx}{2x\sqrt{(2x)^2 - 1}}; \begin{bmatrix} u = 2x \\ du = 2\, dx \end{bmatrix} \to \int \frac{du}{u\sqrt{u^2 - 1}} = \sec^{-1} |u| + C = \sec^{-1} |2x| + C$

79. $\int \frac{7\, dx}{(x-1)\sqrt{x^2 - 2x - 48}} = \int \frac{7\, dx}{(x-1)\sqrt{(x-1)^2 - 49}}; \begin{bmatrix} u = x - 1 \\ du = dx \end{bmatrix} \to \int \frac{7\, du}{u\sqrt{u^2 - 49}} = 7 \cdot \frac{1}{7} \sec^{-1} \left|\frac{u}{7}\right| + C$
$= \sec^{-1} \left|\frac{x-1}{7}\right| + C$

80. $\int \frac{dx}{(2x+1)\sqrt{4x^2 + 4x}} = \int \frac{dx}{(2x+1)\sqrt{(2x+1)^2 - 1}}; \begin{bmatrix} u = 2x + 1 \\ du = 2\, dx \end{bmatrix} \to \int \frac{du}{2u\sqrt{u^2 - 1}} = \frac{1}{2} \sec^{-1} |u| + C$
$= \frac{1}{2} \sec^{-1} |2x + 1| + C$

81. $\int \sec^2 t \tan (\tan t)\, dt; \begin{bmatrix} u = \tan t \\ du = \sec^2 t\, dt \end{bmatrix} \to \int \tan u\, du = -\ln |\cos u| + C = \ln |\sec u| + C = \ln |\sec (\tan t)| + C$

82. $\int \frac{dx}{x\sqrt{3 + x^2}} = -\frac{1}{\sqrt{3}} \operatorname{csch}^{-1}\left|\frac{x}{\sqrt{3}}\right| + C$ (from Table 6.15)

83. (a) $\int \cos^3\theta\, d\theta = \int (\cos\theta)(1-\sin^2\theta)\, d\theta; \begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \to \int (1-u^2)\, du = u - \frac{u^3}{3} + C = \sin\theta - \frac{1}{3}\sin^3\theta + C$

(b) $\int \cos^5\theta\, d\theta = \int (\cos\theta)(1-\sin^2\theta)^2\, d\theta = \int (1-u^2)^2\, du = \int (1-2u^2+u^4)\, du = u - \frac{2}{3}u^3 + \frac{u^5}{5} + C$
$= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + C$

(c) $\int \cos^9\theta\, d\theta = \int (\cos^8\theta)(\cos\theta)\, d\theta = \int (1-\sin^2\theta)^4(\cos\theta)\, d\theta$

84. (a) $\int \sin^3\theta\, d\theta = \int (1-\cos^2\theta)(\sin\theta)\, d\theta; \begin{bmatrix} u = \cos\theta \\ du = -\sin\theta\, d\theta \end{bmatrix} \to \int (1-u^2)(-du) = \frac{u^3}{3} - u + C$
$= -\cos\theta + \frac{1}{3}\cos^3\theta + C$

(b) $\int \sin^5\theta\, d\theta = \int (1-\cos^2\theta)^2(\sin\theta)\, d\theta = \int (1-u^2)^2(-du) = \int (-1+2u^2-u^4)\, du$
$= -\cos\theta + \frac{2}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta + C$

(c) $\int \sin^7\theta\, d\theta = \int (1-u^2)^3(-du) = \int (-1+3u^2-3u^4+u^6)\, du = -\cos\theta + \cos^3\theta - \frac{3}{5}\cos^5\theta + \frac{\cos^7\theta}{7} + C$

(d) $\int \sin^{13}\theta\, d\theta = \int (\sin^{12}\theta)(\sin\theta)\, d\theta = \int (1-\cos^2\theta)^6(\sin\theta)\, d\theta$

85. (a) $\int \tan^3\theta\, d\theta = \int (\sec^2\theta - 1)(\tan\theta)\, d\theta = \int \sec^2\theta\tan\theta\, d\theta - \int \tan\theta\, d\theta = \frac{1}{2}\tan^2\theta - \int \tan\theta\, d\theta$
$= \frac{1}{2}\tan^2\theta + \ln|\cos\theta| + C$

(b) $\int \tan^5\theta\, d\theta = \int (\sec^2\theta - 1)(\tan^3\theta)\, d\theta = \int \tan^3\theta\sec^2\theta\, d\theta - \int \tan^3\theta\, d\theta = \frac{1}{4}\tan^4\theta - \int \tan^3\theta\, d\theta$

(c) $\int \tan^7\theta\, d\theta = \int (\sec^2\theta - 1)(\tan^5\theta)\, d\theta = \int \tan^5\theta\sec^2\theta\, d\theta - \int \tan^5\theta\, d\theta = \frac{1}{6}\tan^6\theta - \int \tan^5\theta\, d\theta$

(d) $\int \tan^{2k+1}\theta\, d\theta = \int (\sec^2\theta - 1)(\tan^{2k-1}\theta)\, d\theta = \int \tan^{2k-1}\theta\sec^2\theta\, d\theta - \int \tan^{2k-1}\theta\, d\theta;$

$\begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{bmatrix} \to \int u^{2k-1}\, du - \int \tan^{2k-1}\theta\, d\theta = \frac{1}{2k}u^{2k} - \int \tan^{2k-1}\theta\, d\theta = \frac{1}{2k}\tan^{2k}\theta - \int \tan^{2k-1}\theta\, d\theta$

86. (a) $\int \cot^3\theta\, d\theta = \int (\csc^2\theta - 1)(\cot\theta)\, d\theta = \int \cot\theta\csc^2\theta\, d\theta - \int \cot\theta\, d\theta = -\frac{1}{2}\cot^2\theta - \int \cot\theta\, d\theta$
$= -\frac{1}{2}\cot^2\theta - \ln|\sin\theta| + C$

(b) $\int \cot^5\theta\, d\theta = \int (\csc^2\theta - 1)(\cot^3\theta)\, d\theta = \int \cot^3\theta\csc^2\theta\, d\theta - \int \cot^3\theta\, d\theta = -\frac{1}{4}\cot^4\theta - \int \cot^3\theta\, d\theta$

(c) $\int \cot^7\theta\, d\theta = \int (\csc^2\theta - 1)(\cot^5\theta)\, d\theta = \int \cot^5\theta\csc^2\theta\, d\theta - \int \cot^5\theta\, d\theta = -\frac{1}{6}\cot^6\theta - \int \cot^5\theta\, d\theta$

(d) $\int \cot^{2k+1}\theta\, d\theta = \int (\csc^2\theta - 1)(\cot^{2k-1}\theta)\, d\theta = \int \cot^{2k-1}\theta \csc^2\theta\, d\theta - \int \cot^{2k-1}\theta\, d\theta;$

$\left[\begin{array}{c} u = \cot\theta \\ du = -\csc^2\theta\, d\theta \end{array}\right] \to -\int u^{2k-1}\, du - \int \cot^{2k-1}\theta\, d\theta = -\frac{1}{2k}u^{2k} - \int \cot^{2k-1}\theta\, d\theta$

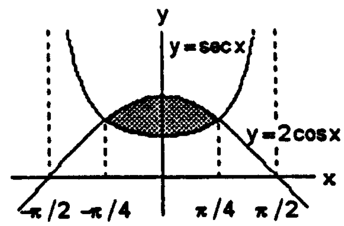
$= -\frac{1}{2k}\cot^{2k}\theta - \int \cot^{2k-1}\theta\, d\theta$

87. $A = \int_{-\pi/4}^{\pi/4} (2\cos x - \sec x)\, dx = [2\sin x - \ln|\sec x + \tan x|]_{-\pi/4}^{\pi/4}$

$= [\sqrt{2} - \ln(\sqrt{2}+1)] - [-\sqrt{2} - \ln(\sqrt{2}-1)]$

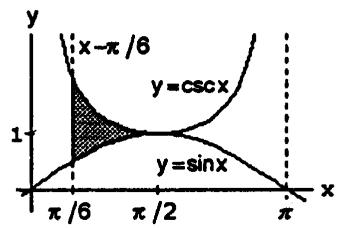
$= 2\sqrt{2} - \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2\sqrt{2} - \ln\left(\frac{(\sqrt{2}+1)^2}{2-1}\right)$

$= 2\sqrt{2} - \ln(3 + 2\sqrt{2})$



88. $A = \int_{\pi/6}^{\pi/2} (\csc x - \sin x)\, dx = [-\ln|\csc x + \cot x| + \cos x]_{\pi/6}^{\pi/2}$

$= -\ln|1+0| + \ln|2+\sqrt{3}| - \frac{\sqrt{3}}{2} = \ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}$



89. $V = \int_{-\pi/4}^{\pi/4} \pi(2\cos x)^2\, dx - \int_{-\pi/4}^{\pi/4} \pi\sec^2 x\, dx = 4\pi \int_{-\pi/4}^{\pi/4} \cos^2 x\, dx - \pi \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx$

$= 2\pi \int_{-\pi/4}^{\pi/4} (1 + \cos 2x)\, dx - \pi[\tan x]_{-\pi/4}^{\pi/4} = 2\pi\left[x + \frac{1}{2}\sin 2x\right]_{-\pi/4}^{\pi/4} - \pi[1 - (-1)]$

$= 2\pi\left[\left(\frac{\pi}{4} + \frac{1}{2}\right) - \left(-\frac{\pi}{4} - \frac{1}{2}\right)\right] - 2\pi = 2\pi\left(\frac{\pi}{2} + 1\right) - 2\pi = \pi^2$

90. $V = \int_{\pi/6}^{\pi/2} \pi\csc^2 x\, dx - \int_{\pi/6}^{\pi/2} \pi\sin^2 x\, dx = \pi \int_{\pi/6}^{\pi/2} \csc^2 x\, dx - \frac{\pi}{2} \int_{\pi/6}^{\pi/2} (1 - \cos 2x)\, dx$

$= \pi[-\cot x]_{\pi/6}^{\pi/2} - \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\pi/6}^{\pi/2} = \pi[0 - (-\sqrt{3})] - \frac{\pi}{2}\left[\left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\right]$

$= \pi\sqrt{3} - \frac{\pi}{2}\left(\frac{2\pi}{6} + \frac{\sqrt{3}}{4}\right) = \pi\left(\frac{7\sqrt{3}}{8} - \frac{\pi}{6}\right)$

91. $y = \ln(\cos x) \Rightarrow \frac{dy}{dx} = -\frac{\sin x}{\cos x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x = \sec^2 x - 1;\ L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$= \int_0^{\pi/3} \sqrt{1+\left(\sec^2 x - 1\right)}\,dx = \int_0^{\pi/3} \sec x\,dx = \left[\ln|\sec x + \tan x|\right]_0^{\pi/3} = \ln\left|2+\sqrt{3}\right| - \ln|1+0| = \ln\left(2+\sqrt{3}\right)$

92. $y = \ln(\sec x) \Rightarrow \frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x = \sec^2 x - 1;\ L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$= \int_0^{\pi/4} \sec x\,dx = \left[\ln|\sec x + \tan x|\right]_0^{\pi/4} = \ln\left|\sqrt{2}+1\right| - \ln|1+0| = \ln\left(\sqrt{2}+1\right)$

93. $\int \csc x\,dx = \int (\csc x)(1)\,dx = \int (\csc x)\left(\frac{\csc x + \cot x}{\csc x + \cot x}\right)dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\,dx;$

$\begin{bmatrix} u = \csc x + \cot x \\ du = \left(-\csc x \cot x - \csc^2 x\right)dx \end{bmatrix} \rightarrow \int \frac{-du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C$

94. $\left[\left(x^2-1\right)(x+1)\right]^{-2/3} = \left[(x-1)(x+1)^2\right]^{-2/3} = (x-1)^{-2/3}(x+1)^{-4/3} = (x+1)^{-2}\left[(x-1)^{-2/3}(x+1)^{2/3}\right]$

$= (x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} = (x+1)^{-2}\left(1-\frac{2}{x+1}\right)^{-2/3}$

(a) $\int \left[\left(x^2-1\right)(x+1)\right]^{-2/3} dx = \int (x+1)^{-2}\left(1-\frac{2}{x+1}\right)^{-2/3} dx;\ \begin{bmatrix} u = \frac{1}{x+1} \\ du = -\frac{1}{(x+1)^2}\,dx \end{bmatrix}$

$\rightarrow \int -(1-2u)^{-2/3}\,du = \frac{3}{2}(1-2u)^{1/3} + C = \frac{3}{2}\left(1-\frac{2}{x+1}\right)^{1/3} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(b) $\int \left[\left(x^2-1\right)(x+1)\right]^{-2/3} dx = \int (x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} dx;\ u = \left(\frac{x-1}{x+1}\right)^k$

$\Rightarrow du = k\left(\frac{x-1}{x+1}\right)^{k-1}\frac{[(x+1)-(x-1)]}{(x+1)^2}\,dx = 2k\frac{(x-1)^{k-1}}{(x+1)^{k+1}}\,dx;\ dx = \frac{(x+1)^2}{2k}\left(\frac{x+1}{x-1}\right)^{k-1} du$

$= \frac{(x+1)^2}{2k}\left(\frac{x-1}{x+1}\right)^{1-k} du;$ then, $\int \left(\frac{x-1}{x+1}\right)^{-2/3}\frac{1}{2k}\left(\frac{x-1}{x+1}\right)^{1-k} du = \frac{1}{2k}\int \left(\frac{x-1}{x+1}\right)^{(1/3-k)} du$

$= \frac{1}{2k}\int \left(\frac{x-1}{x+1}\right)^{k(1/3k-1)} du = \frac{1}{2k}\int u^{(1/3k-1)}\,du = \frac{1}{2k}(3k)\,u^{1/3k} + C = \frac{3}{2}u^{1/3k} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(c) $\int \left[\left(x^2-1\right)(x+1)\right]^{-2/3} dx = \int (x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} dx;$

$\begin{bmatrix} u = \tan^{-1} x \\ x = \tan u \\ dx = \frac{du}{\cos^2 u} \end{bmatrix} \rightarrow \int \frac{1}{(\tan u + 1)^2}\left(\frac{\tan u - 1}{\tan u + 1}\right)^{-2/3}\left(\frac{du}{\cos^2 u}\right) = \int \frac{1}{(\sin u + \cos u)^2}\left(\frac{\sin u - \cos u}{\sin u + \cos u}\right)^{-2/3} du;$

$$\left[\begin{array}{l}\sin u+\cos u=\sin u+\sin\left(\frac{\pi}{2}-u\right)=2\sin\frac{\pi}{4}\cos\left(u-\frac{\pi}{4}\right)\\ \sin u-\cos u=\sin u-\sin\left(\frac{\pi}{2}-u\right)=2\cos\frac{\pi}{4}\sin\left(u-\frac{\pi}{4}\right)\end{array}\right]\to\int\frac{1}{2\cos^2\left(u-\frac{\pi}{4}\right)}\left[\frac{\sin\left(u-\frac{\pi}{4}\right)}{\cos\left(u-\frac{\pi}{4}\right)}\right]^{-2/3}du$$

$$=\frac{1}{2}\int\tan^{-2/3}\left(u-\frac{\pi}{4}\right)\sec^2\left(u-\frac{\pi}{4}\right)du=\frac{3}{2}\tan^{1/3}\left(u-\frac{\pi}{4}\right)+C=\frac{3}{2}\left[\frac{\tan u-\tan\frac{\pi}{4}}{1+\tan u\tan\frac{\pi}{4}}\right]^{1/3}+C$$

$$=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3}+C$$

(d) $u=\tan^{-1}\sqrt{x}\Rightarrow\tan u=\sqrt{x}\Rightarrow\tan^2u=x\Rightarrow dx=2\tan u\left(\frac{1}{\cos^2u}\right)du=\frac{2\sin u}{\cos^3u}du=-\frac{2d(\cos u)}{\cos^3u}$;

$$x-1=\tan^2u-1=\frac{\sin^2u-\cos^2u}{\cos^2u}=\frac{1-2\cos^2u}{\cos^2u};\ x+1=\tan^2u+1=\frac{\cos^2u+\sin^2u}{\cos^2u}=\frac{1}{\cos^2u};$$

$$\int(x-1)^{-2/3}(x+1)^{-4/3}dx=\int\frac{\left(1-2\cos^2u\right)^{-2/3}}{\left(\cos^2u\right)^{-2/3}}\cdot\frac{1}{\left(\cos^2u\right)^{-4/3}}\cdot\frac{-2d(\cos u)}{\cos^3u}$$

$$=\int\left(1-2\cos^2u\right)^{-2/3}\cdot(-2)\cdot\cos u\cdot d(\cos u)=\frac{1}{2}\int\left(1-2\cos^2u\right)^{-2/3}\cdot d\left(1-2\cos^2u\right)$$

$$=\frac{3}{2}\left(1-2\cos^2u\right)^{1/3}+C=\frac{3}{2}\left[\frac{\left(\frac{1-2\cos^2u}{\cos^2u}\right)}{\left(\frac{1}{\cos^2u}\right)}\right]^{1/3}+C=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3}+C$$

(e) $u=\tan^{-1}\left(\frac{x-1}{2}\right)\Rightarrow\frac{x-1}{2}=\tan u\Rightarrow x+1=2(\tan u+1)\Rightarrow dx=\frac{2\,du}{\cos^2u}=2d(\tan u)$;

$$\int(x-1)^{-2/3}(x+1)^{-4/3}dx=\int(\tan u)^{-2/3}(\tan u+1)^{-4/3}\cdot2^{-2}\cdot2\cdot d(\tan u)$$

$$=\frac{1}{2}\int\left(1-\frac{1}{\tan u+1}\right)^{-2/3}d\left(1-\frac{1}{\tan u+1}\right)=\frac{3}{2}\left(1-\frac{1}{\tan u+1}\right)^{1/3}+C=\frac{3}{2}\left(1-\frac{2}{x+1}\right)^{1/3}+C$$

$$=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3}+C$$

(f) $\left[\begin{array}{c}u=\cos^{-1}x\\ x=\cos u\\ dx=-\sin u\,du\end{array}\right]\to-\int\frac{\sin u\,du}{\sqrt[3]{\left(\cos^2u-1\right)^2(\cos u+1)^2}}=-\int\frac{\sin u\,du}{\left(\sin^{4/3}u\right)\left(4\cos\frac{u}{2}\right)^{4/3}}$

$$=-\int\frac{du}{(\sin u)^{1/3}\left(4\cos\frac{u}{2}\right)^{4/3}}=-\int\frac{du}{2\left(\sin\frac{u}{2}\right)^{4/3}\left(\cos\frac{u}{2}\right)^{5/3}}=-\frac{1}{2}\int\left(\frac{\cos\frac{u}{2}}{\sin\frac{u}{2}}\right)^{1/3}\frac{du}{\left(\cos^2\frac{u}{2}\right)}$$

$$=-\int\tan^{-1/3}\left(\frac{u}{2}\right)d\left(\tan\frac{u}{2}\right)=-\frac{3}{2}\tan^{2/3}\frac{u}{2}+C=\frac{3}{2}\left(-\tan^2\frac{u}{2}\right)^{1/3}+C=\frac{3}{2}\left(\frac{\cos u-1}{\cos u+1}\right)^{1/3}+C$$

$$=\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3}+C$$

(g) $\int \left[(x^2-1)(x+1)\right]^{-2/3} dx;\ \begin{bmatrix} u = \cosh^{-1} x \\ x = \cosh u \\ dx = \sinh u \end{bmatrix} \rightarrow \int \frac{\sinh u\, du}{\sqrt[3]{(\cosh^2 u - 1)^2 (\cosh u + 1)^2}}$

$$= \int \frac{\sinh u\, du}{\sqrt[3]{(\sinh^4 u)(\cosh u + 1)^2}} = \int \frac{du}{\sqrt[3]{(\sinh u)\left(4\cosh^4 \frac{u}{2}\right)}} = \frac{1}{2}\int \frac{du}{\sqrt[3]{\sinh\left(\frac{u}{2}\right)\cosh^5\left(\frac{u}{2}\right)}}$$

$$= \int \left(\tanh \frac{u}{2}\right)^{-1/3} d\left(\tanh \frac{u}{2}\right) = \frac{3}{2}\left(\tanh \frac{u}{2}\right)^{2/3} + C = \frac{3}{2}\left(\frac{\cosh x - 1}{\cosh x + 1}\right)^{1/3} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$$

7.2 INTEGRATION BY PARTS

1. $u = x,\ du = dx;\ dv = \sin \frac{x}{2}\, dx,\ v = -2\cos \frac{x}{2};$

$$\int x \sin \frac{x}{2}\, dx = -2x \cos \frac{x}{2} - \int \left(-2\cos \frac{x}{2}\right) dx = -2x\cos\left(\frac{x}{2}\right) + 4\sin\left(\frac{x}{2}\right) + C$$

2. $u = \theta,\ du = d\theta;\ dv = \cos \pi\theta\, d\theta,\ v = \frac{1}{\pi}\sin \pi\theta;$

$$\int \theta \cos \pi\theta\, d\theta = \frac{\theta}{\pi}\sin \pi\theta - \int \frac{1}{\pi}\sin \pi\theta\, d\theta = \frac{\theta}{\pi}\sin \pi\theta + \frac{1}{\pi^2}\cos \pi\theta + C$$

3.
$$\begin{array}{lll} & & \cos t \\ t^2 & \xrightarrow{(+)} & \sin t \\ 2t & \xrightarrow{(-)} & -\cos t \\ 2 & \xrightarrow{(+)} & -\sin t \\ 0 & & \end{array}$$

$$\int t^2 \cos t\, dt = t^2 \sin t + 2t\cos t - 2\sin t + C$$

4.
$$\begin{array}{lll} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$

$$\int x^2 \sin x\, dx = -x^2\cos x + 2x\sin x + 2\cos x + C$$

5. $u = \ln x,\ du = \frac{dx}{x};\ dv = x\, dx,\ v = \frac{x^2}{2};$

$$\int_1^2 x \ln x\, dx = \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2 \frac{x^2}{2}\frac{dx}{x} = 2\ln 2 - \left[\frac{x^2}{4}\right]_1^2 = 2\ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$$

6. $u = \ln x,\ du = \frac{dx}{x};\ dv = x^3\,dx,\ v = \frac{x^4}{4};$

$$\int_1^e x^3 \ln x\,dx = \left[\frac{x^4}{4}\ln x\right]_1^e - \int_1^e \frac{x^4}{4}\frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16}\right]_1^e = \frac{3e^4+1}{16}$$

7. $u = \tan^{-1} y,\ du = \frac{dy}{1+y^2};\ dv = dy,\ v = y;$

$$\int \tan^{-1} y\,dy = y\tan^{-1} y - \int \frac{y\,dy}{(1+y^2)} = y\tan^{-1} y - \tfrac{1}{2}\ln(1+y^2) + C = y\tan^{-1} y - \ln\sqrt{1+y^2} + C$$

8. $u = \sin^{-1} y,\ du = \frac{dy}{\sqrt{1-y^2}};\ dv = dy,\ v = y;$

$$\int \sin^{-1} y\,dy = y\sin^{-1} y - \int \frac{y\,dy}{\sqrt{1-y^2}} = y\sin^{-1} y + \sqrt{1-y^2} + C$$

9. $u = x,\ du = dx;\ dv = \sec^2 x\,dx,\ v = \tan x;$

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos x| + C$$

10. $\int 4x\sec^2 2x\,dx;\ [y = 2x] \rightarrow \int y\sec^2 y\,dy = y\tan y - \int \tan y\,dy = y\tan y - \ln|\sec y| + C$

$= 2x\tan 2x - \ln|\sec 2x| + C$

11. $\qquad e^x$

$x^3 \xrightarrow{(+)} e^x$

$3x^2 \xrightarrow{(-)} e^x$

$6x \xrightarrow{(+)} e^x$

$6 \xrightarrow{(-)} e^x$

0

$$\int x^3e^x\,dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = (x^3 - 3x^2 + 6x - 6)e^x + C$$

12. $\qquad e^{-p}$

$p^4 \xrightarrow{(+)} -e^{-p}$

$4p^3 \xrightarrow{(-)} e^{-p}$

$12p^2 \xrightarrow{(+)} -e^{-p}$

$24p \xrightarrow{(-)} e^{-p}$

$24 \xrightarrow{(+)} e^{-p}$

0

$$\int p^4e^{-p}\,dp = -p^4e^{-p} - 4p^3e^{-p} - 12p^2e^{-p} - 24pe^{-p} - 24e^{-p} + C$$

$$= (-p^4 - 4p^3 - 12p^2 - 24p - 24)e^{-p} + C$$

13. e^x

$x^2 - 5x \xrightarrow{(+)} e^x$

$2x - 5 \xrightarrow{(-)} e^x$

$2 \xrightarrow{(+)} e^x$

0

$$\int (x^2 - 5x)e^x\,dx = (x^2 - 5x)e^x - (2x - 5)e^x + 2e^x + C = x^2e^x - 7xe^x + 7e^x + C$$
$$= (x^2 - 7x + 7)e^x + C$$

14. e^r

$r^2 + r + 1 \xrightarrow{(+)} e^r$

$2r + 1 \xrightarrow{(-)} e^r$

$2 \xrightarrow{(+)} e^r$

0

$$\int (r^2 + r + 1)e^r\,dr = (r^2 + r + 1)e^r - (2r + 1)\,e^r + 2e^r + C$$
$$= \left[(r^2 + r + 1) - (2r + 1) + 2\right]e^r + C = (r^2 - r + 2)e^r + C$$

15. e^x

$x^5 \xrightarrow{(+)} e^x$

$5x^4 \xrightarrow{(-)} e^x$

$20x^3 \xrightarrow{(+)} e^x$

$60x^2 \xrightarrow{(-)} e^x$

$120x \xrightarrow{(+)} e^x$

$120 \xrightarrow{(-)} e^x$

0

$$\int x^5e^x\,dx = x^5e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120xe^x - 120e^x + C$$
$$= (x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)e^x + C$$

16. e^{4t}

$t^2 \xrightarrow{(+)} \frac{1}{4}e^{4t}$

$2t \xrightarrow{(-)} \frac{1}{16}e^{4t}$

$2 \xrightarrow{(+)} \frac{1}{64}e^{4t}$

0

$$\int t^2e^{4t}\,dt = \frac{t^2}{4}e^{4t} - \frac{2t}{16}e^{4t} + \frac{2}{64}e^{4t} + C = \frac{t^2}{4}e^{4t} - \frac{t}{8}e^{4t} + \frac{1}{32}e^{4t} + C$$
$$\left(\frac{t^2}{4} - \frac{t}{8} + \frac{1}{32}\right)e^{4t} + C$$

17. $\sin 2\theta$

$\theta^2 \xrightarrow{(+)} -\frac{1}{2}\cos 2\theta$

$2\theta \xrightarrow{(-)} -\frac{1}{4}\sin 2\theta$

$2 \xrightarrow{(+)} \frac{1}{8}\cos 2\theta$

0

$$\int_0^{\pi/2} \theta^2 \sin 2\theta \, d\theta = \left[-\frac{\theta^2}{2}\cos 2\theta + \frac{\theta}{2}\sin 2\theta + \frac{1}{4}\cos 2\theta\right]_0^{\pi/2}$$

$$= \left[-\frac{\pi^2}{8}\cdot(-1) + \frac{\pi}{4}\cdot 0 + \frac{1}{4}\cdot(-1)\right] - \left[0 + 0 + \frac{1}{4}\cdot 1\right] = \frac{\pi^2}{8} - \frac{1}{2} = \frac{\pi^2 - 4}{8}$$

18. $\cos 2x$

$x^3 \xrightarrow{(+)} \frac{1}{2}\sin 2x$

$3x^2 \xrightarrow{(-)} -\frac{1}{4}\cos 2x$

$6x \xrightarrow{(+)} -\frac{1}{8}\sin 2x$

$6 \xrightarrow{(-)} \frac{1}{16}\cos 2x$

0

$$\int_0^{\pi/2} x^3 \cos 2x \, dx = \left[\frac{x^3}{2}\sin 2x + \frac{3x^2}{4}\cos 2x - \frac{3x}{4}\sin 2x - \frac{3}{8}\cos 2x\right]_0^{\pi/2}$$

$$= \left[\frac{\pi^3}{16}\cdot 0 + \frac{3\pi^2}{16}\cdot(-1) - \frac{3\pi}{8}\cdot 0 - \frac{3}{8}\cdot(-1)\right] - \left[0 + 0 - 0 - \frac{3}{8}\cdot 1\right] = -\frac{3\pi^2}{16} + \frac{3}{4} = \frac{3(4-\pi^2)}{16}$$

19. $u = \sec^{-1} t$, $du = \frac{dt}{t\sqrt{t^2-1}}$; $dv = t\,dt$, $v = \frac{t^2}{2}$;

$$\int_{2/\sqrt{3}}^{2} t \sec^{-1} t \, dt = \left[\frac{t^2}{2}\sec^{-1} t\right]_{2/\sqrt{3}}^{2} - \int_{2/\sqrt{3}}^{2} \left(\frac{t^2}{2}\right)\frac{dt}{t\sqrt{t^2-1}} = \left(2\cdot\frac{\pi}{3} - \frac{2}{3}\cdot\frac{\pi}{6}\right) - \int_{2/\sqrt{3}}^{2} \frac{t\,dt}{2\sqrt{t^2-1}}$$

$$= \frac{5\pi}{9} - \left[\frac{1}{2}\sqrt{t^2-1}\right]_{2/\sqrt{3}}^{2} = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \sqrt{\frac{4}{3}-1}\right) = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right) = \frac{5\pi}{9} - \frac{\sqrt{3}}{3} = \frac{5\pi - 3\sqrt{3}}{9}$$

20. $u = \sin^{-1}(x^2)$, $du = \frac{2x\,dx}{\sqrt{1-x^4}}$; $dv = 2x\,dx$, $v = x^2$;

$$\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx = \left[x^2 \sin^{-1}(x^2)\right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2 \cdot \frac{2x\,dx}{\sqrt{1-x^4}} = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) + \int_0^{1/\sqrt{2}} \frac{d(1-x^4)}{2\sqrt{1-x^4}}$$

$$= \frac{\pi}{12} + \left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}} = \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12}$$

21. $I = \int e^\theta \sin\theta\, d\theta$; $[u = \sin\theta,\ du = \cos\theta\, d\theta;\ dv = e^\theta\, d\theta,\ v = e^\theta] \Rightarrow I = e^\theta \sin\theta - \int e^\theta \cos\theta\, d\theta$;

$[u = \cos\theta,\ du = -\sin\theta\, d\theta;\ dv = e^\theta\, d\theta,\ v = e^\theta] \Rightarrow I = e^\theta \sin\theta - \left(e^\theta \cos\theta + \int e^\theta \sin\theta\, d\theta\right)$

$= e^\theta \sin\theta - e^\theta \cos\theta - I + C' \Rightarrow 2I = (e^\theta \sin\theta - e^\theta \cos\theta) + C' \Rightarrow I = \frac{1}{2}(e^\theta \sin\theta - e^\theta \cos\theta) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

22. $I = \int e^{-y} \cos y \, dy$; $[u = \cos y,\ du = -\sin y \, dy;\ dv = e^{-y} \, dy,\ v = -e^{-y}]$

$\Rightarrow I = -e^{-y} \cos y - \int (-e^{-y})(-\sin y) \, dy = -e^{-y} \cos y - \int e^{-y} \sin y \, dy$; $[u = \sin y,\ du = \cos y \, dy;$

$dv = e^{-y} \, dy,\ v = -e^{-y}] \Rightarrow I = -e^{-y} \cos y - \left(-e^{-y} \sin y - \int (-e^{y}) \cos y \, dy\right) = -e^{-y} \cos y + e^{-y} \sin y - I + C'$

$\Rightarrow 2I = e^{-y}(\sin y - \cos y) + C' \Rightarrow I = \frac{1}{2}(e^{-y} \sin y - e^{-y} \cos y) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

23. $I = \int e^{2x} \cos 3x \, dx$; $\left[u = \cos 3x;\ du = -3 \sin 3x \, dx,\ dv = e^{2x} \, dx;\ v = \frac{1}{2}e^{2x}\right]$

$\Rightarrow I = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{2} \int e^{2x} \sin 3x \, dx$; $\left[u = \sin 3x,\ du = 3 \cos 3x,\ dv = e^{2x} \, dx;\ v = \frac{1}{2}e^{2x}\right]$

$\Rightarrow I = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{2}\left(\frac{1}{2}e^{2x} \sin 3x - \frac{3}{2} \int e^{2x} \cos 3x \, dx\right) = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x - \frac{9}{4}I + C'$

$\Rightarrow \frac{13}{4}I = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{4}e^{2x} \sin 3x + C' \Rightarrow \frac{e^{2x}}{13}(3 \sin 3x + 2 \cos 3x) + C$, where $C = \frac{4}{13}C'$

24. $\int e^{-2x} \sin 2x \, dx$; $[y = 2x] \to \frac{1}{2} \int e^{-y} \sin y \, dy = I$; $[u = \sin y,\ du = \cos y \, dy;\ dv = e^{-y} \, dy,\ v = -e^{-y}]$

$\Rightarrow I = \frac{1}{2}\left(-e^{-y} \sin y + \int e^{-y} \cos y \, dy\right)[u = \cos u,\ du = -\sin y;\ dv = e^{-y} \, dy,\ v = -e^{-y}]$

$\Rightarrow I = -\frac{1}{2}e^{-y} \sin y + \frac{1}{2}\left(-e^{-y} \cos y - \int (-e^{-y})(-\sin y) \, dy\right) = -\frac{1}{2}e^{-y}(\sin y + \cos y) - I + C'$

$\Rightarrow 2I = -\frac{1}{2}e^{-y}(\sin y + \cos y) + C' \Rightarrow I = -\frac{1}{4}e^{-y}(\sin y + \cos y) + C = -\frac{e^{-2x}}{4}(\sin 2x + \cos 2x) + C$, where $C = \frac{C'}{2}$

25. $\int e^{\sqrt{3s+9}} \, ds$; $\begin{bmatrix} 3s + 9 = x^2 \\ ds = \frac{2}{3}x \, dx \end{bmatrix} \to \int e^x \cdot \frac{2}{3}x \, dx = \frac{2}{3} \int xe^x \, dx$; $[u = x,\ du = dx;\ dv = e^x \, dx,\ v = e^x]$;

$\frac{2}{3} \int xe^x \, dx = \frac{2}{3}\left(xe^x - \int e^x \, dx\right) = \frac{2}{3}(xe^x - e^x) + C = \frac{2}{3}\left(\sqrt{3s+9}\, e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}\right) + C$

26. $u = x,\ du = dx;\ dv = \sqrt{1-x} \, dx,\ v = -\frac{2}{3}\sqrt{(1-x)^3}$;

$$\int_0^1 x\sqrt{1-x} \, dx = \left[-\frac{2}{3}\sqrt{(1-x)^3}\, x\right]_0^1 + \frac{2}{3} \int_0^1 \sqrt{(1-x)^3} \, dx = \frac{2}{3}\left[-\frac{2}{5}(1-x)^{5/2}\right]_0^1 = \frac{4}{15}$$

27. $u = x,\ du = dx;\ dv = \tan^2 x \, dx,\ v = \int \tan^2 x \, dx = \int \frac{\sin^2 x}{\cos^2 x} \, dx = \int \frac{1 - \cos^2 x}{\cos^2 x} \, dx = \int \frac{dx}{\cos^2 x} - \int dx$

$= \tan x - x$; $\int_0^{\pi/3} x \tan^2 x \, dx = [x(\tan x - x)]_0^{\pi/3} - \int_0^{\pi/3} (\tan x - x) \, dx = \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \left[\ln|\cos x| + \frac{x^2}{2}\right]_0^{\pi/3}$

$= \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \ln \frac{1}{2} + \frac{\pi^2}{18} = \frac{\pi\sqrt{3}}{3} - \ln 2 - \frac{\pi^2}{18}$

28. $u = \ln(x + x^2)$, $du = \frac{(2x+1)\,dx}{x+x^2}$; $dv = dx$, $v = x$; $\int \ln(x+x^2)\,dx = x\ln(x+x^2) - \int \frac{2x+1}{x(x+1)} \cdot x\,dx$
$= x\ln(x+x^2) - \int \frac{(2x+1)\,dx}{x+1} = x\ln(x+x^2) - \int \frac{2(x+1)-1}{x+1}\,dx = x\ln(x+x^2) - 2x + \ln|x+1| + C$

29. $\int \sin(\ln x)\,dx; \begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \\ dx = e^u\,du \end{bmatrix} \to \int (\sin u)\,e^u\,du$. From Exercise 21, $\int (\sin u)\,e^u\,du = e^u\left(\frac{\sin u - \cos u}{2}\right) + C$
$= \frac{1}{2}\left[-x\cos(\ln x) + x\sin(\ln x)\right] + C$

30. $\int z(\ln z)^2\,dz; \begin{bmatrix} u = \ln z \\ du = \frac{1}{z}\,dz \\ dz = e^u\,du \end{bmatrix} \to \int e^u \cdot u^2 \cdot e^u\,du = \int e^{2u} \cdot u^2\,du;$

e^{2u}
$u^2 \xrightarrow{(+)} \frac{1}{2}e^{2u}$
$2u \xrightarrow{(-)} \frac{1}{4}e^{2u}$
$2 \xrightarrow{(+)} \frac{1}{8}e^{2u}$
0

$\int u^2e^{2u}\,du = \frac{u^2}{2}e^{2u} - \frac{u}{2}e^{2u} + \frac{1}{4}e^{2u} + C = \frac{e^{2u}}{4}\left[2u^2 - 2u + 1\right] + C$
$= \frac{z^2}{4}\left[2(\ln z)^2 - 2\ln z + 1\right] + C$

31. $y = \int x^2e^{4x}\,dx$

Let $u = x^2$ $\quad dv = e^{4x}\,dx$

$du = 2x\,dx$ $\quad v = \frac{1}{4}e^{4x}$

$y = (x^2)\left(\frac{1}{4}e^{4x}\right) - \int \left(\frac{1}{4}e^{4x}\right)(2x\,dx)$
$= \frac{1}{4}x^2e^{4x} - \frac{1}{2}\int xe^{4x}\,dx$

Let $u = x$ $\quad dv = e^{4x}\,dx$

$du = dx$ $\quad v = \frac{1}{4}e^{4x}$

$y = \frac{1}{4}x^2e^{4x} - \frac{1}{2}\left[(x)\left(\frac{1}{4}e^{4x}\right) - \int \left(\frac{1}{4}e^{4x}\right)dx\right]$

$$y = \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C$$

$$y = \left(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32}\right)e^{4x} + C$$

32. $y = \int x^2 \ln x \, dx$

Let $u = \ln x$ $\qquad dv = x^2 \, dx$

$du = \frac{1}{x} \, dx$ $\qquad v = \frac{1}{3}x^3$

$$y = (\ln x)\left(\frac{1}{3}x^3\right) - \int \left(\frac{1}{3}x^3\right)\left(\frac{1}{x} \, dx\right)$$

$$y = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx$$

$$y = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

33. Let $w = \sqrt{\theta}$. Then $dw = \frac{d\theta}{2\sqrt{\theta}}$, so $d\theta = 2\sqrt{\theta} \, dw = 2w \, dw$.

$$\int \sin \sqrt{\theta} \, d\theta = \int (\sin w)(2w \, dw) = 2\int w \sin w \, dw$$

Let $u = w$ $\qquad dv = \sin w \, dw$

$du = dw$ $\qquad v = -\cos w$

$$\int w \sin w \, dw = -w \cos w + \int \cos w \, dw$$

$$= -w \cos w + \sin w + C$$

$$\int \sin \sqrt{\theta} \, d\theta = 2\int w \sin w \, dw$$

$$= -2w \cos w + 2 \sin w + C$$

$$= -2\sqrt{\theta} \cos \sqrt{\theta} + 2 \sin \sqrt{\theta} + C$$

34. $y = \int \theta \sec \theta \tan \theta \, d\theta$

Let $u = \theta$ $\qquad dv = \sec \theta \tan \theta \, d\theta$

$du = d\theta$ $\qquad v = \sec \theta$

$$y = \theta \sec \theta - \int \sec \theta \, d\theta$$

$$y = \theta \sec \theta - \ln|\sec \theta + \tan\theta| + C$$

35. (a) $u = x$, $du = dx$; $dv = \sin x \, dx$, $v = -\cos x$;

$$S_1 = \int_0^{\pi} x \sin x \, dx = [-x \cos x]_0^{\pi} + \int_0^{\pi} \cos x \, dx = \pi + [\sin x]_0^{\pi} = \pi$$

(b) $S_2 = -\int_{\pi}^{2\pi} x \sin x\, dx = \left[[-x \cos x]_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \cos x\, dx\right] = -\left[-3\pi + [\sin x]_{\pi}^{2\pi}\right] = 3\pi$

(c) $S_3 = \int_{2\pi}^{3\pi} x \sin x\, dx = [-x \cos x]_{2\pi}^{3\pi} + \int_{2\pi}^{3\pi} \cos x\, dx = 5\pi + [\sin x]_{2\pi}^{3\pi} = 5\pi$

(d) $S_{n+1} = (-1)^{n+1} \int_{n\pi}^{(n+1)\pi} x \sin x\, dx = (-1)^{n+1}\left[[-x \cos x]_{n\pi}^{(n+1)\pi} + [\sin x]_{n\pi}^{(n+1)\pi}\right]$

$= (-1)^{n+1}[-(n+1)\pi(-1)^n + n\pi(-1)^{n+1}] + 0 = (2n+1)\pi$

36. (a) $u = x$, $du = dx$; $dv = \cos x\, dx$, $v = \sin x$;

$S_1 = -\int_{\pi/2}^{3\pi/2} x \cos x\, dx = -\left[[x \sin x]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} \sin x\, dx\right] = -\left(-\frac{3\pi}{2} - \frac{\pi}{2}\right) - [\cos x]_{\pi/2}^{3\pi/2} = 2\pi$

(b) $S_2 = \int_{3\pi/2}^{5\pi/2} x \cos x\, dx = [x \sin x]_{3\pi/2}^{5\pi/2} - \int_{3\pi/2}^{5\pi/2} \sin x\, dx = \left[\frac{5\pi}{2} - \left(-\frac{3\pi}{2}\right)\right] - [\cos x]_{3\pi/2}^{5\pi/2} = 4\pi$

(c) $S_3 = -\int_{5\pi/2}^{7\pi/2} x \cos x\, dx = -\left[[x \sin x]_{5\pi/2}^{7\pi/2} - \int_{5\pi/2}^{7\pi/2} \sin x\, dx\right] = -\left(-\frac{7\pi}{2} - \frac{5\pi}{2}\right) - [\cos x]_{5\pi/2}^{7\pi/2} = 6\pi$

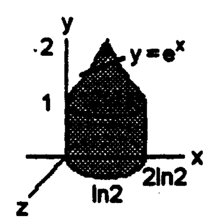
(d) $S_n = (-1)^n \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} x \cos x\, dx = (-1)^n\left[[x \sin x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} - \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} \sin x\, dx\right]$

$= (-1)^n\left[\frac{(2n+1)\pi}{2}(-1)^n - \frac{(2n-1)\pi}{2}(-1)^{n-1}\right] - [\cos x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} = \frac{1}{2}(2n\pi + \pi + 2n\pi - \pi) = 2n\pi$

37. $V = \int_0^{\ln 2} 2\pi(\ln 2 - x)e^x\, dx = 2\pi \ln 2 \int_0^{\ln 2} e^x\, dx - 2\pi \int_0^{\ln 2} xe^x\, dx$

$= (2\pi \ln 2)\left[e^x\right]_0^{\ln 2} - 2\pi\left(\left[xe^x\right]_0^{\ln 2} - \int_0^{\ln 2} e^x\, dx\right)$

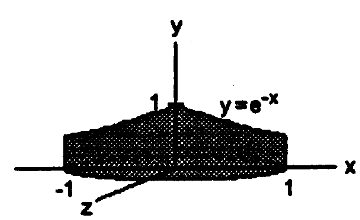
$= 2\pi \ln 2 - 2\pi\left(2 \ln 2 + \left[e^x\right]_0^{\ln 2}\right) = -2\pi \ln 2 + 2 = 2\pi(1 - \ln 2)$



38. (a) $V = \int_0^1 2\pi xe^{-x}\, dx = 2\pi\left(\left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\, dx\right)$

$= 2\pi\left(-\frac{1}{e} + \left[-e^{-x}\right]_0^1\right) = 2\pi\left(-\frac{1}{e} - \frac{1}{e} + 1\right)$

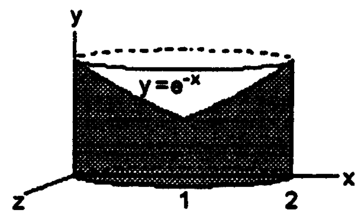
$= 2\pi - \frac{4\pi}{e}$



(b) $V = \int_0^1 2\pi(1-x)e^{-x}\,dx$; $u = 1-x$, $du = -dx$; $dv = e^{-x}\,dx$,

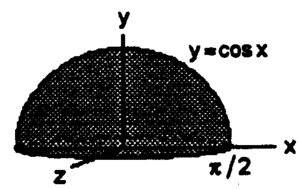
$v = -e^{-x}$; $V = 2\pi\left[\left[(1-x)\left(-e^{-x}\right)\right]_0^1 - \int_0^1 e^{-x}\,dx\right]$

$= 2\pi\left[[0-1(-1)] + \left[e^{-x}\right]_0^1\right] = 2\pi\left(1+\frac{1}{e}-1\right) = \frac{2\pi}{e}$



39. (a) $V = \int_0^{\pi/2} 2\pi x \cos x\,dx = 2\pi\left([x \sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x\,dx\right)$

$= 2\pi\left(\frac{\pi}{2} + [\cos x]_0^{\pi/2}\right) = 2\pi\left(\frac{\pi}{2}+0-1\right) = \pi(\pi-2)$



(b) $V = \int_0^{\pi/2} 2\pi\left(\frac{\pi}{2}-x\right)\cos x\,dx$; $u = \frac{\pi}{2}-x$, $du = -dx$; $dv = \cos x\,dx$, $v = \sin x$;

$V = 2\pi\left[\left(\frac{\pi}{2}-x\right)\sin x\right]_0^{\pi/2} + 2\pi\int_0^{\pi/2} \sin x\,dx = 0 + 2\pi[-\cos x]_0^{\pi/2} = 2\pi(0+1) = 2\pi$

40. (a) $V = \int_0^{\pi} 2\pi x(x \sin x)\,dx$;

$$\begin{array}{lcl} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$

$\Rightarrow V = 2\pi\int_0^{\pi} x^2 \sin x\,dx = 2\pi\left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^{\pi} = 2\pi\left(\pi^2-4\right)$

(b) $V = \int_0^{\pi} 2\pi(\pi-x)x \sin x\,dx = 2\pi^2\int_0^{\pi} x\sin x\,dx - 2\pi\int_0^{\pi} x^2 \sin x\,dx = 2\pi^2\left[-x\cos x + \sin x\right]_0^{\pi} - \left(2\pi^3 - 8\pi\right)$

$= 8\pi$

41. $\text{av}(y) = \frac{1}{2\pi}\int_0^{2\pi} 2e^{-t}\cos t\,dt = \frac{1}{\pi}\left[e^{-t}\left(\frac{\sin t - \cos t}{2}\right)\right]_0^{2\pi}$

(see Exercise 22) $\Rightarrow \text{av}(y) = \frac{1}{2\pi}\left(1-e^{-2\pi}\right)$

42. $\text{av}(y) = \frac{1}{2\pi}\int_0^{2\pi} 4e^{-t}(\sin t - \cos t)\,dt = \frac{2}{\pi}\int_0^{2\pi} e^{-t}\sin t\,dt - \frac{2}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt$

$= \frac{2}{\pi}\left[e^{-t}\left(\frac{-\sin t - \cos t}{2}\right) - e^{-t}\left(\frac{\sin t - \cos t}{2}\right)\right]_0^{2\pi} = \frac{2}{\pi}\left[-e^{-t}\sin t\right]_0^{2\pi} = 0$

43. Let $u = x^n$ $\quad dv = \cos x\, dx$

$du = nx^{n-1}\, dx \quad v = \sin x$

$$\int x^n \cos x\, dx = x^n \sin x - \int (\sin x)(nx^{n-1}\, dx) = x^n \sin x - n\int x^{n-1} \sin x\, dx$$

44. Let $u = x^n$ $\quad dv = \sin x\, dx$

$du = nx^{n-1}\, dx \quad v = -\cos x$

$$\int x^n \sin x\, dx = (x^n)(-\cos x) - \int (-\cos x)(nx^{n-1}\, dx) = -x^n \cos x + n\int x^{n-1} \cos x\, dx$$

45. Let $u = x^n$ $\quad dv = e^{ax}\, dx$

$du = nx^{n-1}\, dx \quad v = \frac{1}{a}e^{ax}$

$$\int x^n e^{ax}\, dx = (x^n)\left(\frac{1}{a}e^{ax}\right) - \int \left(\frac{1}{a}e^{ax}\right)(nx^{n-1}\, dx) = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\, dx,\ a \neq 0$$

46. Let $u = (\ln x)^n$ $\quad dv = dx$

$du = \frac{n(\ln x)^{n-1}}{x}\, dx \quad v = x$

$$\int (\ln x)^n\, dx = x(\ln x)^n - \int x\left[\frac{n(\ln x)^{n-1}}{x}\right] dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx$$

47. (a) Let $y = f^{-1}(x)$. Then $x = f(y)$, so $dx = f'(y)\, dy$.

Hence, $\int f^{-1}(x)\, dx = \int (y)[f'(y)\, dy] = \int yf'(y)\, dy$

(b) Let $u = y$ $\quad dv = f'(y)\, dy$

$du = dy \quad v = f(y)$

$$\int yf'(y)\, dy = yf(y) - \int f(y)\, dy = f^{-1}(x)(x) - \int f(y)\, dy$$

Hence, $\int f^{-1}(x)\, dx = \int yf'(y)\, dy = xf^{-1}(x) - \int f(y)\, dy.$

48. Let $u = f^{-1}(x)$ $\quad dv = dx$

$du = \left(\frac{d}{dx}f^{-1}(x)\right) dx \quad v = x$

$$\int f^{-1}(x)\, dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right) dx$$

49. (a) Using $y = f^{-1}(x) = \sin^{-1} x$ and $f(y) = \sin y$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, we have:

$$\int \sin^{-1} x\, dx = x \sin^{-1} x - \int \sin y\, dy = x \sin^{-1} x + \cos y + C = x \sin^{-1} x + \cos(\sin^{-1} x) + C$$

(b) $\int \sin^{-1} x\, dx = x \sin^{-1} x - \int x\left(\frac{d}{dx} \sin^{-1} x\right) dx = x \sin^{-1} x - \int x \frac{1}{\sqrt{1-x^2}}\, dx$

$u = 1 - x^2,\ du = -2x\, dx = x \sin^{-1} x + \frac{1}{2} \int u^{-1/2}\, du = x \sin^{-1} x + u^{1/2} + C = x \sin^{-1} x + \sqrt{1-x^2} + C$

(c) $\cos(\sin^{-1} x) = \sqrt{1-x^2}$

50. (a) Using $y = f^{-1}(x) = \tan^{-1} x$ and $f(y) = \tan y$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$, we have:

$\int \tan^{-1} x\, dx = x \tan^{-1} x - \int \tan y\, dy = x \tan^{-1} x - \ln|\sec y| + C = x \tan^{-1} x + \ln|\cos y| + C$

$= x \tan^{-1} x + \ln\left|\cos(\tan^{-1}(x))\right| dx + C$

(b) $\int \tan^{-1} x\, dx = x \tan^{-1} x - \int x\left(\frac{d}{dx} \tan^{-1} x\right) dx = x \tan^{-1} x - \int x\left(\frac{1}{1+x^2}\right) dx$

$u = 1 + x^2,\ du = 2x\, dx = x \tan^{-1} x - \frac{1}{2} \int u^{-1}\, du = x \tan^{-1} x - \frac{1}{2} \ln|u| + C = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$

(c) $\ln\left|\cos(\tan^{-1} x)\right| = \ln\left|\frac{1}{\sqrt{1+x^2}}\right| = -\frac{1}{2} \ln(1+x^2)$

51. (a) Using $y = f^{-1}(x) = \cos^{-1} x$ and $f(y) = \cos x$, $0 \le x \le \pi$, we have:

$\int \cos^{-1} x\, dx = x \cos^{-1} x - \int \cos y\, dy = x \cos^{-1} x - \sin y + C = x \cos^{-1} x - \sin(\cos^{-1} x) + C$

(b) $\int \cos^{-1} x\, dx = x \cos^{-1} x - \int x\left(\frac{d}{dx} \cos^{-1} x\right) dx = x \cos^{-1} x - \int x\left(-\frac{1}{\sqrt{1-x^2}}\right) dx$

$u = 1 - x^2,\ du = -2x\, dx = x \cos^{-1} x - \frac{1}{2} \int u^{-1/2}\, du = x \cos^{-1} x - u^{1/2} + C = x \cos^{-1} x - \sqrt{1-x^2} + C$

(c) $\sin(\cos^{-1} x) = \sqrt{1-x^2}$

52. (a) Using $y = f^{-1}(x) = \log_2 x$ and $f(y) = 2^y$, we have

$\int \log_2 x\, dx = x \log_2 x - \int 2^y\, dy = x \log_2 x - \frac{2^y}{\ln 2} + C = x \log_2 x - \frac{1}{\ln 2} 2^{\log_2 x}$

(b) $\int \log_2 x\, dx = x \log_2 x - \int x\left(\frac{d}{dx} \log_2 x\right) dx = x \log_2 x - \int x\left(\frac{1}{x \ln 2}\right) dx = x \log_2 x - \int \frac{dx}{\ln 2}$

$= x \log_2 x - \left(\frac{1}{\ln 2}\right)x + C$

(c) $2^{\log_2 x} = x$

7.3 PARTIAL FRACTIONS

1. $\frac{5x-13}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \Rightarrow 5x - 13 = A(x-2) + B(x-3) = (A+B)x - (2A+3B)$

$\Rightarrow \left.\begin{array}{r} A+B=5 \\ 2A+3B=13 \end{array}\right\} \Rightarrow -B = (10-13) \Rightarrow B = 3 \Rightarrow A = 2$; thus, $\frac{5x-13}{(x-3)(x-2)} = \frac{2}{x-3} + \frac{3}{x-2}$

2. $\frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 5x-7 = A(x-1)+B(x-2) = (A+B)x-(A+2B)$

$\Rightarrow \left.\begin{matrix} A+B=5 \\ A+2B=7 \end{matrix}\right\} \Rightarrow B=2 \Rightarrow A=3$; thus, $\frac{5x-7}{x^2-3x+2} = \frac{3}{x-2} + \frac{2}{x-1}$

3. $\frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow x+4 = A(x+1)+B = Ax+(A+B) \Rightarrow \left.\begin{matrix} A=1 \\ A+B=4 \end{matrix}\right\} \Rightarrow A=1$ and $B=3$;

thus, $\frac{x+4}{(x+1)^2} = \frac{1}{x+1} + \frac{3}{(x+1)^2}$

4. $\frac{2x+2}{x^2-2x+1} = \frac{2x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 2x+2 = A(x-1)+B = Ax+(-A+B) \Rightarrow \left.\begin{matrix} A=2 \\ -A+B=2 \end{matrix}\right\}$

$\Rightarrow A=2$ and $B=4$; thus, $\frac{2x+2}{x^2-2x+1} = \frac{2}{x-1} + \frac{4}{(x-1)^2}$

5. $\frac{z+1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} \Rightarrow z+1 = Az(z-1)+B(z-1)+Cz^2 \Rightarrow z+1 = (A+C)z^2+(-A+B)z-B$

$\Rightarrow \left.\begin{matrix} A+C=0 \\ -A+B=1 \\ -B=1 \end{matrix}\right\} \Rightarrow B=-1 \Rightarrow A=-2 \Rightarrow C=2$; thus, $\frac{z+1}{z^2(z-1)} = \frac{-2}{z} + \frac{-1}{z^2} + \frac{2}{z-1}$

6. $\frac{z}{z^3-z^2-6z} = \frac{1}{z^2-z-6} = \frac{1}{(z-3)(z+2)} = \frac{A}{z-3} + \frac{B}{z+2} \Rightarrow 1 = A(z+2)+B(z-3) = (A+B)z+(2A-3B)$

$\Rightarrow \left.\begin{matrix} A+B=0 \\ 2A-3B=1 \end{matrix}\right\} \Rightarrow -5B=1 \Rightarrow B=-\frac{1}{5} \Rightarrow A=\frac{1}{5}$; thus, $\frac{z}{z^3-z^2-6z} = \frac{\frac{1}{5}}{z-3} + \frac{-\frac{1}{5}}{z+2}$

7. $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{5t+2}{t^2-5t+6}$ (after long division); $\frac{5t+2}{t^2-5t+6} = \frac{5t+2}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2}$

$\Rightarrow 5t+2 = A(t-2)+B(t-3) = (A+B)t+(-2A-3B) \Rightarrow \left.\begin{matrix} A+B=5 \\ -2A-3B=2 \end{matrix}\right\} \Rightarrow -B = (10+2) = 12$

$\Rightarrow B=-12 \Rightarrow A=17$; thus, $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{17}{t-3} + \frac{-12}{t-2}$

8. $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^2(t^2+9)}$ (after long division); $\frac{-9t^2+9}{t^2(t^2+9)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+9}$

$\Rightarrow -9t^2+9 = At(t^2+9)+B(t^2+9)+(Ct+D)t^2 = (A+C)t^3+(B+D)t^2+9At+9B$

$\Rightarrow \left.\begin{matrix} A+C=0 \\ B+D=-9 \\ 9A=0 \\ 9B=9 \end{matrix}\right\} \Rightarrow A=0 \Rightarrow C=0;\ B=1 \Rightarrow D=-10$; thus, $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{1}{t^2} + \frac{-10}{t^2+9}$

9. $\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \Rightarrow 1 = A(1+x)+B(1-x)$; $x=1 \Rightarrow A=\frac{1}{2}$; $x=-1 \Rightarrow B=\frac{1}{2}$;

$\int \frac{dx}{1-x^2} = \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x} = \frac{1}{2}[\ln|1+x| - \ln|1-x|] + C$

10. $\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow 1 = A(x+2) + Bx;\ x = 0 \Rightarrow A = \frac{1}{2};\ x = -2 \Rightarrow B = -\frac{1}{2};$

$\int \frac{dx}{x^2+2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2} = \frac{1}{2}[\ln|x| - \ln|x+2|] + C$

11. $\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} \Rightarrow x+4 = A(x-1) + B(x+6);\ x = 1 \Rightarrow B = \frac{5}{7};\ x = -6 \Rightarrow A = \frac{-2}{-7} = \frac{2}{7};$

$\int \frac{x+4}{x^2+5x-6}\,dx = \frac{2}{7}\int \frac{dx}{x+6} + \frac{5}{7}\int \frac{dx}{x-1} = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C = \frac{1}{7}\ln\left|(x+6)^2(x-1)^5\right| + C$

12. $\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} \Rightarrow 2x+1 = A(x-3) + B(x-4);\ x = 3 \Rightarrow B = \frac{7}{-1} = -7;\ x = 4 \Rightarrow A = \frac{9}{1} = 9;$

$\int \frac{2x+1}{x^2-7x+12}\,dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3} = 9\ln|x-4| - 7\ln|x-3| + C = \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C$

13. $\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1) + B(y-3);\ y = -1 \Rightarrow B = \frac{-1}{-4} = \frac{1}{4};\ y = 3 \Rightarrow A = \frac{3}{4};$

$\int_4^8 \frac{y\,dy}{y^2-2y-3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3} + \frac{1}{4}\int_4^8 \frac{dy}{y+1} = \left[\frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1|\right]_4^8 = \left(\frac{3}{4}\ln 5 + \frac{1}{4}\ln 9\right) - \left(\frac{3}{4}\ln 1 + \frac{1}{4}\ln 5\right)$

$= \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3 = \frac{\ln 15}{2}$

14. $\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y+4 = A(y+1) + By;\ y = 0 \Rightarrow A = 4;\ y = -1 \Rightarrow B = \frac{3}{-1} = -3;$

$\int_{1/2}^1 \frac{y+4}{y^2+y}\,dy = 4\int_{1/2}^1 \frac{dy}{y} - 3\int_{1/2}^1 \frac{dy}{y+1} = [4\ln|y| - 3\ln|y+1|]_{1/2}^1 = (4\ln 1 - 3\ln 2) - \left(4\ln\frac{1}{2} - 3\ln\frac{3}{2}\right)$

$= \ln\frac{1}{8} - \ln\frac{1}{16} + \ln\frac{27}{8} = \ln\left(\frac{27}{8}\cdot\frac{1}{8}\cdot 16\right) = \ln\frac{27}{4}$

15. $\frac{1}{t^3+t^2-2t} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1} \Rightarrow 1 = A(t+2)(t-1) + Bt(t-1) + Ct(t+2);\ t = 0 \Rightarrow A = -\frac{1}{2};\ t = -2$

$\Rightarrow B = \frac{1}{6};\ t = 1 \Rightarrow C = \frac{1}{3};\ \int \frac{dt}{t^3+t^2-2t} = -\frac{1}{2}\int \frac{dt}{t} + \frac{1}{6}\int \frac{dt}{t+2} + \frac{1}{3}\int \frac{dt}{t-1}$

$= -\frac{1}{2}\ln|t| + \frac{1}{6}\ln|t+2| + \frac{1}{3}\ln|t-1| + C$

16. $\frac{x+3}{2x^3-8x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \Rightarrow \frac{1}{2}(x+3) = A(x+2)(x-2) + Bx(x-2) = Cx(x+2);\ x = 0 \Rightarrow A = \frac{3}{-8};\ x = -2$

$\Rightarrow B = \frac{1}{16};\ x = 2 \Rightarrow C = \frac{5}{16};\ \int \frac{x+3}{2x^3-8x}\,dx = -\frac{3}{8}\int \frac{dx}{x} + \frac{1}{16}\int \frac{dx}{x+2} + \frac{5}{16}\int \frac{dx}{x-2}$

$= -\frac{3}{8}\ln|x| + \frac{1}{16}\ln|x+2| + \frac{5}{16}\ln|x-2| + C = \frac{1}{16}\ln\left|\frac{(x-2)^5(x+2)}{x^6}\right| + C$

17. $\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$ (after long division); $\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+3 = A(x+1)+B$

$= Ax + (A+B) \Rightarrow A = 3,\ A+B = 2 \Rightarrow A = 3,\ B = -1;\ \int_0^1 \frac{x^3\,dx}{x^2+2x+1}$

$= \int_0^1 (x-2)\,dx + 3\int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1$

$= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) = 3\ln 2 - 2$

18. $\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x+2}{(x-1)^2}$ (after long division); $\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 3x-2 = A(x-1)+B$

$= Ax + (-A+B) \Rightarrow A = 3,\ -A+B = -2 \Rightarrow A = 3,\ B = 1;\ \int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$

$= \int_{-1}^0 (x+2)\,dx + 3\int_{-1}^0 \frac{dx}{x-1} + \int_{-1}^0 \frac{dx}{(x-1)^2} = \left[\frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1}\right]_{-1}^0$

$= \left(0 + 0 + 3\ln 1 - \frac{1}{(-1)}\right) - \left(\frac{1}{2} - 2 + 3\ln 2 - \frac{1}{(-2)}\right) = 2 - 3\ln 2$

19. $\frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{(x-1)^2} \Rightarrow 1 = A(x+1)(x-1)^2 + B(x-1)(x+1)^2 + C(x-1)^2 + D(x+1)^2;$

$x = -1 \Rightarrow C = \frac{1}{4};\ x = 1 \Rightarrow D = \frac{1}{4};$ coefficient of $x^3 = A + B \Rightarrow A + B = 0;$ constant $= A - B + C + D$

$\Rightarrow A - B + C + D = 1 \Rightarrow A - B = \frac{1}{2};$ thus, $A = \frac{1}{4} \Rightarrow B = -\frac{1}{4};\ \int \frac{dx}{(x^2-1)^2}$

$= \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x-1} + \frac{1}{4}\int \frac{dx}{(x+1)^2} + \frac{1}{4}\int \frac{dx}{(x-1)^2} = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2(x^2-1)} + C$

20. $\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Rightarrow x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1);\ x = -1$

$\Rightarrow C = -\frac{1}{2};\ x = 1 \Rightarrow A = \frac{1}{4};$ coefficient of $x^2 = A + B \Rightarrow A + B = 1 \Rightarrow B = \frac{3}{4};\ \int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$

$= \frac{1}{4}\int \frac{dx}{x-1} + \frac{3}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{(x+1)^2} = \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)} + C$

$= \frac{\ln\left|(x-1)(x+1)^3\right|}{4} + \frac{1}{2(x+1)} + C$

21. $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + (Bx+C)(x+1);\ x = -1 \Rightarrow A = \frac{1}{2};$ coefficient of x^2

$= A + B \Rightarrow A + B = 0 \Rightarrow B = -\frac{1}{2};$ constant $= A + C \Rightarrow A + C = 1 \Rightarrow C = \frac{1}{2};\ \int_0^1 \frac{dx}{(x+1)(x^2+1)}$

$$=\frac{1}{2}\int_0^1 \frac{dx}{x+1}+\frac{1}{2}\int_0^1 \frac{(-x+1)}{x^2+1}\,dx=\left[\frac{1}{2}\ln|x+1|-\frac{1}{4}\ln(x^2+1)+\frac{1}{2}\tan^{-1}x\right]_0^1$$

$$=\left(\frac{1}{2}\ln 2-\frac{1}{4}\ln 2+\frac{1}{2}\tan^{-1}1\right)-\left(\frac{1}{2}\ln 1-\frac{1}{4}\ln 1+\frac{1}{2}\tan^{-1}0\right)=\frac{1}{4}\ln 2+\frac{1}{2}\left(\frac{\pi}{4}\right)=\frac{(\pi+2\ln 2)}{8}$$

22. $\frac{3t^2+t+4}{t^3+t}=\frac{A}{t}+\frac{Bt+C}{t^2+1}\Rightarrow 3t^2+t+4=A(t^2+1)+(Bt+C)t$; $t=0\Rightarrow A=4$; coefficient of t^2 $=A+B\Rightarrow A+B=3\Rightarrow B=-1$; coefficient of $t=C\Rightarrow C=1$; $\int_1^{\sqrt{3}}\frac{3t^2+t+4}{t^3+t}\,dt$

$$=4\int_1^{\sqrt{3}}\frac{dt}{t}+\int_1^{\sqrt{3}}\frac{(-t+1)}{t^2+1}\,dt=\left[4\ln|t|-\frac{1}{2}\ln(t^2+1)+\tan^{-1}t\right]_1^{\sqrt{3}}$$

$$=\left(4\ln\sqrt{3}-\frac{1}{2}\ln 4+\tan^{-1}\sqrt{3}\right)-\left(4\ln 1-\frac{1}{2}\ln 2+\tan^{-1}1\right)=2\ln 3-\ln 2+\frac{\pi}{3}+\frac{1}{2}\ln 2+\frac{\pi}{4}$$

$$=2\ln 3-\frac{1}{2}\ln 2+\frac{\pi}{12}=\ln\left(\frac{9}{\sqrt{2}}\right)+\frac{\pi}{12}$$

23. $\frac{y^2+2y+1}{(y^2+1)^2}=\frac{Ay+B}{y^2+1}+\frac{Cy+D}{(y^2+1)^2}\Rightarrow y^2+2y+1=(Ay+B)(y^2+1)+Cy+D$

$=Ay^3+By^2+(A+C)y+(B+D)\Rightarrow A=0,\ B=1;\ A+C=2\Rightarrow C=2;\ B+D=1\Rightarrow D=0;$

$$\int\frac{y^2+2y+1}{(y^2+1)^2}\,dy=\int\frac{1}{y^2+1}\,dy+2\int\frac{y}{(y^2+1)^2}\,dy=\tan^{-1}y-\frac{1}{y^2+1}+C$$

24. $\frac{8x^2+8x+2}{(4x^2+1)^2}=\frac{Ax+B}{4x^2+1}+\frac{Cx+D}{(4x^2+1)^2}\Rightarrow 8x^2+8x+2=(Ax+B)(4x^2+1)+Cx+D$

$=4Ax^3+4Bx^2+(A+C)x+(B+D);\ A=0,\ B=2;\ A+C=8\Rightarrow C=8;\ B+D=2\Rightarrow D=0;$

$$\int\frac{8x^2+8x+2}{(4x^2+1)^2}\,dx=2\int\frac{dx}{4x^2+1}+8\int\frac{x\,dx}{(4x^2+1)^2}=\tan^{-1}2x-\frac{1}{4x^2+1}+C$$

25. $\frac{2s+2}{(s^2+1)(s-1)^3}=\frac{As+B}{s^2+1}+\frac{C}{s-1}+\frac{D}{(s-1)^2}+\frac{E}{(s-1)^3}\Rightarrow 2s+2$

$=(As+B)(s-1)^3+C(s^2+1)(s-1)^2+D(s^2+1)(s-1)+E(s^2+1)$

$=\left[As^4+(-3A+B)s^3+(3A-3B)s^2+(-A+3B)s-B\right]+C(s^4-2s^3+2s^2-2s+1)+D(s^3-s^2+s-1)$
$\quad+E(s^2+1)$

$=(A+C)s^4+(-3A+B-2C+D)s^3+(3A-3B+2C-D+E)s^2+(-A+3B-2C+D)s+(-B+C-D+E)$

$$\Rightarrow\left.\begin{array}{r}A\qquad+C\qquad\quad=0\\ -3A+\ B-2C+D\qquad=0\\ 3A-3B+2C-D+E=0\\ -A+3B-2C+D\qquad=2\\ -B+\ C-D+E=2\end{array}\right\}\text{ summing all equations}\Rightarrow 2E=4\Rightarrow E=2;$$

summing eqs (2) and (3) $\Rightarrow -2B+2=0 \Rightarrow B=1$; summing eqs (3) and (4) $\Rightarrow 2A+2=2 \Rightarrow A=0$; $C=0$ from eq (1); then $-1+0-D+2=2$ from eq (5) $\Rightarrow D=-1$;

$$\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds = \int \frac{ds}{s^2+1} - \int \frac{ds}{(s-1)^2} + 2\int \frac{ds}{(s-1)^3} = -(s-1)^{-2} + (s-1)^{-1} + \tan^{-1} s + C$$

26. $\frac{s^4+81}{s(s^2+9)^2} = \frac{A}{s} + \frac{Bs+C}{s^2+9} + \frac{Ds+E}{(s^2+9)^2} \Rightarrow s^4+81 = A(s^2+9)^2 + (Bs+C)s(s^2+9) + (Ds+E)s$
$= A(s^4+18s^2+81) + (Bs^4+Cs^3+9Bs^2+9Cs) + Ds^2 + Es$
$= (A+B)s^4 + Cs^3 + (18A+9B+D)s^2 + (9C+E)s + 81A \Rightarrow 81A = 81$ or $A=1$; $A+B=1 \Rightarrow B=0$;
$C=0$; $9C+E=0 \Rightarrow E=0$; $18A+9B+D=0 \Rightarrow D=-18$; $\int \frac{s^4+81}{s(s^2+9)^2}\,ds = \int \frac{ds}{s} - 18\int \frac{s\,ds}{(s^2+9)^2}$
$= \ln|s| + \frac{9}{(s^2+9)} + C$

27. $\frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} = \frac{A\theta+B}{\theta^2+2\theta+2} + \frac{C\theta+D}{(\theta^2+2\theta+2)^2} \Rightarrow 2\theta^3+5\theta^2+8\theta+4 = (A\theta+B)(\theta^2+2\theta+2) + C\theta + D$
$= A\theta^3 + (2A+B)\theta^2 + (2A+2B+C)\theta + (2B+D) \Rightarrow A=2$; $2A+B=5 \Rightarrow B=1$; $2A+2B+C=8 \Rightarrow C=2$;
$2B+D=4 \Rightarrow D=2$; $\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta = \int \frac{2\theta+1}{(\theta^2+2\theta+2)}\,d\theta + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2}\,d\theta$
$= \int \frac{2\theta+2}{\theta^2+2\theta+2}\,d\theta - \int \frac{d\theta}{\theta^2+2\theta+2} + \int \frac{d(\theta^2+2\theta+2)}{(\theta^2+2\theta+2)^2} = \int \frac{d(\theta^2+2\theta+2)}{\theta^2+2\theta+2} - \int \frac{d\theta}{(\theta+1)^2+1} - \frac{1}{\theta^2+2\theta+2}$
$= \frac{-1}{\theta^2+2\theta+2} + \ln(\theta^2+2\theta+2) - \tan^{-1}(\theta+1) + C$

28. $\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3} = \frac{A\theta+B}{\theta^2+1} + \frac{C\theta+D}{(\theta^2+1)^2} + \frac{E\theta+F}{(\theta^2+1)^3} \Rightarrow \theta^4-4\theta^3+2\theta^2-3\theta+1$
$= (A\theta+B)(\theta^2+1)^2 + (C\theta+D)(\theta^2+1) + E\theta + F = (A\theta+B)(\theta^4+2\theta^2+1) + (C\theta^3+D\theta^2+C\theta+D) + E\theta + F$
$= (A\theta^5+B\theta^4+2A\theta^3+2B\theta^2+A\theta+B) + (C\theta^3+D\theta^2+C\theta+D) + E\theta + F$
$= A\theta^5 + B\theta^4 + (2A+C)\theta^3 + (2B+D)\theta^2 + (A+C+E)\theta + (B+D+F) \Rightarrow A=0$; $B=1$; $2A+C=-4$
$\Rightarrow C=-4$; $2B+D=2 \Rightarrow D=0$; $A+C+E=-3 \Rightarrow E=1$; $B+D+F=1 \Rightarrow F=0$;
$\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta = \int \frac{d\theta}{\theta^2+1} - 4\int \frac{\theta\,d\theta}{(\theta^2+1)^2} + \int \frac{\theta\,d\theta}{(\theta^2+1)^3} = \tan^{-1}\theta + 2(\theta^2+1)^{-1} - \frac{1}{4}(\theta^2+1)^{-2} + C$

29. $\frac{2x^3-2x^2+1}{x^2-x} = 2x + \frac{1}{x^2-x} = 2x + \frac{1}{x(x-1)}$; $\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + Bx$; $x=0 \Rightarrow A=-1$;
$x=1 \Rightarrow B=1$; $\int \frac{2x^3-2x^2+1}{x^2-x} = \int 2x\,dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} = x^2 - \ln|x| + \ln|x-1| + C = x^2 + \ln\left|\frac{x-1}{x}\right| + C$

30. $\frac{x^4}{x^2-1} = (x^2+1) + \frac{1}{x^2-1} = (x^2+1) + \frac{1}{(x+1)(x-1)}$; $\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x+1)$;

$x = -1 \Rightarrow A = -\frac{1}{2}$; $x = 1 \Rightarrow B = \frac{1}{2}$; $\int \frac{x^4}{x^2-1}\,dx = \int (x^2+1)\,dx - \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x-1}$

$= \frac{1}{3}x^3 + x - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C = \frac{x^3}{3} + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

31. $\frac{9x^3-3x+1}{x^3-x^2} = 9 + \frac{9x^2-3x+1}{x^2(x-1)}$ (after long division); $\frac{9x^2-3x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$

$\Rightarrow 9x^2 - 3x + 1 = Ax(x-1) + B(x-1) + Cx^2$; $x = 1 \Rightarrow C = 7$; $x = 0 \Rightarrow B = -1$; $A + C = 9 \Rightarrow A = 2$;

$\int \frac{9x^3-3x+1}{x^3-x^2}\,dx = \int 9\,dx + 2\int \frac{dx}{x} - \int \frac{dx}{x^2} + 7\int \frac{dx}{x-1} = 9x + 2\ln|x| + \frac{1}{x} + 7\ln|x-1| + C$

32. $\frac{16x^3}{4x^2-4x+1} = (4x+4) + \frac{12x-4}{4x^2-4x+1}$; $\frac{12x-4}{(2x-1)^2} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} \Rightarrow 12x - 4 = A(2x-1) + B$

$\Rightarrow A = 6$; $-A + B = -4 \Rightarrow B = 2$; $\int \frac{16x^3}{4x^2-4x+1}\,dx = 4\int (x+1)\,dx + 6\int \frac{dx}{2x-1} + 2\int \frac{dx}{(2x-1)^2}$

$= 2(x+1)^2 + 3\ln|2x-1| - \frac{1}{2x-1} + C_1 = 2x^2 + 4x + 3\ln|2x-1| - (2x-1)^{-1} + C$, where $C = 2 + C_1$

33. $\frac{y^4+y^2-1}{y^3+y} = y - \frac{1}{y(y^2+1)}$; $\frac{1}{y(y^2+1)} = \frac{A}{y} + \frac{By+C}{y^2+1} \Rightarrow 1 = A(y^2+1) + (By+C)y = (A+B)y^2 + Cy + A$

$\Rightarrow A = 1$; $A + B = 0 \Rightarrow B = -1$; $C = 0$; $\int \frac{y^4+y^2-1}{y^3+y}\,dy = \int y\,dy - \int \frac{dy}{y} + \int \frac{y\,dy}{y^2+1}$

$= \frac{y^2}{2} - \ln|y| + \frac{1}{2}\ln(1+y^2) + C$

34. $\frac{2y^4}{y^3-y^2+y-1} = 2y + 2 + \frac{2}{y^3-y^2+y-1}$; $\frac{2}{y^3-y^2+y-1} = \frac{2}{(y^2+1)(y-1)} = \frac{A}{y-1} + \frac{By+C}{y^2+1}$

$\Rightarrow 2 = A(y^2+1) + (By+C)(y-1) = (Ay^2+A) + (By^2+Cy-By-C) = (A+B)y^2 + (-B+C)y + (A-C)$

$\Rightarrow A + B = 0$, $-B + C = 0$ or $C = B$, $A - C = A - B = 2 \Rightarrow A = 1$, $B = -1$, $C = -1$;

$\int \frac{2y^4}{y^3-y^2+y-1}\,dy = 2\int (y+1)\,dy + \int \frac{dy}{y-1} - \int \frac{y}{y^2+1}\,dy - \int \frac{dy}{y^2+1}$

$= (y+1)^2 + \ln|y-1| - \frac{1}{2}\ln(y^2+1) - \tan^{-1}y + C_1 = y^2 + 2y + \ln|y-1| - \frac{1}{2}\ln(y^2+1) - \tan^{-1}y + C$,
where $C = C_1 + 1$

35. $\int \frac{e^t\,dt}{e^{2t}+3e^t+2} = [e^t = y]\int \frac{dy}{y^2+3y+2} = \int \frac{dy}{y+1} - \int \frac{dy}{y+2} = \ln\left|\frac{y+1}{y+2}\right| + C = \ln\left|\frac{e^t+1}{e^t+2}\right| + C$

36. $\int \frac{e^{4t}+2e^{2t}-e^t}{e^{2t}+1}\,dt$; $[e^t = y] \to \int \frac{y^3+2y-1}{y^2+1}\,dy = \int \left(y + \frac{y-1}{y^2+1}\right)dy = \frac{y^2}{2} + \frac{1}{2}\ln(y^2+1) - \tan^{-1}y + C$

$= \frac{1}{2}e^{2t} - \tan^{-1}(e^t) + \frac{1}{2}\ln(e^{2t}+1) + C$

37. $\int \frac{\cos y\, dy}{\sin^2 y + \sin y - 6}$; $[\sin y = t, \cos y\, dy = dt] \to \int \frac{dy}{t^2+t-6} = \frac{1}{5}\int \left(\frac{1}{t-2} - \frac{1}{t+3}\right) dt = \frac{1}{5}\ln\left|\frac{t-2}{t+3}\right| + C$

$= \frac{1}{5}\ln\left|\frac{\sin y - 2}{\sin y + 3}\right| + C$

38. $\int \frac{\sin\theta\, d\theta}{\cos^2\theta + \cos\theta - 2}$; $[\cos\theta = y] \to -\int \frac{dy}{y^2+y-2} = \frac{1}{3}\int \frac{dy}{y+2} - \frac{1}{3}\int \frac{dy}{y-1} = \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C = \frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right| + C$

$= \frac{1}{3}\ln\left|\frac{2+\cos\theta}{1-\cos\theta}\right| + C = -\frac{1}{3}\ln\left|\frac{\cos\theta-1}{\cos\theta+2}\right| + C$

39. $\int \frac{(x-2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\, dx = \int \frac{\tan^{-1}(2x)}{4x^2+1}\, dx - 3\int \frac{x}{(x-2)^2}\, dx$

$= \frac{1}{2}\int \tan^{-1}(2x)\, d(\tan^{-1}(2x)) - 3\int \frac{dx}{x-2} - 6\int \frac{dx}{(x-2)^2} = \frac{(\tan^{-1}2x)^2}{4} - 3\ln|x-2| + \frac{6}{x-2} + C$

40. $\int \frac{(x+1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2}\, dx = \int \frac{\tan^{-1}(3x)}{9x^2+1}\, dx + \int \frac{x}{(x+1)^2}\, dx$

$= \frac{1}{3}\int \tan^{-1}(3x)\, d(\tan^{-1}(3x)) + \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2} = \frac{(\tan^{-1}3x)^2}{6} + \ln|x+1| + \frac{1}{x+1} + C$

41. $(t^2-3t+2)\frac{dx}{dt} = 1$; $x = \int \frac{dt}{t^2-3t+2} = \int \frac{dt}{t-2} - \int \frac{dt}{t-1} = \ln\left|\frac{t-2}{t-1}\right| + C$; $\frac{t-2}{t-1} = Ce^x$; $t = 3$ and $x = 0$

$\Rightarrow \frac{1}{2} = C \Rightarrow \frac{t-2}{t-1} = \frac{1}{2}e^x \Rightarrow x = \ln\left|2\left(\frac{t-2}{t-1}\right)\right| = \ln|t-2| - \ln|t-1| + \ln 2$

42. $(3t^4+4t^2+1)\frac{dx}{dt} = 2\sqrt{3}$; $x = 2\sqrt{3}\int \frac{dt}{3t^4+4t^2+1} = \sqrt{3}\int \frac{dt}{t^2+\frac{1}{3}} - \sqrt{3}\int \frac{dt}{t^2+1}$

$= 3\tan^{-1}(\sqrt{3}t) - \sqrt{3}\tan^{-1}t + C$; $t = 1$ and $x = \frac{-\pi\sqrt{3}}{4} \Rightarrow -\frac{\sqrt{3}\pi}{4} = \pi - \frac{\sqrt{3}}{4}\pi + C \Rightarrow C = -\pi$

$\Rightarrow x = 3\tan^{-1}(\sqrt{3}t) - \sqrt{3}\tan^{-1}t - \pi$

43. $(t^2+2t)\frac{dx}{dt} = 2x+2$; $\frac{1}{2}\int \frac{dx}{x+1} = \int \frac{dt}{t^2+2t} \Rightarrow \frac{1}{2}\ln|x+1| = \frac{1}{2}\int \frac{dt}{t} - \frac{1}{2}\int \frac{dt}{t+2} \Rightarrow \ln|x+1| = \ln\left|\frac{t}{t+2}\right| + C$;

$t = 1$ and $x = 1 \Rightarrow \ln 2 = \ln\frac{1}{3} + C \Rightarrow C = \ln 2 + \ln 3 = \ln 6 \Rightarrow \ln|x+1| = \ln 6\left|\frac{t}{t+2}\right| \Rightarrow x+1 = \frac{6t}{t+2}$

$\Rightarrow x = \frac{6t}{t+2} - 1,\ t > 0$

44. $(t+1)\frac{dx}{dt} = x^2 + 1 \Rightarrow \int \frac{dx}{x^2+1} = \int \frac{dt}{t+1} \Rightarrow \tan^{-1} x = \ln|t+1| + C$; $t = 0$ and $x = \frac{\pi}{4} \Rightarrow \tan^{-1}\frac{\pi}{4} = \ln|1| + C$
$\Rightarrow C = \tan^{-1}\frac{\pi}{4} = 1 \Rightarrow \tan^{-1} x = \ln|t+1| + 1 \Rightarrow x = \tan(\ln(t+1)+1),\ t > -1$

45. $\frac{1}{y^2-y}\,dy = e^x\,dx \Rightarrow \int \frac{1}{y(y-1)}\,dy = \int e^x\,dx = e^x + C$

$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1} \Rightarrow 1 = A(y-1) + B(y) = (A+B)y - A$

Equating coefficients of like terms gives

$A + B = 0$ and $-A = 1$

Solving the system simultaneously yields $A = -1$, $B = 1$.

$\int \frac{1}{y(y-1)}\,dy = \int -\frac{1}{y}\,dy + \int \frac{1}{y-1}\,dy = -\ln|y| + \ln|y-1| + C_2 \Rightarrow -\ln|y| + \ln|y-1| = e^x + C$

Substitute $x = 0$, $y = 2$.

$-\ln 2 + 0 = 1 + C$ or $C = -1 - \ln 2$

The solution to the initial value problem is

$-\ln|y| + \ln|y-1| = e^x - 1 - \ln 2.$

46. $\frac{1}{(y+1)^2}\,dy = \sin\theta\,d\theta \Rightarrow \int \frac{1}{(y+1)^2}\,dy = \int \sin\theta\,d\theta \Rightarrow -\frac{1}{y+1} = -\cos\theta + C$

Substitute $\theta = \frac{\pi}{2}$, $y = 0 \Rightarrow -1 = 0 + C$ or $C = -1$

The solution to the initial value problem is

$-\frac{1}{y+1} = -\cos\theta - 1 \Rightarrow y + 1 = \frac{1}{\cos\theta + 1} \Rightarrow y = \frac{1}{\cos\theta + 1} - 1$

47. $dy = \frac{dx}{x^2-3x+2}$; $x^2 - 3x + 2 = (x-2)(x-1) \Rightarrow \frac{1}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x-2)$
$\Rightarrow 1 = (A+B)x - A - 2B$

Equating coefficients of like terms gives

$A + B = 0,\ -A - 2B = 1$

Solving the system simultaneously yields $A = 1$, $B = -1$.

$\int dy = \int \frac{dx}{x^2-3x+2} = \int \frac{dx}{x-2} - \int \frac{dx}{x-1}$

$y = \ln|x-2| - \ln|x-1| + C$

Substitute $x = 3$, $y = 0 \Rightarrow 0 = 0 - \ln 2 + C$ or $C = \ln 2$

The solution to the initial value problems is

$y = \ln|x-2| - \ln|x-1| + \ln 2$

48. $\frac{ds}{2s+2} = \frac{dt}{t^2+2t} \Rightarrow \int \frac{ds}{2s+2} = \frac{1}{2}\int \frac{ds}{s+1} = \frac{1}{2}\ln|s+1| + C_1$

$t^2+2t = t(t+2) \Rightarrow \frac{1}{t^2+2t} = \frac{A}{t} + \frac{B}{t+2} \Rightarrow 1 = A(t+2) + Bt \Rightarrow 1 = (A+B)t + 2A$

Equating coefficients of like terms gives $A + B = 0$ and $2A = 1$

Solving the system simultaneously yields $A = \frac{1}{2}$, $B = -\frac{1}{2}$.

$\int \frac{dt}{t^2+2t} = \int \frac{1/2}{t}\,dt - \int \frac{1/2}{t+2}\,dt = \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t+2| + C_2 \Rightarrow \frac{1}{2}\ln|s+1| = \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t+2| + C_3$

$\Rightarrow \ln|s+1| = \ln|t| - \ln|t+2| + C$

Substitute $t = 1$, $s = 1 \Rightarrow \ln 2 = 0 - \ln 3 + C$ or $C = \ln 2 + \ln 3 = \ln 6$

The solution to the initial value problem is

$\ln|s+1| = \ln|t| - \ln|t+2| + \ln 6 \Rightarrow \ln|s+1| = \ln\left|\frac{6t}{t+2}\right| \Rightarrow |s+1| = \left|\frac{6t}{t+2}\right|.$

49. $V = \pi \int_{0.5}^{2.5} y^2\,dx = \pi \int_{0.5}^{2.5} \frac{9}{3x-x^2}\,dx = 3\pi\left(\int_{0.5}^{2.5} \left(-\frac{1}{x-3} + \frac{1}{x}\right)\right)dx = \left[3\pi \ln\left|\frac{x}{x-3}\right|\right]_{0.5}^{2.5} = 3\pi \ln 25$

50. $V = 2\pi \int_0^1 xy\,dx = 2\pi \int_0^1 \frac{2x}{(x+1)(2-x)}\,dx = 4\pi \int_0^1 \left(-\frac{1}{3}\left(\frac{1}{x+1}\right) + \frac{2}{3}\left(\frac{1}{2-x}\right)\right)dx$

$= \left[-\frac{4\pi}{3}(\ln|x+1| + 2\ln|2-x|)\right]_0^1 = \frac{4\pi}{3}(\ln 2)$

51. (a) $\frac{dx}{dt} = kx(N-x) \Rightarrow \int \frac{dx}{x(N-x)} = \int k\,dt \Rightarrow \frac{1}{N}\int \frac{dx}{x} + \frac{1}{N}\int \frac{dx}{N-x} = \int k\,dt \Rightarrow \frac{1}{N}\ln\left|\frac{x}{N-x}\right| = kt + C;$

$k = \frac{1}{250}$, $N = 1000$, $t = 0$ and $x = 2 \Rightarrow \frac{1}{1000}\ln\left|\frac{2}{998}\right| = C \Rightarrow \frac{1}{1000}\ln\left|\frac{x}{1000-x}\right| = \frac{t}{250} + \frac{1}{1000}\ln\left(\frac{1}{499}\right)$

$\Rightarrow \ln\left|\frac{499x}{1000-x}\right| = 4t \Rightarrow \frac{499x}{1000-x} = e^{4t} \Rightarrow 499x = e^{4t}(1000-x) \Rightarrow (499+e^{4t})x = 1000e^{4t} \Rightarrow x = \frac{1000e^{4t}}{499+e^{4t}}$

(b) $x = \frac{1}{2}N = 500 \Rightarrow 500 = \frac{1000e^{4t}}{499+e^{4t}} \Rightarrow 500 \cdot 499 + 500e^{4t} = 1000e^{4t} \Rightarrow e^{4t} = 499 \Rightarrow t = \frac{1}{4}\ln 499 \approx 1.55$ days

52. $\frac{dx}{dt} = k(a-x)(b-x) \Rightarrow \frac{dx}{(a-x)(b-x)} = k\,dt$

(a) $a = b$: $\int \frac{dx}{(a-x)^2} = \int k\,dt \Rightarrow \frac{1}{a-x} = kt + C$; $t = 0$ and $x = 0 \Rightarrow \frac{1}{a} = C \Rightarrow \frac{1}{a-x} = kt + \frac{1}{a}$

$\Rightarrow \frac{1}{a-x} = \frac{akt+1}{a} \Rightarrow a - x = \frac{a}{akt+1} \Rightarrow x = a - \frac{a}{akt+1} = \frac{a^2kt}{akt+1}$

(b) $a \neq b$: $\int \frac{dx}{(a-x)(b-x)} = \int k\,dt \Rightarrow \frac{1}{b-a}\int \frac{dx}{a-x} - \frac{1}{b-a}\int \frac{dx}{b-x} = \int k\,dt \Rightarrow \frac{1}{b-a}\ln\left|\frac{b-x}{a-x}\right| = kt + C;$

$t = 0$ and $x = 0 \Rightarrow \frac{1}{b-a}\ln\frac{b}{a} = C \Rightarrow \ln\left|\frac{b-x}{a-x}\right| = (b-a)kt + \ln\left(\frac{b}{a}\right) \Rightarrow \frac{b-x}{a-x} = \frac{b}{a}e^{(b-a)kt}$

$\Rightarrow x = \frac{ab\left[1 - e^{(a-b)kt}\right]}{a - be^{(a-b)kt}}$

7.4 TRIGONOMETRIC SUBSTITUTIONS

1. $y = 3\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dy = \frac{3\,d\theta}{\cos^2\theta},\ 9 + y^2 = 9(1 + \tan^2\theta) = \frac{9}{\cos^2\theta} \Rightarrow \frac{1}{\sqrt{9+y^2}} = \frac{|\cos\theta|}{3} = \frac{\cos\theta}{3}$

$\left(\text{because } \cos\theta > 0 \text{ when } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right);$

$\int \frac{dy}{\sqrt{9+y^2}} = 3\int \frac{\cos\theta\,d\theta}{3\cos^2\theta} = \int \frac{d\theta}{\cos\theta} = \ln|\sec\theta + \tan\theta| + C' = \ln\left|\frac{\sqrt{9+y^2}}{3} + \frac{y}{3}\right| + C' = \ln\left|\sqrt{9+y^2} + y\right| + C$

2. $\int \frac{3\,dy}{\sqrt{1+9y^2}};\ [3y = x] \to \int \frac{dx}{\sqrt{1+x^2}};\ x = \tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2},\ dx = \frac{dt}{\cos^2 t},\ \sqrt{1+x^2} = \frac{1}{\cos t};$

$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{dt}{\cos^2 t\left(\frac{1}{\cos t}\right)} = \ln|\sec t + \tan t| + C = \ln\left|\sqrt{x^2+1} + x\right| + C = \ln\left|\sqrt{1+9y^2} + 3y\right| + C$

3. $t = 5\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = 5\cos\theta\,d\theta,\ \sqrt{25 - t^2} = 5\cos\theta;$

$\int \sqrt{25-t^2}\,dt = \int (5\cos\theta)(5\cos\theta)\,d\theta = 25\int \cos^2\theta\,d\theta = 25\int \frac{1+\cos 2\theta}{2}\,d\theta = 25\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + C$

$= \frac{25}{2}(\theta + \sin\theta\cos\theta) + C = \frac{25}{2}\left[\sin^{-1}\left(\frac{t}{5}\right) + \left(\frac{t}{5}\right)\left(\frac{\sqrt{25-t^2}}{5}\right)\right] + C = \frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right) + \frac{t\sqrt{25-t^2}}{2} + C$

4. $t = \frac{1}{3}\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\cos\theta\,d\theta,\ \sqrt{1-9t^2} = \cos\theta;$

$\int \sqrt{1-9t^2}\,dt = \frac{1}{3}\int (\cos\theta)(\cos\theta)\,d\theta = \frac{1}{3}\int \cos^2\theta\,d\theta = \frac{1}{6}(\theta + \sin\theta\cos\theta) + C = \frac{1}{6}\left[\sin^{-1}(3t) + 3t\sqrt{1-9t^2}\right] + C$

5. $x = \frac{7}{2}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{7}{2}\sec\theta\tan\theta\,d\theta,\ \sqrt{4x^2 - 49} = \sqrt{49\sec^2\theta - 49} = 7\tan\theta;$

$\int \frac{dx}{\sqrt{4x^2-49}} = \int \frac{\left(\frac{7}{2}\sec\theta\tan\theta\right)d\theta}{7\tan\theta} = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}\ln\left|\frac{2x}{7} + \frac{\sqrt{4x^2-49}}{7}\right| + C$

6. $x = \frac{3}{5}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{3}{5}\sec\theta\tan\theta\,d\theta,\ \sqrt{25x^2 - 9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta;$

$\int \frac{5\,dx}{\sqrt{25x^2-9}} = \int \frac{5\left(\frac{3}{5}\sec\theta\tan\theta\right)d\theta}{3\tan\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C$

7. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\,d\theta,\ \sqrt{x^2-1} = \tan\theta;$

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\tan\theta} = \int \frac{d\theta}{\sec\theta} = \sin\theta + C = \tan\theta\cos\theta + C = \frac{\sqrt{x^2-1}}{x} + C$$

8. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\,d\theta,\ \sqrt{x^2-1} = \tan\theta;$

$$\int \frac{2\,dx}{x^3\sqrt{x^2-1}} = \int \frac{2\tan\theta\sec\theta\,d\theta}{\sec^3\theta\tan\theta} = 2\int \cos^2\theta\,d\theta = 2\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \theta + \sin\theta\cos\theta + C$$

$$= \theta + \tan\theta\cos^2\theta + C = \sec^{-1}x + \sqrt{x^2-1}\left(\frac{1}{x}\right)^2 + C = \sec^{-1}x + \frac{\sqrt{x^2-1}}{x^2} + C$$

9. $x = 2\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{2\,d\theta}{\cos^2\theta},\ \sqrt{x^2+4} = \frac{2}{\cos\theta};$

$$\int \frac{x^3\,dx}{\sqrt{x^2+4}} = \int \frac{(8\tan^3\theta)(\cos\theta)\,d\theta}{\cos^2\theta} = 8\int \frac{\sin^3\theta\,d\theta}{\cos^4\theta} = 8\int \frac{(\cos^2\theta-1)(-\sin\theta)\,d\theta}{\cos^4\theta};$$

$$[t = \cos\theta] \to 8\int \frac{t^2-1}{t^4}\,dt = 8\int\left(\frac{1}{t^2}-\frac{1}{t^4}\right)dt = 8\left(-\frac{1}{t}+\frac{1}{3t^3}\right)+C = 8\left(-\sec\theta + \frac{\sec^3\theta}{3}\right)+C$$

$$= 8\left(-\frac{\sqrt{x^2+4}}{2} + \frac{(x^2+4)^{3/2}}{8\cdot 3}\right) + C = \frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C$$

10. $x = \tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \sec^2\theta\,d\theta,\ \sqrt{x^2+1} = \sec\theta;$

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sec\theta} = \int \frac{\cos\theta\,d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = \frac{-\sqrt{x^2+1}}{x} + C$$

11. $w = 2\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dw = 2\cos\theta\,d\theta,\ \sqrt{4-w^2} = 2\cos\theta;$

$$\int \frac{8\,dw}{w^2\sqrt{4-w^2}} = \int \frac{8\cdot 2\cos\theta\,d\theta}{4\sin^2\theta\cdot 2\cos\theta} = 2\int \frac{d\theta}{\sin^2\theta} = -2\cot\theta + C = \frac{-2\sqrt{4-w^2}}{w} + C$$

12. $w = 3\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dw = 3\cos\theta\,d\theta,\ \sqrt{9-w^2} = 3\cos\theta;$

$$\int \frac{\sqrt{9-w^2}}{w^2}\,dw = \int \frac{3\cos\theta\cdot 3\cos\theta\,d\theta}{9\sin^2\theta} = \int \cot^2\theta\,d\theta = \int\left(\frac{1-\sin^2\theta}{\sin^2\theta}\right)d\theta = \int (\csc^2\theta - 1)\,d\theta$$

$$= -\cot\theta - \theta + C = -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + C$$

13. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\,d\theta,\ (x^2-1)^{3/2} = \tan^3\theta;$

$$\int \frac{dx}{(x^2-1)^{3/2}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\tan^3\theta} = \int \frac{\cos\theta\,d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = -\left(\frac{1}{\tan\theta}\right)\left(\frac{1}{\cos\theta}\right) + C$$

$$= -\left(\frac{1}{\sqrt{x^2-1}}\right)(x) + C = -\frac{x}{\sqrt{x^2-1}} + C$$

14. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\,d\theta,\ \left(x^2-1\right)^{5/2} = \tan^5\theta;$

$$\int \frac{x^2\,dx}{\left(x^2-1\right)^{5/2}} = \int \frac{\sec^2\theta\cdot\sec\theta\tan\theta\,d\theta}{\tan^5\theta} = \int \frac{\cos\theta}{\sin^4\theta}\,d\theta = -\frac{1}{3\sin^3\theta} + C = -\frac{1}{3}\left(\frac{1}{\tan^3\theta}\right)\left(\frac{1}{\cos^3\theta}\right) + C$$

$$= -\frac{1}{3}\left[\frac{1}{\left(x^2-1\right)^{3/2}}\right]\left(x^3\right) + C = -\frac{x^3}{3\left(x^2-1\right)^{3/2}} + C$$

15. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ \left(1-x^2\right)^{3/2} = \cos^3\theta;$

$$\int \frac{\left(1-x^2\right)^{3/2}\,dx}{x^6} = \int \frac{\cos^3\theta\cdot\cos\theta\,d\theta}{\sin^6\theta} = \int \cot^4\theta\csc^2\theta\,d\theta = -\frac{\cot^5\theta}{5} + C = -\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + C$$

16. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ \left(1-x^2\right)^{1/2} = \cos\theta;$

$$\int \frac{\left(1-x^2\right)^{1/2}\,dx}{x^4} = \int \frac{\cos\theta\cdot\cos\theta\,d\theta}{\sin^4\theta} = \int \cot^2\theta\csc^2\theta\,d\theta = -\frac{\cot^3\theta}{3} + C = -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C$$

17. $x = \frac{1}{2}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{1}{2}\sec^2\theta\,d\theta,\ \left(4x^2+1\right)^2 = \sec^4\theta;$

$$\int \frac{8\,dx}{\left(4x^2+1\right)^2} = \int \frac{8\left(\frac{1}{2}\sec^2\theta\right)d\theta}{\sec^4\theta} = 4\int \cos^2\theta\,d\theta = 2(\theta + \sin\theta\cos\theta) + C = 2\left(\theta + \tan\theta + \cos^2\theta\right) + C$$

$$= 2\tan^{-1}2x + \frac{4x}{\left(4x^2+1\right)} + C$$

18. $t = \frac{1}{3}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\sec^2\theta\,d\theta,\ 9t^2+1 = \sec^2\theta;$

$$\int \frac{6\,dt}{\left(9t^2+1\right)^2} = \int \frac{6\left(\frac{1}{3}\sec^2\theta\right)d\theta}{\sec^4\theta} = 2\int \cos^2\theta\,d\theta = \theta + \sin\theta\cos\theta + C = \theta + \tan\theta\cos^2\theta + C$$

$$= \tan^{-1}3t + \frac{3t}{\left(9t^2+1\right)} + C$$

19. Let $e^t = 3\tan\theta,\ t = \ln(3\tan\theta),\ dt = \frac{\sec^2\theta}{\tan\theta}\,d\theta,\ \sqrt{e^{2t}+9} = \sqrt{9\tan^2\theta+9} = 3\sec\theta;$

$$\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3\tan\theta\cdot\sec^2\theta\,d\theta}{\tan\theta\cdot 3\sec\theta} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \sec\theta\,d\theta = \left[\ln|\sec\theta+\tan\theta|\right]_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)}$$

$$= \ln\left(\frac{5}{3}+\frac{4}{3}\right) - \ln\left(\frac{\sqrt{10}}{3}+\frac{1}{3}\right) = \ln 9 - \ln\left(1+\sqrt{10}\right)$$

20. Let $e^t = \tan\theta$, $t = \ln(\tan\theta)$, $\frac{3}{4} \le \theta \le \frac{4}{3}$, $dt = \frac{\sec^2\theta}{\tan\theta}\,d\theta$, $1 + e^{2t} = 1 + \tan^2\theta = \sec^2\theta$;

$$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{\left(1+e^{2t}\right)^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{(\tan\theta)\left(\frac{\sec^2\theta}{\tan\theta}\right)d\theta}{\sec^3\theta} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos\theta\,d\theta = [\sin\theta]_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$$

21. $\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$; $\left[u = 2\sqrt{t},\ du = \frac{1}{\sqrt{t}}\,dt\right] \to \int_{1/\sqrt{3}}^{1} \frac{2\,du}{1+u^2}$; $u = \tan\theta$, $\frac{\pi}{6} < \theta < \frac{\pi}{4}$, $du = \sec^2\theta\,d\theta$, $1 + u^2 = \sec^2\theta$;

$$\int_{1/\sqrt{3}}^{1} \frac{4\,du}{u\left(1+u^2\right)} = \int_{\pi/6}^{\pi/4} \frac{2\sec^2\theta\,d\theta}{\sec^2\theta} = [2\theta]_{\pi/6}^{\pi/4} = 2\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\pi}{6}$$

22. $y = e^{\tan\theta}$, $dy = e^{\tan\theta}\sec^2\theta\,d\theta$, $\sqrt{1+(\ln y)^2} = \sqrt{1+\tan^2\theta} = \sec\theta$;

$$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta}\,d\theta = \int_0^{\pi/4} \sec\theta\,d\theta = \left[\ln|\sec\theta + \tan\theta|\right]_0^{\pi/4} = \ln\left(1+\sqrt{2}\right)$$

23. $x = \sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$;

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec\theta\tan\theta} = \theta + C = \sec^{-1}|x| + C$$

24. $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $1 + x^2 = \sec^2\theta$;

$$\int \frac{dx}{x^2+1} = \int \frac{\sec^2\theta\,d\theta}{\sec^2\theta} = \theta + C = \tan^{-1}x + C$$

25. $x = \sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$;

$$\int \frac{x\,dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta\cdot\sec\theta\tan\theta\,d\theta}{\tan\theta} = \int \sec^2\theta\,d\theta = \tan\theta + C = \sqrt{x^2-1} + C$$

26. $x = \sin\theta$, $dx = \cos\theta\,d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$;

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\,d\theta}{\cos\theta} = \theta + C = \sin^{-1}x + C$$

27. $x\frac{dy}{dx} = \sqrt{x^2-4}$; $dy = \sqrt{x^2-4}\,\frac{dx}{x}$; $y = \int \frac{\sqrt{x^2-4}}{x}\,dx$; $\left[\begin{array}{c} x = 2\sec\theta,\ 0 < \theta < \frac{\pi}{2} \\ dx = 2\sec\theta\tan\theta\,d\theta \\ \sqrt{x^2-4} = 2\tan\theta \end{array}\right]$

$$\to y = \int \frac{(2\tan\theta)(2\sec\theta\tan\theta)\,d\theta}{2\sec\theta} = 2\int \tan^2\theta\,d\theta = 2\int \left(\sec^2\theta - 1\right)d\theta = 2(\tan\theta - \theta) + C$$

$= 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right] + C$; $x = 2$ and $y = 0 \Rightarrow 0 = 0 + C \Rightarrow C = 0 \Rightarrow y = 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\frac{x}{2}\right]$

28. $\sqrt{x^2-9}\,\frac{dy}{dx} = 1$, $dy = \frac{dx}{\sqrt{x^2-9}}$; $y = \int \frac{dx}{\sqrt{x^2-9}}$; $\left[\begin{array}{c} x = 3\sec\theta,\ 0 < \theta < \frac{\pi}{2} \\ dx = 3\sec\theta\tan\theta\, d\theta \\ \sqrt{x^2-9} = 3\tan\theta \end{array}\right] \to y = \int \frac{3\sec\theta\tan\theta\, d\theta}{3\tan\theta}$

$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$; $x = 5$ and $y = \ln 3 \Rightarrow \ln 3 = \ln 3 + C \Rightarrow C = 0$

$\Rightarrow y = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|$

29. $(x^2+4)\frac{dy}{dx} = 3$, $dy = \frac{3\,dx}{x^2+4}$; $y = 3\int \frac{dx}{x^2+4} = \frac{3}{2}\tan^{-1}\frac{x}{2} + C$; $x = 2$ and $y = 0 \Rightarrow 0 = \frac{3}{2}\tan^{-1} 1 + C$

$\Rightarrow C = -\frac{3\pi}{8} \Rightarrow y = \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{3\pi}{8}$

30. $(x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}$, $dy = \frac{dx}{(x^2+1)^{3/2}}$; $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, $(x^2+1)^{3/2} = \sec^3\theta$;

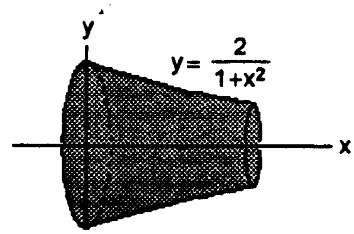
$y = \int \frac{\sec^2\theta\, d\theta}{\sec^3\theta} = \int \cos\theta\, d\theta = \sin\theta + C = \tan\theta\cos\theta + C = \frac{\tan\theta}{\sec\theta} + C = \frac{x}{\sqrt{x^2+1}} + C$; $x = 0$ and $y = 1$

$\Rightarrow 1 = 0 + C \Rightarrow y = \frac{x}{\sqrt{x^2+1}} + 1$

31. $A = \int_0^3 \frac{\sqrt{9-x^2}}{3}\, dx$; $x = 3\sin\theta$, $0 \le \theta \le \frac{\pi}{2}$, $dx = 3\cos\theta\, d\theta$, $\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta} = 3\cos\theta$;

$A = \int_0^{\pi/2} \frac{3\cos\theta \cdot 3\cos\theta\, d\theta}{3} = 3\int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{3}{2}\left[\theta + \sin\theta\cos\theta\right]_0^{\pi/2} = \frac{3\pi}{4}$

32. $V = \int_0^1 \pi\left(\frac{2}{1+x^2}\right)^2 dx = 4\pi\int_0^1 \frac{dx}{(x^2+1)^2}$;



$x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, $x^2 + 1 = \sec^2\theta$;

$V = 4\pi\int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = 4\pi\int_0^{\pi/4} \cos^2\theta\, d\theta = 2\pi\int_0^{\pi/4} (1 + \cos 2\theta)\, d\theta = 2\pi\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \pi\left(\frac{\pi}{2} + 1\right)$

33. (a) From the figure, $\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x}$

(b) From part (a), $z = \frac{\sin x}{1+\cos x} \Rightarrow z(1+\cos x) = \sin x \Rightarrow z^2(1+\cos x)^2 = \sin^2 x$

$\Rightarrow z^2(1+\cos x)^2-(1-\cos x)(1+\cos x)=0 \Rightarrow (1+\cos x)(z^2+z^2\cos x-1+\cos x)=0$

$1+\cos x=0$ or $(z^2+1)\cos x=1-z^2$

$\cos x=-1$ $\qquad \cos x=\frac{1-z^2}{1+z^2}$

$\cos x=-1$ does not make sense in this case.

(c) From part (b), $\cos x=\frac{1-z^2}{1+z^2} \Rightarrow \sin^2 x=1-\frac{(1-z^2)^2}{(1+z^2)^2}=\frac{(1+z^2)^2-(1-z^2)^2}{(1+z^2)^2}$

$=\frac{1+2z^2+z^4-1+2z^2-z^4}{(1+z^2)^2}=\frac{4z^2}{(1+z^2)^2} \Rightarrow \sin x=\pm\frac{2z}{1+z^2}$

Only $\sin x=\frac{2z}{1+z^2}$ makes sense in this case.

(d) $z=\tan\frac{x}{2}$, $dz=\left(\frac{1}{2}\sec^2\frac{x}{2}\right)dx \Rightarrow dz=\frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx \Rightarrow dz=\frac{1}{2}(1+z^2)\,dx \Rightarrow dx=\frac{2\,dz}{1+z^2}$

34. $\int \frac{dx}{1+\sin x}=\int \frac{\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}}=\int \frac{2\,dz}{z^2+2z+1}=\int \frac{2\,dz}{(z+1)^2}=-\frac{2}{z+1}+C=-\frac{2}{\tan\frac{x}{2}+1}+C$

35. $\int \frac{dx}{1-\cos x}=\int \frac{\frac{2\,dz}{1+z^2}}{1-\frac{1-z^2}{1+z^2}}=\int \frac{dz}{z^2}=-\frac{1}{z}+C=-\frac{1}{\tan\frac{x}{2}}+C$

36. $\int \frac{d\theta}{1-\sin\theta}=\int \frac{\frac{2\,dz}{1+z^2}}{1-\frac{2z}{1+z^2}}=\int \frac{2\,dz}{z^2-2z+1}=\int \frac{2\,dz}{(z-1)^2}=-\frac{2}{z-1}+C=-\frac{2}{\tan\frac{\theta}{2}-1}+C$

$=\frac{2}{1-\tan\frac{\theta}{2}}+C$

37. $\int \frac{dt}{1+\sin t+\cos t}=\int \frac{\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}}=\int \frac{dz}{z+1}=\ln|z+1|+C=\ln\left|\tan\frac{t}{2}+1\right|+C$

38. $\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta}=\int_0^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{2+\left(\frac{1-z^2}{1+z^2}\right)}=\int_0^1 \frac{2\,dz}{2+2z^2+1-z^2}=\int_0^1 \frac{2\,dz}{z^2+3}=\frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{z}{\sqrt{3}}\right]_0^1=\frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}$

$=\frac{\pi}{3\sqrt{3}}=\frac{\sqrt{3}\pi}{9}$

39. $\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta+\sin\theta}=\int_1^{\sqrt{3}} \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2\,dz}{1+z^2}\right)}{\left[\frac{2z(1-z^2)}{(1+z^2)^2}+\left(\frac{2z}{1+z^2}\right)\right]}=\int_1^{\sqrt{3}} \frac{2(1-z^2)\,dz}{2z-2z^3+2z+2z^3}=\int_1^{\sqrt{3}} \frac{1-z^2}{2z}\,dz$

$$=\left[\frac{1}{2}\ln z-\frac{z^2}{4}\right]_1^{\sqrt{3}}=\left(\frac{1}{2}\ \ln\ \sqrt{3}-\frac{3}{4}\right)-\left(0-\frac{1}{4}\right)=\frac{\ln 3}{4}-\frac{1}{2}=\frac{1}{4}(\ln 3-2)=\frac{1}{2}(\ln\sqrt{3}-1)$$

40. $$\int\frac{dt}{\sin t-\cos t}=\int\frac{\left(\frac{2\,dz}{1+z^2}\right)}{\left(\frac{2z}{1+z^2}-\frac{1-z^2}{1+z^2}\right)}=\int\frac{2\,dz}{2z-1+z^2}=\int\frac{2\,dz}{(z+1)^2-2}=\frac{1}{\sqrt{2}}\ln\left|\frac{z+1-\sqrt{2}}{z+1+\sqrt{2}}\right|+C$$

$$=\frac{1}{\sqrt{2}}\ln\left|\frac{\tan\left(\frac{t}{2}\right)+1-\sqrt{2}}{\tan\left(\frac{t}{2}\right)+1+\sqrt{2}}\right|+C$$

41. $$\int\frac{\cos t\,dt}{1-\cos t}=\int\frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2\,dz}{1+z^2}\right)}{1-\left(\frac{1-z^2}{1+z^2}\right)}=\int\frac{2\left(1-z^2\right)dz}{\left(1+z^2\right)^2-\left(1+z^2\right)\left(1-z^2\right)}=\int\frac{2\left(1-z^2\right)dz}{\left(1+z^2\right)\left(1+z^2-1+z^2\right)}$$

$$=\int\frac{\left(1-z^2\right)dz}{\left(1+z^2\right)z^2}=\int\frac{dz}{z^2\left(1+z^2\right)}-\int\frac{dz}{1+z^2}=\int\frac{dz}{z^2}-2\int\frac{dz}{z^2+1}=-\frac{1}{z}-2\tan^{-1}z+C=-\cot\left(\frac{t}{2}\right)-t+C$$

7.5 INTEGRAL TABLES, COMPUTER ALGEBRA SYSTEMS, AND MONTE CARLO INTEGRATION

1. $$\int\frac{dx}{x\sqrt{x-3}}=\frac{2}{\sqrt{3}}\tan^{-1}\sqrt{\frac{x-3}{3}}+C$$

(We used FORMULA 13(a) with $a=1$, $b=-3$)

2. $$\int\frac{x\,dx}{\sqrt{x-2}}=\int\frac{(x-2)\,dx}{\sqrt{x-2}}+2\int\frac{dx}{\sqrt{x-2}}=\int\left(\sqrt{x-2}\right)^1dx+2\int\left(\sqrt{x-2}\right)^{-1}dx$$

$$=\left(\frac{2}{1}\right)\frac{\left(\sqrt{x-2}\right)^3}{3}+2\left(\frac{2}{1}\right)\frac{\left(\sqrt{x-2}\right)^1}{1}=\sqrt{x-2}\left[\frac{2(x-2)}{3}+4\right]+C$$

(We used FORMULA 11 with $a=1$, $b=-2$, $n=1$ and $a=1$, $b=-2$, $n=-1$)

3. $$\int x\sqrt{2x-3}\,dx=\frac{1}{2}\int(2x-3)\sqrt{2x-3}\,dx+\frac{3}{2}\int\sqrt{2x-3}\,dx=\frac{1}{2}\int\left(\sqrt{2x-3}\right)^3dx+\frac{3}{2}\int\left(\sqrt{2x-3}\right)^1dx$$

$$=\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\frac{\left(\sqrt{2x-3}\right)^5}{5}+\left(\frac{3}{2}\right)\left(\frac{2}{2}\right)\frac{\left(\sqrt{2x-3}\right)^3}{3}+C=\frac{(2x-3)^{3/2}}{2}\left[\frac{2x-3}{5}+1\right]+C=\frac{(2x-3)^{3/2}(x+1)}{5}+C$$

(We used FORMULA 11 with $a=2$, $b=-3$, $n=3$ and $a=2$, $b=-3$, $n=1$)

4. $$\int\frac{\sqrt{9-4x}}{x^2}\,dx=-\frac{\sqrt{9-4x}}{x}+\frac{(-4)}{2}\int\frac{dx}{x\sqrt{9-4x}}+C$$

(We used FORMULA 14 with $a=-4$, $b=9$)

$$= -\frac{\sqrt{9-4x}}{x} - 2\left(\frac{1}{\sqrt{9}}\right) \ln \left| \frac{\sqrt{9-4x}-\sqrt{9}}{\sqrt{9-4x}+\sqrt{9}} \right| + C$$

(We used FORMULA 13(b) with $a = -4$, $b = 9$)

$$= \frac{-\sqrt{9-4x}}{x} - \frac{2}{3} \ln \left| \frac{\sqrt{9-4x}-3}{\sqrt{9-4x}+3} \right| + C$$

5. $\int x\sqrt{4x-x^2}\, dx = \int x\sqrt{2\cdot 2x - x^2}\, dx = \frac{(x+2)(2x-3\cdot 2)\sqrt{2\cdot 2\cdot x - x^2}}{6} + \frac{2^3}{2} \sin^{-1}\left(\frac{x-2}{2}\right) + C$

$$= \frac{(x+2)(2x-6)\sqrt{4x-x^2}}{6} + 4 \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

(We used FORMULA 51 with $a = 2$)

6. $\int \frac{dx}{x\sqrt{7+x^2}} = \int \frac{dx}{x\sqrt{\left(\sqrt{7}\right)^2 + x^2}} = -\frac{1}{\sqrt{7}} \ln \left| \frac{\sqrt{7} + \sqrt{\left(\sqrt{7}\right)^2 + x^2}}{x} \right| + C = -\frac{1}{\sqrt{7}} \ln \left| \frac{\sqrt{7} + \sqrt{7+x^2}}{x} \right| + C$

(We used FORMULA 26 with $a = \sqrt{7}$)

7. $\int \frac{\sqrt{4-x^2}}{x}\, dx = \int \frac{\sqrt{2^2-x^2}}{x}\, dx = \sqrt{2^2-x^2} - 2 \ln \left| \frac{2+\sqrt{2^2-x^2}}{x} \right| + C = \sqrt{4-x^2} - 2 \ln \left| \frac{2+\sqrt{4-x^2}}{x} \right| + C$

(We used FORMULA 31 with $a = 2$)

8. $\int \sqrt{25-p^2}\, dp = \int \sqrt{5^2-p^2}\, dp = \frac{p}{2}\sqrt{5^2-p^2} + \frac{5^2}{2} \sin^{-1} \frac{p}{5} + C = \frac{p}{2}\sqrt{25-p^2} + \frac{25}{2} \sin^{-1} \frac{p}{5} + C$

(We used FORMULA 29 with $a = 5$)

9. $\int \frac{r^2}{\sqrt{4-r^2}}\, dr = \int \frac{r^2}{\sqrt{2^2-r^2}}\, dr = \frac{2^2}{2} \sin^{-1}\left(\frac{r}{2}\right) - \frac{1}{2} r\sqrt{2^2-r^2} + C = 2 \sin^{-1}\left(\frac{r}{2}\right) - \frac{1}{2} r\sqrt{4-r^2} + C$

(We used FORMULA 33 with $a = 2$)

10. $\int \frac{d\theta}{5+4\sin 2\theta} = \frac{-2}{2\sqrt{25-16}} \tan^{-1}\left[\sqrt{\frac{5-4}{5+4}} \tan\left(\frac{\pi}{4} - \frac{2\theta}{2}\right)\right] + C = -\frac{1}{3} \tan^{-1}\left[\frac{1}{3} \tan\left(\frac{\pi}{4} - \theta\right)\right] + C$

(We used FORMULA 70 with $b = 5$, $c = 4$, $a = 2$)

11. $\int e^{2t} \cos 3t\, dt = \frac{e^{2t}}{2^2+3^2}(2\cos 3t + 3\sin 3t) + C = \frac{e^{2t}}{13}(2\cos 3t + 3\sin 3t) + C$

(We used FORMULA 108 with $a = 2$, $b = 3$)

12. $\int x \cos^{-1} x\, dx = \int x^1 \cos^{-1} x\, dx = \frac{x^{1+1}}{1+1} \cos^{-1} x + \frac{1}{1+1} \int \frac{x^{1+1}\, dx}{\sqrt{1-x^2}} = \frac{x^2}{2} \cos^{-1} x + \frac{1}{2} \int \frac{x^2\, dx}{\sqrt{1-x^2}}$

(We used FORMULA 100 with $a = 1$, $n = 1$)

$= \frac{x^2}{2}\cos^{-1}x + \frac{1}{2}\left(\frac{1}{2}\sin^{-1}x\right) - \frac{1}{2}\left(\frac{1}{2}x\sqrt{1-x^2}\right) + C = \frac{x^2}{2}\cos^{-1}x + \frac{1}{4}\sin^{-1}x - \frac{1}{4}x\sqrt{1-x^2} + C$

(We used FORMULA 33 with $a = 1$)

13. $\int \frac{ds}{(9-s^2)^2} = \int \frac{ds}{(3^3-s^2)^2} = \frac{s}{2\cdot 3^2\cdot(3^2-s^2)} + \frac{1}{2\cdot 3^2}\int \frac{ds}{3^2-s^2}$

(We used FORMULA 19 with $a = 3$)

$= \frac{s}{18(9-s^2)} + \frac{1}{18}\left(\frac{1}{2\cdot 3}\ln\left|\frac{s+3}{s-3}\right|\right) + C = \frac{s}{18(9-s^2)} + \frac{1}{108}\ln\left|\frac{s+3}{s-3}\right| + C$

(We used FORMULA 18 with $a = 3$)

14. $\int \frac{\sqrt{4x+9}}{x^2}\,dx = -\frac{\sqrt{4x+9}}{x} + \frac{4}{2}\int \frac{dx}{x\sqrt{4x+9}}$

(We used FORMULA 14 with $a = 4$, $b = 9$)

$= -\frac{\sqrt{4x+9}}{x} + 2\left(\frac{1}{\sqrt{9}}\ln\left|\frac{\sqrt{4x+9}-\sqrt{9}}{\sqrt{4x+9}+\sqrt{9}}\right|\right) + C = -\frac{\sqrt{4x+9}}{x} + \frac{2}{3}\ln\left|\frac{\sqrt{4x+9}-3}{\sqrt{4x+9}+3}\right| + C$

(We used FORMULA 13(b) with $a = 4$, $b = 9$)

15. $\int \frac{\sqrt{3t-4}}{t}\,dt = 2\sqrt{3t-4} + (-4)\int \frac{dt}{t\sqrt{3t-4}}$

(We used FORMULA 12 with $a = 3$, $b = -4$)

$= 2\sqrt{3t-4} - 4\left(\frac{2}{\sqrt{4}}\tan^{-1}\sqrt{\frac{3t-4}{4}}\right) + C = 2\sqrt{3t-4} - 4\tan^{-1}\sqrt{\frac{3t-4}{4}} + C$

(We used FORMULA 13(a) with $a = 3$, $b = -4$)

16. $\int x^2\tan^{-1}x\,dx = \frac{x^{2+1}}{2+1}\tan^{-1}x - \frac{1}{2+1}\int \frac{x^{2+1}}{1+x^2}\,dx = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx$

(We used FORMULA 101 with $a = 1$, $n = 2$);

$\int \frac{x^3}{1+x^2}\,dx = \int x\,dx - \int \frac{x\,dx}{1+x^2} = \frac{x^2}{2} - \frac{1}{2}\ln(1+x^2) + C \Rightarrow \int x^2\tan^{-1}x\,dx$

$= \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\ln(1+x^2) + C$

17. $\int \sin 3x\cos 2x\,dx = -\frac{\cos 5x}{10} - \frac{\cos x}{2} + C$

(We used FORMULA 62(a) with $a = 3$, $b = 2$)

18. $\int 8\sin 4t\sin\frac{t}{2}\,dx = \frac{8}{7}\sin\left(\frac{7t}{2}\right) - \frac{8}{9}\sin\left(\frac{9t}{2}\right) + C = 8\left[\frac{\sin\left(\frac{7t}{2}\right)}{7} - \frac{\sin\left(\frac{9t}{2}\right)}{9}\right] + C$

(We used FORMULA 62(b) with $a = 4$, $b = \frac{1}{2}$)

19. $\int \cos\frac{\theta}{3}\cos\frac{\theta}{4}\,d\theta = 6\sin\left(\frac{\theta}{12}\right)+\frac{6}{7}\sin\left(\frac{7\theta}{12}\right)+C$

(We used FORMULA 62(c) with $a=\frac{1}{3}$, $b=\frac{1}{4}$)

20. $\int \cos\frac{\theta}{2}\cos 7\theta\,d\theta = \frac{1}{13}\sin\left(\frac{13\theta}{2}\right)+\frac{1}{15}\sin\left(\frac{15\theta}{2}\right)+C = \frac{\sin\left(\frac{13\theta}{2}\right)}{13}+\frac{\sin\left(\frac{15\theta}{2}\right)}{15}+C$

(We used FORMULA 62(c) with $a=\frac{1}{2}$, $b=7$)

21. $\int \frac{x^3+x+1}{(x^2+1)^2}\,dx = \int \frac{x\,dx}{x^2+1}+\int \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1}+\int \frac{dx}{(x^2+1)^2}$

$= \frac{1}{2}\ln|x^2+1|+\frac{x}{2(1+x^2)}+\frac{1}{2}\tan^{-1}x+C$

(For the second integral we used FORMULA 17 with $a=1$)

22. $\int \frac{x^2+6x}{(x^2+3)^2}\,dx = \int \frac{dx}{x^2+3}+\int \frac{6x\,dx}{(x^2+3)^2}-\int \frac{3\,dx}{(x^2+3)^2} = \int \frac{dx}{x^2+(\sqrt{3})^2}+3\int \frac{d(x^2+3)}{(x^2+3)^2}-3\int \frac{dx}{\left[x^2+(\sqrt{3})^2\right]^2}$

$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-\frac{3}{(x^2+3)}-3\left(\frac{x}{2(\sqrt{3})^2\left((\sqrt{3})^2+x^2\right)}+\frac{1}{2(\sqrt{3})^3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right)+C$

(For the first integral we used FORMULA 16 with $a=\sqrt{3}$; for the third integral we used FORMULA 17 with $a=\sqrt{3}$)

$= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-\frac{3}{x^2+3}-\frac{x}{2(x^2+3)}+C$

23. $\int \sin^{-1}\sqrt{x}\,dx;\ \begin{bmatrix} u=\sqrt{x} \\ x=u^2 \\ dx=2u\,du \end{bmatrix} \to 2\int u^1\sin^{-1}u\,du = 2\left(\frac{u^{1+1}}{1+1}\sin^{-1}u-\frac{1}{1+1}\int \frac{u^{1+1}}{\sqrt{1-u^2}}\,du\right)$

$= u^2\sin^{-1}u-\int \frac{u^2\,du}{\sqrt{1-u^2}}$

(We used FORMULA 99 with $a=1$, $n=1$)

$= u^2\sin^{-1}u-\left(\frac{1}{2}\sin^{-1}u-\frac{1}{2}u\sqrt{1-u^2}\right)+C = \left(u^2-\frac{1}{2}\right)\sin^{-1}u+\frac{1}{2}u\sqrt{1-u^2}+C$

(We used FORMULA 33 with $a=1$)

$= \left(x-\frac{1}{2}\right)\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x-x^2}+C$

24. $\int \frac{\cos^{-1}\sqrt{x}}{\sqrt{x}}\,dx;\ \begin{bmatrix} u=\sqrt{x} \\ x=u^2 \\ dx=2u\,du \end{bmatrix} \to \int \frac{\cos^{-1}u}{u}\cdot 2u\,du = 2\int \cos^{-1}u\,du = 2\left(u\cos^{-1}u-\frac{1}{1}\sqrt{1-u^2}\right)+C$

(We used FORMULA 97 with $a=1$)

$= 2\left(\sqrt{x}\cos^{-1}\sqrt{x} - \sqrt{1-x}\right) + C$

25. $\int (\cot t)\sqrt{1-\sin^2 t}\, dt = \int \frac{\sqrt{1-\sin^2 t}\,(\cos t)\, dt}{\sin t};\ \begin{bmatrix} u = \sin t \\ du = \cos t\, dt \end{bmatrix} \to \int \frac{\sqrt{1-u^2}\, du}{u}$

$= \sqrt{1-u^2} - \ln\left|\frac{1+\sqrt{1-u^2}}{u}\right| + C$

(We used FORMULA 31 with $a = 1$)

$= \sqrt{1-\sin^2 t} - \ln\left|\frac{1+\sqrt{1-\sin^2 t}}{\sin t}\right| + C$

26. $\int \frac{dt}{(\tan t)\sqrt{4-\sin^2 t}} = \int \frac{\cos t\, dt}{(\sin t)\sqrt{4-\sin^2 t}};\ \begin{bmatrix} u = \sin t \\ du = \cos t\, dt \end{bmatrix} \to \int \frac{du}{u\sqrt{4-u^2}} = -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-u^2}}{u}\right| + C$

(We used FORMULA 34 with $a = 2$)

$= -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-\sin^2 t}}{\sin t}\right| + C$

27. $\int \frac{dy}{y\sqrt{3+(\ln y)^2}};\ \begin{bmatrix} u = \ln y \\ y = e^u \\ dy = e^u\, du \end{bmatrix} \to \int \frac{e^u\, du}{e^u\sqrt{3+u^2}} = \int \frac{du}{\sqrt{3+u^2}} = \ln\left|u + \sqrt{3+u^2}\right| + C$

$= \ln\left|\ln y + \sqrt{3+(\ln y)^2}\right| + C$

(We used FORMULA 20 with $a = \sqrt{3}$)

28. $\int \frac{\cos\theta\, d\theta}{\sqrt{5+\sin^2\theta}};\ \begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \to \int \frac{du}{\sqrt{5+u^2}} = \ln\left|u + \sqrt{5+u^2}\right| + C = \ln\left|\sin\theta + \sqrt{5+\sin^2\theta}\right| + C$

(We used FORMULA 20 with $a = \sqrt{5}$)

29. $\int \frac{3\, dr}{\sqrt{9r^2-1}};\ \begin{bmatrix} u = 3r \\ du = 3\, dr \end{bmatrix} \to \int \frac{du}{\sqrt{u^2-1}} = \ln\left|u + \sqrt{u^2-1}\right| + C = \ln\left|3r + \sqrt{9r^2-1}\right| + C$

(We used FORMULA 36 with $a = 1$)

30. $\int \frac{3\, dy}{\sqrt{1+9y^2}};\ \begin{bmatrix} u = 3y \\ du = 3\, dy \end{bmatrix} \to \int \frac{du}{\sqrt{1+u^2}} = \ln\left|u + \sqrt{1+u^2}\right| + C = \ln\left|3y + \sqrt{1+9y^2}\right| + C$

(We used FORMULA 20 with $a = 1$)

31. $\int \cos^{-1}\sqrt{x}\,dx;\left[\begin{array}{c} t=\sqrt{x} \\ x=t^2 \\ dx=2t\,dt \end{array}\right] \to 2\int t\cos^{-1}t\,dt = 2\left(\frac{t^2}{2}\cos^{-1}t+\frac{1}{2}\int \frac{t^2}{\sqrt{1-t^2}}\,dt\right) = t^2\cos^{-1}t+\int \frac{t^2}{\sqrt{1-t^2}}\,dt$

(We used FORMULA 100 with $a=1$, $n=1$)

$= t^2\cos^{-1}t+\frac{1}{2}\sin^{-1}t-\frac{1}{2}t\sqrt{1-t^2}+C$

(We used FORMULA 33 with $a=1$)

$= x\cos^{-1}\sqrt{x}+\frac{1}{2}\sin^{-1}\sqrt{x}-\frac{1}{2}\sqrt{x}\sqrt{1-x}+C = x\cos^{-1}\sqrt{x}+\frac{1}{2}\sin^{-1}\sqrt{x}-\frac{1}{2}\sqrt{x-x^2}+C$

32. $\int \tan^{-1}\sqrt{y}\,dy;\left[\begin{array}{c} t=\sqrt{y} \\ y=t^2 \\ dy=2t\,dt \end{array}\right] \to 2\int t\tan^{-1}t\,dt = 2\left[\frac{t^2}{2}\tan^{-1}t-\frac{1}{2}\int \frac{t^2}{1+t^2}\,dt\right] = t^2\tan^{-1}t-\int \frac{t^2}{1+t^2}\,dt$

(We used FORMULA 101 with $n=1$, $a=1$)

$= t^2\tan^{-1}t-\int \frac{t^2+1}{t^2+1}\,dt+\int \frac{dt}{1+t^2} = t^2\tan^{-1}t-t+\tan^{-1}t+C = y\tan^{-1}\sqrt{y}+\tan^{-1}\sqrt{y}-\sqrt{y}+C$

33. $\int xe^{3x}\,dx = \frac{e^{3x}}{3^2}(3x-1)+C = \frac{e^{3x}}{9}(3x-1)+C$

(We used FORMULA 104 with $a=3$)

34. $\int x^3e^{x/2}\,dx = 2x^3e^{x/2}-3\cdot 2\int x^2e^{x/2}\,dx = 2x^3e^{x/2}-6\left(2x^2e^{x/2}-2\cdot 2\int xe^{x/2}\,dx\right)$

$= 2x^3e^{x/2}-12x^2e^{x/2}+24\cdot 4e^{x/2}\left(\frac{x}{2}-1\right)+C = 2x^3e^{x/2}-12x^2e^{x/2}+96e^{x/2}\left(\frac{x}{2}-1\right)+C$

(We used FORMULA 105 with $a=\frac{1}{2}$ twice and FORMULA 104 with $a=\frac{1}{2}$)

35. $\int x^2 2^x\,dx = \frac{x^2 2^x}{\ln 2}-\frac{2}{\ln 2}\int x2^x\,dx = \frac{x^2 2^x}{\ln 2}-\frac{2}{\ln 2}\left(\frac{x2^x}{\ln 2}-\frac{1}{\ln 2}\int 2^x\,dx\right) = \frac{x^2 2^x}{\ln 2}-\frac{2}{\ln 2}\left[\frac{x2^x}{\ln 2}-\frac{2^x}{(\ln 2)^2}\right]+C$

(We used FORMULA 106 with $a=1$, $b=2$)

36. $\int x\pi^x\,dx = \frac{x\pi^x}{\ln \pi}-\frac{1}{\ln \pi}\int \pi^x\,dx = \frac{x\pi^x}{\ln \pi}-\frac{1}{\ln \pi}\left(\frac{\pi^x}{\ln \pi}\right)+C = \frac{x\pi^x}{\ln \pi}-\frac{\pi^x}{\ln \pi}-\frac{\pi^x}{(\ln \pi)^2}+C$

(We used FORMULA 106 with $n=1$, $b=\pi$, $a=1$)

37. $\int \frac{1}{8}\sinh^5 3x\,dx = \frac{1}{8}\left(\frac{\sinh^4 3x\cosh 3x}{5\cdot 3}-\frac{5-1}{5}\int \sinh^3 3x\,dx\right)$

$= \frac{\sinh^4 3x\cosh 3x}{120}-\frac{1}{10}\left(\frac{\sinh^2 3x\cosh 3x}{3\cdot 3}-\frac{3-1}{3}\int \sinh 3x\,dx\right)$

(We used FORMULA 117 with $a=3$, $n=5$ and $a=1$, $n=3$)

$= \frac{\sinh^4 3x \cosh 3x}{120} - \frac{\sinh^2 3x \cosh 3x}{90} + \frac{2}{30}\left(\frac{1}{3} \cosh 3x\right) + C$

$= \frac{1}{120} \sinh^4 3x \cosh 3x - \frac{1}{90} \sinh^2 3x \cosh 3x + \frac{2}{90} \cosh 3x + C$

38. $\int \frac{\cosh^4 \sqrt{x}}{\sqrt{x}}\, dx;\ \begin{bmatrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to 2 \int \cosh^4 u\, du = 2\left(\frac{\cosh^3 u \sinh u}{4} + \frac{4-1}{4} \int \cosh^2 u\, du\right)$

$= \frac{\cosh^3 u \sinh u}{2} + \frac{3}{2}\left(\frac{\sinh 2u}{4} + \frac{u}{2}\right) + C$

(We used FORMULA 118 with $a = 1$, $n = 2$ and FORMULA 116 with $a = 1$)

$= \frac{1}{2} \cosh^3 \sqrt{x} \sinh \sqrt{x} + \frac{3}{8} \sinh 2\sqrt{x} + \frac{3}{4}\sqrt{x} + C$

39. $\int x^2 \cosh 3x\, dx = \frac{x^2}{3} \sinh 3x - \frac{2}{3} \int x \sinh 3x\, dx = \frac{x^2}{3} \sinh 3x - \frac{2}{3}\left(\frac{x}{3} \cosh 3x - \frac{1}{3} \int \cosh 3x\, dx\right)$

(We used FORMULA 122 with $a = 3$, $n = 2$ and FORMULA 121 with $a = 3$, $n = 1$)

$= \frac{x^2}{3} \sinh 3x - \frac{2x}{9} \cosh 3x + \frac{2}{27} \sinh 3x + C$

40. $\int x \sinh 5x\, dx = \frac{x}{5} \cosh 5x - \frac{1}{25} \sinh 5x + C$

(We used FORMULA 119 with $a = 5$, $n = 1$)

41. $u = ax + b \Rightarrow x = \frac{u-b}{a} \Rightarrow dx = \frac{du}{a}$;

$\int \frac{x\, dx}{(ax+b)^2} = \int \frac{(u-b)}{au^2} \frac{du}{a} = \frac{1}{a^2} \int \left(\frac{1}{u} - \frac{b}{u^2}\right) du = \frac{1}{a^2}\left[\ln |u| + \frac{b}{u}\right] + C = \frac{1}{a^2}\left[\ln |ax+b| + \frac{b}{ax+b}\right] + C$

42. $x = a \sin \theta \Rightarrow a^2 - x^2 = a^2 \cos^2 \theta \Rightarrow -2x\, dx = -2a^2 \cos \theta \sin \theta\, d\theta \Rightarrow dx = a \cos \theta\, d\theta$;

$\int \sqrt{a^2 - x^2}\, dx = \int a \cos \theta (a \cos \theta)\, d\theta = a^2 \int \cos^2 \theta\, d\theta = \frac{a^2}{2} \int (1 + \cos 2\theta)\, d\theta = \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$

$= \frac{a^2}{2}(\theta + \cos \theta \sin \theta) + C = \frac{a^2}{2}\left(\theta + \sqrt{1 - \sin^2 \theta} \cdot \sin \theta\right) + C = \frac{a^2}{2}\left(\sin^{-1} \frac{x}{a} + \frac{\sqrt{a^2 - x^2}}{a} \cdot \frac{x}{a}\right) + C$

$= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C$

43. $\int x^n (\ln ax)^m\, dx = \int (\ln ax)^m\, d\left(\frac{x^{n+1}}{n+1}\right) = \frac{x^{n+1}(\ln ax)^m}{n+1} - \int \left(\frac{x^{n+1}}{n+1}\right) m (\ln ax)^{m-1}\left(\frac{1}{x}\right) dx$

$= \frac{x^{n+1}(\ln ax)^m}{n+1} - \frac{m}{n+1} \int x^n (\ln ax)^{m-1}\, dx$

$\left(\text{We used integration by parts } \int u\, dv = uv - \int v\, du \text{ with } u = (\ln ax)^m,\ v = \frac{x^{n+1}}{n+1}\right)$

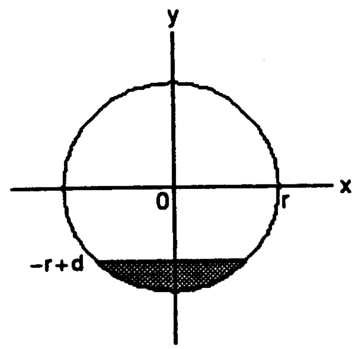
44. $\int x^n \sin^{-1} ax\,dx = \int \sin^{-1} ax\, d\left(\frac{x^{n+1}}{n+1}\right) = \frac{x^{n+1}}{n+1}\sin^{-1} ax - \int \left(\frac{x^{n+1}}{n+1}\right)\frac{a}{\sqrt{1-(ax)^2}}\,dx$

$= \frac{x^{n+1}}{n+1}\sin^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}},\ n \neq -1$

$\left(\text{We used integration by parts } \int u\,dv = uv - \int v\,du \text{ with } u = \sin^{-1} ax,\ v = \frac{x^{n+1}}{n+1}\right)$

45. (a) The volume of the filled part equals the length of the tank times the area of the shaded region shown in the accompanying figure. Consider a layer of gasoline of thickness dy located at height y where $-r < y < -r+d$. The width of this layer is $2\sqrt{r^2-y^2}$. Therefore, $A = 2\int_{-r}^{-r+d}\sqrt{r^2-y^2}\,dy$

and $V = L\cdot A = 2L\int_{-r}^{-r+d}\sqrt{r^2-y^2}\,dy$



(b) $2L\int_{-r}^{-r+d}\sqrt{r^2-y^2}\,dy = 2L\left[\frac{y\sqrt{r^2-y^2}}{2}+\frac{r^2}{2}\sin^{-1}\frac{y}{r}\right]_{-r}^{-r+d}$

(We used FORMULA 29 with $a = r$)

$= 2L\left[\frac{(d-r)}{2}\sqrt{2rd-d^2}+\frac{r^2}{2}\sin^{-1}\left(\frac{d-r}{r}\right)+\frac{r^2}{2}\left(\frac{\pi}{2}\right)\right] = 2L\left[\left(\frac{d-r}{2}\right)\sqrt{2rd-d^2}+\left(\frac{r^2}{2}\right)\left(\sin^{-1}\left(\frac{d-r}{r}\right)+\frac{\pi}{2}\right)\right]$

46. The integrand $f(x) = \sqrt{x-x^2}$ is nonnegative, so the integral is maximized by integrating over the function's entire domain, which runs from $x = 0$ to $x = 1$

$\Rightarrow \int_0^1 \sqrt{x-x^2}\,dx = \int_0^1 \sqrt{2\cdot\frac{1}{2}x-x^2}\,dx = \left[\frac{\left(x-\frac{1}{2}\right)}{2}\sqrt{2\cdot\frac{1}{2}x-x^2}+\frac{\left(\frac{1}{2}\right)^2}{2}\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$

$\left(\text{We used FORMULA 48 with } a = \frac{1}{2}\right)$

$= \left[\frac{\left(x-\frac{1}{2}\right)}{2}\sqrt{x-x^2}+\frac{1}{8}\sin^{-1}(2x-1)\right]_0^1 = \frac{1}{8}\cdot\frac{\pi}{2}-\frac{1}{8}\left(-\frac{\pi}{2}\right) = \frac{\pi}{8}$

CAS EXPLORATIONS

For MAPLE use the int(f(x),x) command, and for MATHEMATICA use the command Integrate[f(x),x], as discussed in the text.

47. (e) $\int x^n \ln x\,dx = \frac{x^{n+1}\ln x}{n+1} - \frac{1}{n+1}\int x^n\,dx,\ n \neq -1$

(We used FORMULA 110 with $a = 1$, $m = 1$)

$$= \frac{x^{n+1} \ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C$$

48. (e) $\int x^{-n} \ln x \, dx = \frac{x^{-n+1} \ln x}{-n+1} - \frac{1}{(-n)+1} \int x^{-n} \, dx, \; n \neq 1$

(We used FORMULA 110 with $a = 1$, $m = 1$, $n = -n$)

$$= \frac{x^{1-n} \ln x}{1-n} - \frac{1}{1-n}\left(\frac{x^{1-n}}{1-n}\right) + C = \frac{x^{1-n}}{1-n}\left(\ln x - \frac{1}{1-n}\right) + C$$

49. (a) Neither MAPLE nor MATHEMATICA can find this integral for arbitrary n.

(b) MAPLE and MATHEMATICA get stuck at about $n = 5$.

(c) Let $x = \frac{\pi}{2} - u \Rightarrow dx = -du$; $x = 0 \Rightarrow u = \frac{\pi}{2}$, $x = \frac{\pi}{2} \Rightarrow u = 0$;

$$I = \int_0^{\pi/2} \frac{\sin^n x \, dx}{\sin^n x + \cos^n x} = \int_{\pi/2}^{0} \frac{-\sin^n\left(\frac{\pi}{2} - u\right) du}{\sin^n\left(\frac{\pi}{2} - u\right) + \cos^n\left(\frac{\pi}{2} - u\right)} = \int_0^{\pi/2} \frac{\cos^n u \, du}{\cos^n u + \sin^n u} = \int_0^{\pi/2} \frac{\cos^n x \, dx}{\cos^n x + \sin^n x}$$

$$\Rightarrow I + I = \int_0^{\pi/2} \left(\frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x}\right) dx = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

The following *Mathematica* module is used to obtain the Monte Carlo estimates of area in Problems 50 through 55.

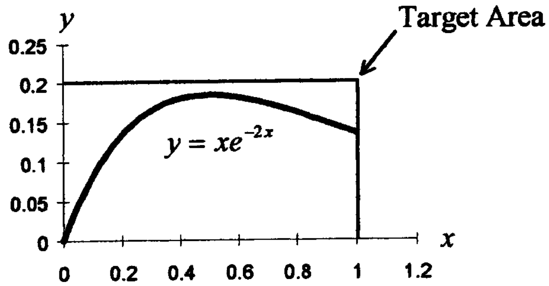
```
monte[f_, indvar_, m_, a_, b_, n_List] :=
 Module[{g, x, xr, yr, area, lim, areaavg, y1, y2},
  g = f/. indvar -> x;
  lim = Length[n];
  area - Table[0, {k, 1, lim}];
  For[k = 1, k <= lim, k++,
    For[counter = 0; i = 1, i <= n[[k]], i++,
     xr = a+ (b - a)*Random[];
     yr = m*Random[];
    If[yr <= g/. x -> xr, counter = counter + 1];];
  area[[k]] = m*(b - a)*counter/n[[k]]];
  areaavg = (Sum[n[[i]]*area[[i]], {i, 1, lim}]) /
    Sum[n[i]], {i, 1, lim}];
  y1 = Integrate[g, {x, a, b}] // N;
  y2 = Integrate[g, {x, a, b}];
  Print[area];
  Print[areaavg];
  Print[y2];
  Print y1  ;
```

The following command executes the preceding module. The arguments are the integrand function, the independent variable, an upper bound on the integrand function, the lower limit of integration, the upper limit of integration, and a list of the numbers of random points to generate in each estimation.

```
monte[z*Sqrt[1 - z], z, 0.5, 0, 1, {100, 200, 300, 400,
  500, 600, 700, 800, 900, 1000, 2000, 3000, 4000,
  5000, 6000, 8000, 10000, 15000, 20000, 30000
```

The preceding command is for Problem 51.

50.



Select M = 0.2
The area approximations will vary depending on the random number generator and seed value that is used

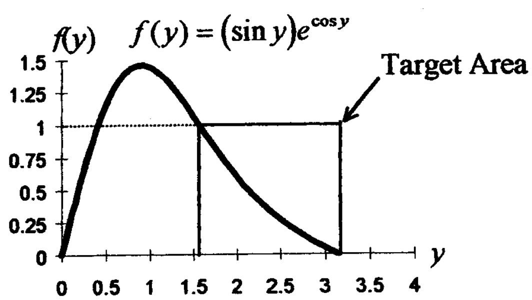
Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.154	2000	0.1492
200	0.151	3000	0.147867
300	0.148	4000	0.1497
400	0.149	5000	0.14712
500	0.1528	6000	0.148433
600	0.151667	8000	0.147925
700	0.149429	10,000	0.14796
800	0.1435	15,000	0.148147
900	0.146444	20,000	0.14824
1000	0.1408	30,000	0.147687

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_0^1 xe^{-2x}\,dx \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area}(i)\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.147987 \text{ by Monte Carlo.}$$

The actual value of the integral is $\frac{(1-3e^2)}{4} \approx 0.148499$.

51.



Select M = 1
The area approximations will vary depending on the random number generator and seed value that is used

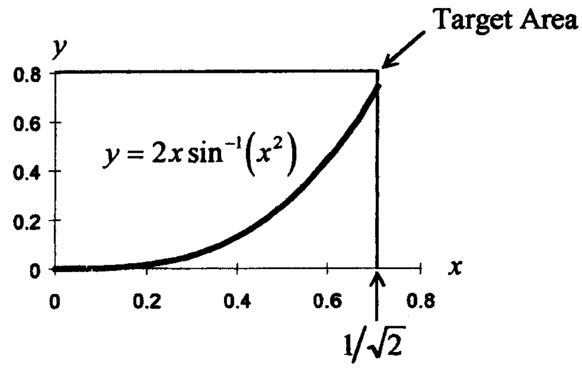
Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.722566	2000	0.628319
200	0.628319	3000	0.646121
300	0.586431	4000	0.642456
400	0.581195	5000	0.636487
500	0.637743	6000	0.627533
600	0.560251	8000	0.643437
700	0.583439	10,000	0.62235
800	0.577268	15,000	0.625386
900	0.5621337	20,000	0.635073
1000	0.655022	30,000	0.638895

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_{\pi/2}^{\pi} (\sin y)e^{\cos y}\, dy \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area}(i)\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.63298 \text{ by Monte Carlo.}$$

The actual value of the integral is $1 - \frac{1}{e} \approx 0.632121$.

52.



Select M = 0.8

The area approximations will vary depending on the random number generator and seed value that is used

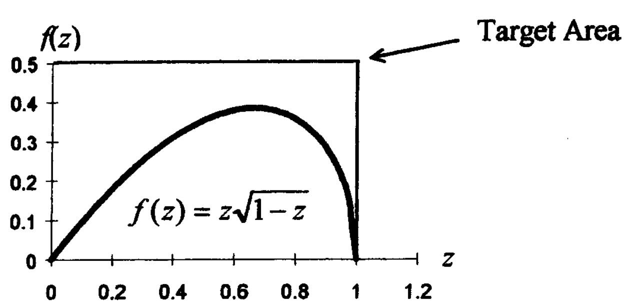
Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.152735	2000	0.129542
200	0.10748	3000	0.133879
300	0.118794	4000	0.125724
400	0.130108	5000	0.123206
500	0.139159	6000	0.130956
600	0.129165	8000	0.128693
700	0.118794	10,000	0.127279
800	0.123744	15,000	0.129844
900	0.121308	20,000	0.129712
1000	0.122188	30,000	0.128335

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_{0}^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area}(i)\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.128523 \text{ by Monte Carlo.}$$

The actual value of the integral is $\frac{\pi - 12 + 6\sqrt{3}}{12} \approx 0.127825$.

53.



Select M = 0.5

The area approximations will vary depending on the random number generator and seed value that is used

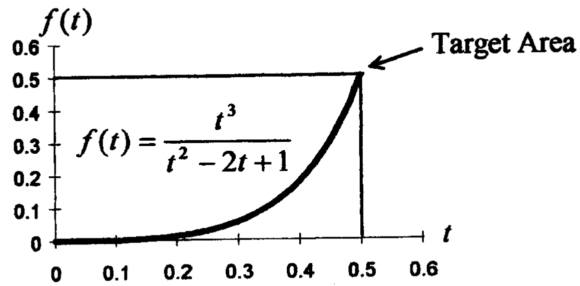
Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.28	2000	0.259
200	0.265	3000	0.262167
300	0.278333	4000	0.259625
400	0.2625	5000	0.2724
500	0.261	6000	0.270583
600	0.27	8000	0.265875
700	0.254286	10,000	0.26495
800	0.270625	15,000	0.2668
900	0.277778	20,000	0.268275
1000	0.2685	30,000	0.265875

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_0^1 z\sqrt{1-z}\,dz \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area}(i)\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.266465 \text{ by Monte Carlo.}$$

The actual value of the integral is $\frac{4}{15} \approx 0.266667$.

54.



Select M = 0.5

The area approximations will vary depending on the random number generator and seed value that is used

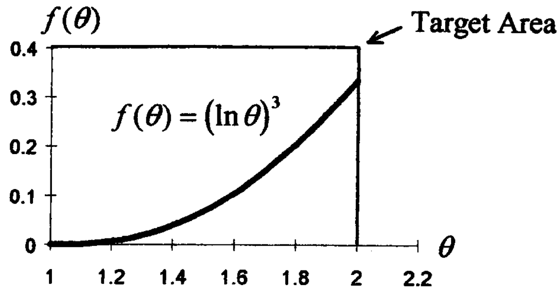
Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.0375	2000	0.04725
200	0.06	3000	0.0435
300	0.06	4000	0.0480625
400	0.0425	5000	0.046
500	0.0435	6000	0.04525
600	0.05125	8000	0.0445937
700	0.0439286	10,000	0.047375
800	0.053125	15,000	0.0449
900	0.0472222	20,000	0.0446375
1000	0.0425	30,000	0.0458

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_0^{1/2} \frac{t^3\,dt}{t^2-2t+1} \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area}(i)\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.0456313 \text{ by Monte Carlo.}$$

The actual value of the integral is $\frac{17}{8} - 3\ln 2 \approx 0.0455585$.

55.



Select M = 0.4

The area approximations will vary depending on the random number generator and seed value that is used

Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.096	2000	0.095
200	0.104	3000	0.103467
300	0.0986667	4000	0.0999
400	0.095	5000	0.10096
500	0.0992	6000	0.1048
600	0.096	8000	0.10105
700	0.0908571	10,000	0.104
800	0.0985	15,000	0.0995733
900	0.1	20,000	0.1013
1000	0.104	30,000	0.100707

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_1^2 (\ln\theta)^3\,d\theta \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area}(i)\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.101054 \text{ by Monte Carlo.}$$

The actual value of the integral is $6 + 2\left[(\ln 2)^3 - 3(\ln 2)^2 + 6\ln 2 - 6\right] \approx 0.101097$.

7.6 L'HÔPITAL'S RULE

1. l'Hôpital: $\lim\limits_{x\to 2} \frac{x-2}{x^2-4} = \frac{1}{2x}\Big|_{x=2} = \frac{1}{4}$ or $\lim\limits_{x\to 2} \frac{x-2}{x^2-4} = \lim\limits_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim\limits_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

2. l'Hôpital: $\lim\limits_{x\to 0} \frac{\sin 5x}{x} = \frac{5\cos 5x}{1}\Big|_{x=0} = 5$ or $\lim\limits_{x\to 0} \frac{\sin 5x}{x} = 5 \lim\limits_{\substack{x\to 0\\ 5x\to 0}} \frac{\sin 5x}{5x} = 5\cdot 1 = 5$

3. l'Hôpital: $\lim\limits_{x\to\infty} \frac{5x^3-3x}{7x^2+1} = \lim\limits_{x\to\infty} \frac{10x-3}{14x} = \lim\limits_{x\to\infty} \frac{10}{14} = \frac{5}{7}$ or $\lim\limits_{x\to\infty} \frac{5x^2-3x}{7x^2+1} = \lim\limits_{x\to\infty} \frac{5-\frac{3}{x}}{7+\frac{1}{x}} = \frac{5}{7}$

4. l'Hôpital: $\lim\limits_{x\to 1} \frac{x^3-1}{4x^3-x-3} = \lim\limits_{x\to 1} \frac{3x^2}{12x^2-1} = \frac{3}{11}$ or $\lim\limits_{x\to 1} \frac{x^3-1}{4x^3-x-3} = \lim\limits_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(4x^2+4x+3)}$

$$= \lim_{x\to 1} \frac{(x^2+x+1)}{(4x^2+4x+3)} = \frac{3}{11}$$

5. l'Hôpital: $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$ or $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \left[\frac{1-\cos x}{x^2}\left(\frac{1+\cos x}{1+\cos x}\right)\right]$

$= \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} = \lim_{x\to 0} \left[\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{x}\right)\left(\frac{1}{1+\cos x}\right)\right] = \frac{1}{2}$

6. l'Hôpital: $\lim_{x\to\infty} \frac{2x^2+3x}{x^3+x+1} = \lim_{x\to\infty} \frac{4x+3}{3x^2+1} = \lim_{x\to\infty} \frac{4}{6x^2} = 0$ or $\lim_{x\to\infty} \frac{2x^2+3x}{x^3+x+1} = \lim_{x\to\infty} \frac{\frac{2}{x}+\frac{3}{x^2}}{1+\frac{1}{x^2}+\frac{1}{x^3}} = \frac{0}{1} = 0$

7. $\lim_{\theta\to 0} \frac{\sin \theta^2}{\theta} = \lim_{\theta\to 0} \frac{2\theta \cos \theta^2}{1} = (2)(0) \cos (0)^2 = 0$

8. $\lim_{\theta\to\pi/2} \frac{1-\sin \theta}{1+\cos 2\theta} = \lim_{\theta\to\pi/2} \frac{-\cos \theta}{-2 \sin 2\theta} = \lim_{\theta\to\pi/2} \frac{\sin \theta}{-4 \cos 2\theta} = \frac{\sin \pi/2}{-4 \cos \pi} = \frac{1}{4}$

9. $\lim_{t\to 0} \frac{\cos t - 1}{e^t - t - 1} = \lim_{t\to 0} \frac{-\sin t}{e^t - 1} = \lim_{t\to 0} \frac{-\cos t}{e^t} = -1$

10. $\lim_{t\to 1} \frac{t-1}{\ln t - \sin \pi t} = \lim_{t\to 1} \frac{1}{\frac{1}{t} - \pi \cos \pi t} = \frac{1}{1-\pi(-1)} = \frac{1}{\pi+1}$

11. $\lim_{x\to\infty} \frac{\ln(x+1)}{\log_2 t} = \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x \ln 2}} = \lim_{x\to\infty} \frac{x \ln 2}{x+1} = \lim_{x\to\infty} \ln 2 = \ln 2$

12. $\lim_{x\to\infty} \frac{\log_2 x}{\log_3 (x+3)} = \lim_{x\to\infty} \frac{\frac{1}{x \ln 2}}{\frac{1}{(x+3) \ln 3}} = \lim_{x\to\infty} \frac{(x+3)\ln 3}{x \ln 2} = \lim_{x\to\infty} \frac{x \ln 3 + 3 \ln 3}{x \ln 2} = \lim_{x\to\infty} \frac{\ln 3}{\ln 2} = \frac{\ln 3}{\ln 2}$

13. $\lim_{y\to 0^+} \frac{\ln (y^2+2y)}{\ln y} = \lim_{y\to 0^+} \frac{\frac{2y+2}{y^2+2y}}{\frac{1}{y}} = \lim_{y\to 0^+} \frac{y(2y+2)}{y^2+2y} = \lim_{y\to 0^+} \frac{2y^2+2y}{y^2+2y} = \lim_{y\to 0^+} \frac{4y+2}{2y+2} = \frac{4(0)+2}{2(0)+2} = \frac{2}{2} = 1$

14. $\lim_{y\to\pi/2} \left(\frac{\pi}{2} - y\right) \tan y = \lim_{y\to\pi/2} \frac{\left(\frac{\pi}{2} - y\right) \sin y}{\cos y} = \lim_{y\to\pi/2} \frac{\left(\frac{\pi}{2} - y\right) \cos y + (-1) \sin y}{-\sin y} = \frac{\left(\frac{\pi}{2} - \frac{\pi}{2}\right) \cos \frac{\pi}{2} + (-1) \sin \frac{\pi}{2}}{-\sin \frac{\pi}{2}}$

$= \frac{(-1)(1)}{-(1)} = 1$

15. $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{-x^2}{x} = \lim_{x\to 0^+} -x = 0$

16. $\lim\limits_{x\to\infty} x \tan \frac{1}{x} = \lim\limits_{x\to\infty} \frac{\tan \frac{1}{x}}{\frac{1}{x}} = \lim\limits_{x\to\infty} \frac{-\frac{1}{x^2}\sec^2 \frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty} \sec^2 \frac{1}{x} = \sec^2 0 = 1$

17. $\lim\limits_{x\to 0^+} (\csc x - \cot x + \cos x) = \lim\limits_{x\to 0^+} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} + \cos x\right) = \lim\limits_{x\to 0^+} \frac{1 - \cos x + \cos x \sin x}{\sin x}$

$= \lim\limits_{x\to 0^+} \frac{\sin x + \cos x \cos x - \sin x \sin x}{\cos x} = 1$

18. $\lim\limits_{x\to\infty} (\ln 2x - \ln(x+1)) = \lim\limits_{x\to\infty} \ln\left(\frac{2x}{x+1}\right)$; Let $f(x) = \frac{2x}{x+1} \Rightarrow \lim\limits_{x\to\infty} \frac{2x}{x+1} = \lim\limits_{x\to\infty} \frac{2}{1} = 2$. Therefore,

$\lim\limits_{x\to\infty} (\ln 2x - \ln \sin x) = \lim\limits_{x\to\infty} \ln f(x) = \ln 2$

19. $\lim\limits_{x\to 0^+} (\ln x - \ln \sin x) = \lim\limits_{x\to 0^+} \ln \frac{x}{\sin x}$; let $f(x) = \frac{x}{\sin x} \Rightarrow \lim\limits_{x\to 0^+} \frac{x}{\sin x} = \lim\limits_{x\to 0^+} \frac{1}{\cos x} = 1$. Therefore,

$\lim\limits_{x\to 0^+} (\ln x - \ln \sin x) = \lim\limits_{x\to 0^+} \ln f(x) = \ln 1 = 0$

20. $\lim\limits_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right) = \lim\limits_{x\to 0^+} \frac{1-\sqrt{x}}{x} = \infty$

21. The limit leads to the indeterminate form 1^∞. Let $f(x) = (e^x + x)^{1/x} \Rightarrow \ln (e^x + x)^{1/x} = \frac{\ln (e^x + x)}{x}$

$\Rightarrow \lim\limits_{x\to 0} \frac{\ln (e^x + x)}{x} = \lim\limits_{x\to 0} \frac{\frac{e^x+1}{e^x+x}}{1} = 2 \Rightarrow \lim\limits_{x\to 0} (e^x + x)^{1/x} \lim\limits_{x\to 0} e^{\ln f(x)} = e^2$

22. The limit leads to the indeterminate form ∞^0. Let $f(x) = \left(\frac{1}{x^2}\right)^x \Rightarrow \ln\left(\frac{1}{x^2}\right)^x = x \ln\left(\frac{1}{x^2}\right) = \frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}}$

$\Rightarrow \lim\limits_{x\to 0} \frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}} = \lim\limits_{x\to 0} \frac{\frac{-2/x^3}{1/x^2}}{-1/x^2} = \lim\limits_{x\to 0} 2x = 0 \Rightarrow \lim\limits_{x\to 0} \left(\frac{1}{x^2}\right)^x = \lim\limits_{x\to 0} e^{\ln f(x)} = e^0 = 1$

23. $\lim\limits_{x\to \pm\infty} \frac{3x-5}{2x^2 - x + 2} = \lim\limits_{x\to \pm\infty} \frac{3}{4x-1} = 0$

24. $\lim\limits_{x\to 0} \frac{\sin 7x}{\tan 11x} = \lim\limits_{x\to 0} \frac{7 \cos 7x}{11 \sec^2 11x} = \frac{7}{11}$

25. The limit leads to the indeterminate form ∞^0. Let $f(x) = (\ln x)^{1/x} \Rightarrow \ln (\ln x)^{1/x} = \frac{\ln (\ln x)}{x}$

$\Rightarrow \lim\limits_{x\to\infty} \frac{\ln (\ln x)}{x} = \lim\limits_{x\to\infty} \frac{\frac{1/x}{\ln x}}{1} = \lim\limits_{x\to\infty} \frac{1}{x \ln x} = 0 \Rightarrow \lim\limits_{x\to\infty} (\ln x)^{1/x} = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^0 = 1$

26. The limit leads to the indeterminate form ∞^0. Let $f(x) = (1+2x)^{1/(2\ln x)} \Rightarrow \ln(1+2x)^{1/(2\ln x)} = \frac{\ln(1+2x)}{2\ln x}$

$$\Rightarrow \lim_{x\to\infty} \frac{\ln(1+2x)}{2\ln x} = \lim_{x\to\infty} \frac{\frac{2}{1+2x}}{\frac{2}{x}} = \lim_{x\to\infty} \frac{x}{1+2x} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2} \Rightarrow \lim_{x\to\infty} (1+2x)^{1/(2\ln x)}$$

$$= \lim_{x\to\infty} e^{\ln f(x)} = e^{1/2} = \sqrt{e}$$

27. The limit leads to the indeterminate form 0^0. Let $f(x) = (x^2-2x+1)^{x-1}$

$$\Rightarrow \ln(x^2-2x+1)^{x-1} = (x-1)\ln(x^2-2x+1) = \frac{\ln(x^2-2x+1)}{\frac{1}{x-1}} \Rightarrow \lim_{x\to1} \frac{\ln(x^2-2x+1)}{\frac{1}{x-1}} = \lim_{x\to1} \frac{\frac{2x-2}{x^2-2x+1}}{-\frac{1}{(x-1)^2}}$$

$$= \lim_{x\to1} \frac{\frac{2(x-1)}{(x-1)^2}}{-\frac{1}{(x-1)^2}} = \lim_{x\to1} -2(x-1) = 0 \Rightarrow \lim_{x\to1} (x^2-2x+1)^{x-1} = \lim_{x\to1} e^{\ln f(x)} = e^0 = 1$$

28. The limit leads to the indeterminate form 0^0. Let $f(x) = (\cos x)^{\cos x} \Rightarrow \ln(\cos x)^{\cos x}$

$$= (\cos x)\ln(\cos x) = \frac{\ln(\cos x)}{\sec x} \Rightarrow \lim_{x\to\pi/2^-} \frac{\ln(\cos x)}{\sec x} = \lim_{x\to\pi/2^-} \frac{\frac{-\sin x}{\cos x}}{\sec x\tan x} = \lim_{x\to\pi/2^-} \frac{-\tan x}{\sec x\tan x}$$

$$= \lim_{x\to\pi/2^-} -\cos x = 0 \Rightarrow \lim_{x\to\pi/2^-} (\cos x)^{\cos x} = \lim_{x\to\pi/2^-} e^{\ln f(x)} = e^0 = 1$$

29. The limit leads to the indeterminate form 1^∞. Let $f(x) = (1+x)^{1/x} \Rightarrow \ln(1+x)^{1/x} = \frac{\ln(1+x)}{x}$

$$\Rightarrow \lim_{x\to0^+} \frac{\ln(1+x)}{x} = \lim_{x\to0^+} \frac{\frac{1}{1+x}}{1} = 1 \Rightarrow \lim_{x\to0^+} (1+x)^{1/x} = \lim_{x\to0^+} e^{\ln f(x)} = e^1 = e$$

30. The limit leads to the indeterminate form 1^∞. Let $f(x) = x^{1/(x-1)} \Rightarrow \ln x^{1/(x-1)} = \frac{\ln x}{x-1}$

$$\Rightarrow \lim_{x\to1} \frac{\ln x}{x-1} = \lim_{x\to1} \frac{1/x}{1} = 1 \Rightarrow \lim_{x\to1} x^{1/(x-1)} = \lim_{x\to1} e^{\ln f(x)} = e^1 = e$$

31. The limit leads to the indeterminate form 0^0. Let $f(x) = (\sin x)^x \Rightarrow \ln(\sin x)^x = x\ln(\sin x) = \frac{\ln(\sin x)}{\frac{1}{x}}$

$$\Rightarrow \lim_{x\to0^+} \frac{\ln(\sin x)}{\frac{1}{x}} = \lim_{x\to0^+} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} = \lim_{x\to0^+} \frac{-x^2\cos x}{\sin x} = \lim_{x\to0^+} \frac{x^2\sin x - 2x\cos x}{\cos x} = 0$$

$$\Rightarrow \lim_{x\to0^+} (\sin x)^x = \lim_{x\to0^+} e^{\ln f(x)} = e^0 = 1$$

32. The limit leads to the indeterminate form 0^0. Let $f(x) = (\sin x)^{\tan x} \Rightarrow \ln(\sin x)^{\tan x}$

$$= \tan x \ln(\sin x) = \frac{\ln(\sin x)}{\cot x} \Rightarrow \lim_{x\to 0^+} \frac{\ln(\sin x)}{\cot x} = \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x} = \lim_{x\to 0^+} (-\sin x \cos x) = 0$$

$$\Rightarrow \lim_{x\to 0^+} (\sin x)^{\tan x} = \lim_{x\to 0^+} e^{\ln f(x)} = e^0 = 1$$

33. The limit leads to the indeterminate form $1^{-\infty}$. Let $f(x) = x^{1/(1-x)} \Rightarrow \ln x^{1/(1-x)} = \frac{\ln x}{1-x}$

$$\Rightarrow \lim_{x\to 1^+} \frac{\ln x}{1-x} = \lim_{x\to 1^+} \frac{\frac{1}{x}}{-1} = -1 \Rightarrow \lim_{x\to 1^+} x^{1/(1-x)} = \lim_{x\to 1^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$$

34. $\lim\limits_{x\to\infty} x^2 e^{-x} = \lim\limits_{x\to\infty} \frac{x^2}{e^x} = \lim\limits_{x\to\infty} \frac{2x}{e^x} = \lim\limits_{x\to\infty} \frac{2}{e^x} = 0$

35. $\lim\limits_{x\to\infty} \int_x^{2x} \frac{1}{t}\,dt = \lim\limits_{x\to\infty} \left[\ln|t|\right]_x^{2x} = \lim\limits_{x\to\infty} \ln\left(\frac{2x}{x}\right) = \ln 2$

36. $\lim\limits_{x\to\infty} \frac{\int_1^x \ln t\,dt}{x \ln x} = \lim\limits_{x\to\infty} \frac{\ln x}{\ln x + 1} = \lim\limits_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = 1$

37. $\lim\limits_{\theta\to 0} \frac{\cos\theta - 1}{e^\theta - \theta - 1} = \lim\limits_{\theta\to 0} \frac{-\sin\theta}{e^\theta - 1} = \lim\limits_{\theta\to 0} \frac{-\cos\theta}{e^\theta} = -1$

38. $\lim\limits_{t\to\infty} \frac{e^t + t^2}{e^t - 1} = \lim\limits_{t\to\infty} \frac{e^t + 2t}{e^t} = \lim\limits_{t\to\infty} \frac{e^t + 2}{e^t} = \lim\limits_{t\to\infty} \frac{e^t}{e^t} = 1$

39. $\lim\limits_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}} = \sqrt{\lim\limits_{x\to\infty} \frac{9x+1}{x+1}} = \sqrt{\lim\limits_{x\to\infty} \frac{9}{1}} = \sqrt{9} = 3$

40. $\lim\limits_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}} = \sqrt{\frac{1}{\lim\limits_{x\to 0^+} \frac{\sin x}{x}}} = \sqrt{\frac{1}{1}} = 1$

41. $\lim\limits_{x\to \pi/2^-} \frac{\sec x}{\tan x} = \lim\limits_{x\to \pi/2^-} \left(\frac{1}{\cos x}\right)\left(\frac{\cos x}{\sin x}\right) = \lim\limits_{x\to \pi/2^-} \frac{1}{\sin x} = 1$

42. $\lim\limits_{x\to 0^+} \frac{\cot x}{\csc x} = \lim\limits_{x\to 0^+} \frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)} = \lim\limits_{x\to 0^+} \cos x = 1$

43. Part (b) is correct because part (a) is neither in the $\frac{0}{0}$ nor $\frac{\infty}{\infty}$ form and so l'Hôpital's rule may not be used.

44. Answers may vary.

(a) $f(x) = 3x + 1$; $g(x) = x$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{3x+1}{x} = \lim_{x\to\infty} \frac{3}{1} = 3$$

(b) $f(x) = x + 1$; $g(x) = x^2$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x+1}{x^2} = \lim_{x\to\infty} \frac{1}{2x} = 0$$

(c) $f(x) = x^2$; $g(x) = x + 1$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{x+1} = \lim_{x\to\infty} \frac{2x}{1} = \infty$$

45. If f(x) is to be continuous at $x = 0$, then $\lim_{x\to 0} f(x) = f(0) \Rightarrow c = f(0) = \lim_{x\to 0} \frac{9x - 3\sin 3x}{5x^3} = \lim_{x\to 0} \frac{9 - 9\cos 3x}{15x^2}$

$= \lim_{x\to 0} \frac{27\sin 3x}{30x} = \lim_{x\to 0} \frac{81\cos 3x}{30} = \frac{27}{10}$.

46. (a) For $x \neq 0$, $f'(x) = \frac{d}{dx}(x+2) = 1$ and $g'(x) = \frac{d}{dx}(x+1) = 1$. Therefore, $\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \frac{1}{1} = 1$, while

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{x+2}{x+1} = \frac{0+2}{0+1} = 2.$$

(b) This does not contradict l'Hôpital's rule because neither f nor g is differentiable at $x = 0$ (as evidenced by the fact that neither is continuous at $x = 0$), so l'Hôpital's rule does not apply.

47. (a) The limit leads to the indeterminate form 1^∞. Let $f(k) = \left(1 + \frac{r}{k}\right)^{kt} \Rightarrow \ln f(k) = kt \ln\left(1 + \frac{r}{k}\right) = \frac{t\ln\left(1+\frac{r}{k}\right)}{\frac{1}{k}}$

$$\Rightarrow \lim_{k\to\infty} \frac{t\ln\left(1+\frac{r}{k}\right)}{\frac{1}{k}} = \lim_{k\to\infty} \frac{t\left(-\frac{r}{k^2}\right)\left(1+\frac{r}{k}\right)^{-1}}{-\frac{1}{k^2}} = \lim_{k\to\infty} \frac{rt}{1+\frac{r}{k}} = \frac{rt}{1} = rt$$

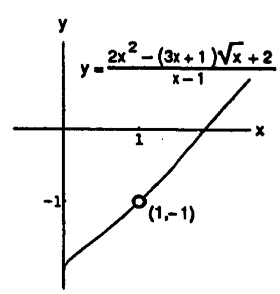
$$\Rightarrow \lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt} = A_0 \lim_{k\to\infty} \left(1+\frac{r}{k}\right)^{kt} = A_0 \lim_{k\to\infty} e^{\ln f(k)} = A_0 e^{rt}$$

(b) Part (a) shows that as the number of compoundings per year increases toward infinity, the limit of interest compounded k times per year is interest compounded continuously.

48. The graph indicates a limit near -1. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to 1} \frac{2x^2 - (3x+1)\sqrt{x} + 2}{x-1}$

$$= \lim_{x\to 1} \frac{2x^2 - 3x^{3/2} - x^{1/2} + 2}{x-1} = \lim_{x\to 1} \frac{4x - \frac{9}{2}x^{1/2} - \frac{1}{2}x^{-1/2}}{1}$$

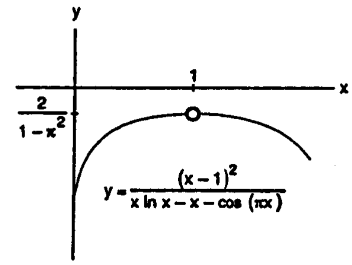
$$= \frac{4 - \frac{9}{2} - \frac{1}{2}}{1} = \frac{4-5}{1} = -1$$



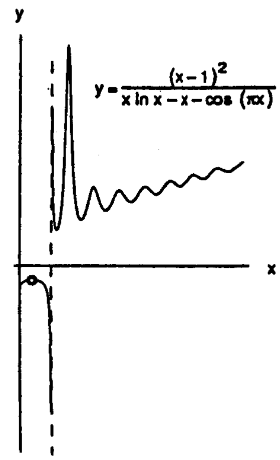
49. (a) The graph indicates a limit near -0.225. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to 1} \frac{(x-1)^2}{x \ln x - x - \cos(\pi x)}$

$$= \lim_{x\to 1} \frac{2(x-1)}{\ln x + 1 - 1 + \pi \sin(\pi x)} = \lim_{x\to 1} \frac{2}{\frac{1}{x} + \pi^2 \cos(\pi x)}$$

$$= \frac{2}{1+\pi^2(-1)} = \frac{2}{1-\pi^2}$$



(b) The graph of $y = \frac{(x-1)^2}{x \ln x - x - \cos(\pi x)}$ has a vertical asymptote near $x = 2.552$.



50. (a) $\ln f(x)^{g(x)} = g(x) \ln f(x)$

$$\lim_{x\to c} (g(x) \ln f(x)) = \left(\lim_{x\to c} g(x)\right)\left(\lim_{x\to c} \ln f(x)\right) = \infty(-\infty) = -\infty$$

$$\lim_{x\to c} f(x)^{g(x)} = \lim_{x\to c} e^{\ln f(x)^{g(x)}} = e^{-\infty} = 0$$

(b) $\lim_{x\to c} (g(x) \ln f(x)) = \left(\lim_{x\to c} g(x)\right)\left(\lim_{x\to c} \ln f(x)\right) = (-\infty)(-\infty) = \infty$

$$\lim_{x\to c} f(x)^{g(x)} = \lim_{x\to c} e^{\ln f(x)^{g(x)}} = e^{\infty} = \infty$$

51. (a) Because the difference in the numerator is so small compared to the values being subtracted, any calculator or computer with limited precision will give the incorrect result that $1 - \cos x^6$ is 0 for even moderately small values of x. For example, at $x = 0.1$, $\cos x^6 \approx 0.9999999999995$ (13 places), so on a 10-place calculator, $\cos x^6 = 1$ and $1 - \cos x^6 = 0$.

(b) Same reason as in part (a) applies.

(c) $\lim_{x\to 0} \frac{1-\cos x^6}{x^{12}} = \lim_{x\to 0} \frac{6x^5 \sin x^6}{12x^{11}} = \lim_{x\to 0} \frac{\sin x^6}{2x^6} = \lim_{x\to 0} \frac{6x^5 \cos x^6}{12x^5} = \lim_{x\to 0} \frac{\cos x^6}{2} = \frac{1}{2}$

(d) The graph and/or table on a grapher shows the value of the function to be 0 for x-values moderately close to 0, but the limit is 1/2. The calculator is giving unreliable information because there is significant round-off error in computing values of this function on a limited precision device.

52. (b) The limit leads to the indeterminate form $\infty - \infty$:

$$\lim_{x\to\infty} \left(x - \sqrt{x^2+x}\right) = \lim_{x\to\infty} \left(x - \sqrt{x^2+x}\right)\left(\frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}}\right) = \lim_{x\to\infty} \left(\frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}}\right) = \lim_{x\to\infty} \left(\frac{-x}{x+\sqrt{x^2+x}}\right)$$

$$= \lim_{x\to\infty} \left(\frac{-1}{1+\sqrt{1+\frac{1}{x}}}\right) = \frac{-1}{1+\sqrt{1+0}} = -\frac{1}{2}$$

53. (a) $f(x) = e^{x\,\ln(1+1/x)}$

$1 + \frac{1}{x} > 0$ when $x < -1$ or $x > 0$

Domain: $(-\infty, -1) \cup (0, \infty)$

(b) The form is 0^{-1}, so $\lim_{x\to -1} f(x) = \infty$

(c) $$\lim_{x\to -\infty} x \ln\left(1+\tfrac{1}{x}\right) = \lim_{x\to -\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to -\infty} \frac{\left(-\frac{1}{x^2}\right)\left(1+\frac{1}{x}\right)^{-1}}{-\frac{1}{x^2}} = \lim_{x\to -\infty} \frac{1}{1+\frac{1}{x}} = 1$$

$$\Rightarrow \lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} e^{x\,\ln(1+1/x)} = e$$

54. (a) $y = x^{1/x} \Rightarrow \ln y = \frac{\ln x}{x} \Rightarrow \frac{y'}{y} = \frac{\left(\frac{1}{x}\right)(x) - \ln x}{x^2} \Rightarrow y' = \left(\frac{1-\ln x}{x^2}\right)\left(x^{1/x}\right)$. The sign pattern is

$y' = \underset{0}{|} + + + + + \underset{e}{|} - - - -$ which indicates a maximum value of $y = e^{1/e}$ when $x = e$

(b) $y = x^{1/x^2} \Rightarrow \ln y = \frac{\ln x}{x^2} \Rightarrow \frac{y'}{y} = \frac{\left(\frac{1}{x}\right)(x^2) - 2x\ln x}{x^4} \Rightarrow y' = \left(\frac{1-2\ln x}{x^3}\right)\left(x^{1/x^2}\right)$. The sign pattern is

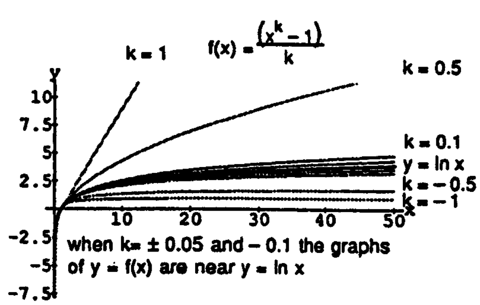
$y' = \underset{0}{|} + + + \underset{\sqrt{e}}{|} - - - -$ which indicates a maximum of $y = e^{1/2e}$ when $x = \sqrt{e}$

(c) $y = x^{1/x^n} \Rightarrow \ln y = \frac{\ln x}{x^n} = \frac{\left(\frac{1}{x}\right)(x^n) - (\ln x)\left(nx^{n-1}\right)}{x^{2n}} \Rightarrow y' = \frac{x^{n-1}(1 - n\ln x)}{x^{2n}} \cdot x^{1/x^n}$. The sign pattern is

$y' = \underset{0}{|} + + + \underset{\sqrt[n]{e}}{|} - - - -$ which indicates a maximum of $y = e^{1/ne}$ when $x = \sqrt[n]{e}$

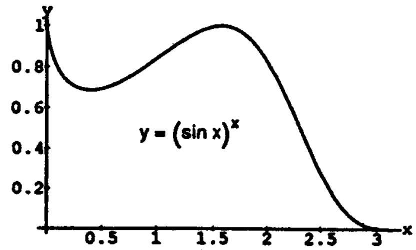
(d) $$\lim_{x\to\infty} x^{1/x^n} = \lim_{x\to\infty} \left(e^{\ln x}\right)^{1/x^n} = \lim_{x\to\infty} e^{(\ln x)/x^n} = \exp\left(\lim_{x\to\infty} \frac{\ln x}{x^n}\right) = \exp\left(\lim_{x\to\infty} \left(\frac{1}{nx^n}\right)\right) = e^0 = 1$$

55. (a)



(b) $\lim_{k\to 0} \frac{x^k - 1}{k} = \lim_{k\to 0} \frac{x^k \ln x}{1} = \ln x$

56. (a) We should assign the value 1 to $f(x) = (\sin x)^x$ to make it continuous at $x = 0$.

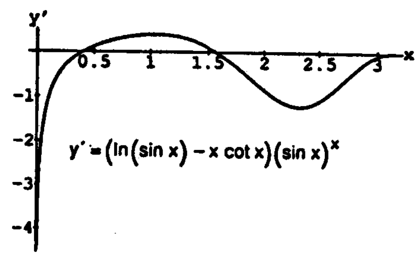


(b) $\ln f(x) = x \ln(\sin x) = \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)} \Rightarrow \lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)} = \lim_{x\to 0^+} \frac{\left(\frac{1}{\sin x}\right)(\cos x)}{\left(-\frac{1}{x^2}\right)}$

$= \lim_{x\to 0} \frac{-x^2}{\tan x} = \lim_{x\to 0} \frac{-2x}{\sec^2 x} = 0 \Rightarrow \lim_{x\to 0} f(x) = e^0 = 1$

(c) The maximum value of f(x) is close to 2 near the point $x \approx 1.55$ (see the graph in part (a)).

(d) The root in question is near 1.57.



(e) $y' = 0 \Rightarrow (\ln(\sin x) - x \cot x)(\sin x)^x = 0 \Rightarrow \ln(\sin x) - x \cot x = 0$. Let $g(x) = \ln(\sin x) - x \cot x$ $\Rightarrow g'(x) = \cot x - \cot x + x \csc^2 x = x \csc^2 x$. Using Newton's method, $g(x) = 0 \Rightarrow x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$ $= x_n - \frac{\ln(\sin x_n) - x_n \cot x_n}{x_n \csc^2 x_n}$. Then $x_1 = 1.55 \Rightarrow x_2 = 1.57093 \Rightarrow x_3 = 1.57080 \Rightarrow x_4 = 1.57080$ $\Rightarrow x_k = 1.57080,\ k \ge 3$.

(f)

x	1.55	1.57	1.57080
$(\sin x)^x$	0.999664854	0.999999502	1

7.7 IMPROPER INTEGRALS

1. (a) The integral is improper because of an infinite limit of integration.

(b) $\int_0^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \left[\tan^{-1} x\right]_0^b = \lim_{b\to\infty} (\tan^{-1} b - 0) = \frac{\pi}{2}$

The integral converges.

(c) $\frac{\pi}{2}$

2. (a) The integral is improper because the integrand has an infinite discontinuity at $x = 0$.

(b) $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^+} \int_b^1 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^+} \left[2\sqrt{x}\right]_b^1 = \lim_{b\to 0^+} \left(2 - 2\sqrt{b}\right) = 2$

The integral converges.

(c) 2

3. (a) The integral involves improper integrals because the integrand has an infinite discontinuity at $x = 0$.

(b) $\int_{-8}^{1} \frac{dx}{x^{1/3}} = \int_{-8}^{0} \frac{dx}{x^{1/3}} + \int_{0}^{1} \frac{dx}{x^{1/3}}$

$\int_{-8}^{0} \frac{dx}{x^{1/3}} = \lim_{b\to 0^-} \int_{-8}^{b} \frac{dx}{x^{1/3}} = \lim_{b\to 0^-} \left[\frac{3}{2}x^{2/3}\right]_{-8}^{b} = \lim_{b\to 0^-} \left(\frac{3}{2}b^{2/3} - 6\right) = -6$

$\int_{0}^{1} \frac{dx}{x^{1/3}} = \lim_{b\to 0^+} \int_{b}^{1} \frac{dx}{x^{1/3}} = \lim_{b\to 0^+} \left[\frac{3}{2}x^{2/3}\right]_{b}^{1} = \lim_{b\to 0^+} \left(\frac{3}{2} - \frac{3}{2}b^{2/3}\right) = \frac{3}{2}$

$\int_{-8}^{1} \frac{dx}{x^{1/3}} = -6 + \frac{3}{2} = -\frac{9}{2}$

The integral converges.

(c) $-\frac{9}{2}$

4. (a) The integral is improper because of two infinite limits of integration.

(b) $\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2} = \int_{-\infty}^{0} \frac{2x\,dx}{(x^2+1)^2} + \int_{0}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$

$\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2} = \lim_{b\to -\infty} \int_{b}^{0} \frac{2x\,dx}{(x^2+1)^2} = \lim_{b\to -\infty} \left[-(x^2+1)^{-1}\right]_{b}^{0} = \lim_{b\to -\infty} \left[-1 + (b^2+1)^{-1}\right] = -1$

$\int_{0}^{\infty} \frac{2x\,dx}{(x^2+1)^2} = \lim_{b\to \infty} \int_{0}^{b} \frac{2x\,dx}{(x^2+1)^2} = \lim_{b\to \infty} \left[-(x^2+1)^{-1}\right]_{0}^{b}$

$\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2} = -1 + 1 = 0$

The integral converges.

(c) 0

5. (a) The integral is improper because the integrand has an infinite discontinuity at 0.

(b) $\int_0^{\ln 2} x^{-2}e^{1/x}\,dx = \lim_{b\to 0^+}\int_b^{\ln 2} x^{-2}e^{1/x}\,dx = \lim_{b\to 0^+}\left[-e^{1/x}\right]_b^{\ln 2} = \lim_{b\to 0^+}\left[-e^{1/\ln 2} + e^{1/b}\right] = \infty$

The integral diverges.

(c) No value

6. (a) The integral is improper because the integrand has an infinite discontinuity at $x = 0$.

(b) $\int_0^{\pi/2} \cot\theta\,d\theta = \lim_{b\to 0^+}\int_b^{\pi/2} \cot\theta\,d\theta = \lim_{b\to 0^+}\int_b^{\pi/2} \frac{\cos\theta\,d\theta}{\sin\theta} = \lim_{b\to 0^+}\left[\ln|\sin\theta|\right]_b^{\pi/2} = \lim_{b\to 0^+}(0 - \ln|\sin b|) = \infty$

The integral diverges.

(c) No value

7. $\int_1^{\infty} \frac{dx}{x^{1.001}} = \lim_{b\to\infty}\int_1^{b} \frac{dx}{x^{1.001}} = \lim_{b\to\infty}\left[-1000x^{-0.001}\right]_1^b = \lim_{b\to\infty}\left(\frac{-1000}{b^{0.001}} + 1000\right) = 1000$

8. $\int_{-1}^{1} \frac{dx}{x^{2/3}} = \int_{-1}^{0} \frac{dx}{x^{2/3}} + \int_0^1 \frac{dx}{x^{2/3}} = \lim_{b\to 0^-}\left[3x^{1/3}\right]_{-1}^b + \lim_{c\to 0^+}\left[3x^{1/3}\right]_c^1$

$= \lim_{b\to 0^-}\left[3b^{1/3} - 3(-1)^{1/3}\right] + \lim_{c\to 0^+}\left[3(1)^{1/3} - 3c^{1/3}\right] = (0+3) + (3-0) = 6$

9. $\int_0^4 \frac{dr}{\sqrt{4-r}} = \lim_{b\to 4^-}\left[-2\sqrt{4-r}\right]_0^b = \lim_{b\to 4^-}\left[-2\sqrt{4-b} - (-2\sqrt{4})\right] = 0 + 4 = 4$

10. $\int_0^1 \frac{dr}{r^{0.999}} = \lim_{b\to 0^+}\left[1000r^{0.001}\right]_b^1 = \lim_{b\to 0^+}\left(1000 - 1000b^{0.001}\right) = 1000 - 0 = 1000$

11. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-}\left[\sin^{-1}x\right]_0^b = \lim_{b\to 1^-}\left(\sin^{-1}b - \sin^{-1}0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

12. $\int_{-\infty}^{2} \frac{2\,dx}{x^2+4} = \lim_{b\to -\infty}\left[\tan^{-1}\frac{x}{2}\right]_b^2 = \lim_{b\to -\infty}\left(\tan^{-1}1 - \tan^{-1}b\right) = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{4}$

13. $\int_{-\infty}^{-2} \frac{2\,dx}{x^2+1} = \int_{-\infty}^{-2} \frac{dx}{x-1} - \int_{-\infty}^{-2} \frac{dx}{x+1} = \lim_{b\to -\infty}\left[\ln|x-1|\right]_b^{-2} - \lim_{b\to -\infty}\left[\ln|x+1|\right]_b^{-2} = \lim_{b\to -\infty}\left[\ln\left|\frac{x-1}{x+1}\right|\right]_b^{-2}$

$= \lim_{b\to -\infty}\left(\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right|\right) = \ln 3 - \ln\left(\lim_{b\to -\infty}\frac{b-1}{b+1}\right) = \ln 3 - \ln 1 = \ln 3$

14. $\int_2^{\infty} \frac{3\,dt}{t^2-1} = \int_2^{\infty} \frac{3\,dt}{t-1} - \int_2^{\infty} \frac{3\,dt}{t} = \lim_{b\to\infty} \left[3\ln(t-1) - 3\ln t\right]_2^b = \lim_{b\to\infty} \left[3\ln\left(\frac{t-1}{t}\right)\right]_2^b$

$= 3\lim_{b\to\infty} \left[\ln\left(\frac{b-1}{b}\right) - \ln\left(\frac{1}{2}\right)\right] = 3\lim_{b\to\infty} \left[\ln\left(\frac{1-\frac{1}{b}}{1}\right) + \ln 2\right] = 3(\ln 1 + \ln 2) = 3\ln 2$

15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta; \begin{bmatrix} u = \theta^2+2\theta \\ du = 2(\theta+1)\,d\theta \end{bmatrix} \to \int_0^3 \frac{du}{2\sqrt{u}} = \lim_{b\to 0^+} \int_b^3 \frac{du}{2\sqrt{u}} = \lim_{b\to 0^+} \left[\sqrt{u}\right]_b^3 = \lim_{b\to 0^+} \left(\sqrt{3}-\sqrt{b}\right)$

$= \sqrt{3} - 0 = \sqrt{3}$

16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds = \frac{1}{2}\int_0^2 \frac{2s\,ds}{\sqrt{4-s^2}} + \int_0^2 \frac{ds}{\sqrt{4-s^2}}; \begin{bmatrix} u = 4-s^2 \\ du = -2s\,ds \end{bmatrix} \to -\frac{1}{2}\int_4^0 \frac{du}{\sqrt{u}} + \lim_{c\to 2^-} \int_0^c \frac{ds}{\sqrt{4-s^2}}$

$= \lim_{b\to 0^+} \int_b^4 \frac{du}{2\sqrt{u}} + \lim_{c\to 2^-} \int_0^c \frac{ds}{\sqrt{4-s^2}} = \lim_{b\to 0^+} \left[\sqrt{u}\right]_b^4 + \lim_{c\to 2^-} \left[\sin^{-1}\frac{s}{2}\right]_0^c$

$= \lim_{b\to 0^+} \left(2-\sqrt{b}\right) + \lim_{c\to 2^-} \left(\sin^{-1}\frac{c}{2} - \sin^{-1} 0\right) = (2-0) + \left(\frac{\pi}{2} - 0\right) = \frac{4+\pi}{2}$

17. $\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}; \begin{bmatrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int_0^{\infty} \frac{2\,du}{u^2+1} = \lim_{b\to\infty} \int_0^b \frac{2\,du}{u^2+1} = \lim_{b\to\infty} \left[2\tan^{-1} u\right]_0^b$

$= \lim_{b\to\infty} \left(2\tan^{-1} b - 1\tan^{-1} 0\right) = 2\left(\frac{\pi}{2}\right) - 2(0) = \pi$

18. $\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \lim_{b\to 1^+} \int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c\to\infty} \int_2^c \frac{dx}{x\sqrt{x^2-1}}$

$= \lim_{b\to 1^+} \left[\sec^{-1}|x|\right]_b^2 + \lim_{c\to\infty} \left[\sec^{-1}|x|\right]_2^c = \lim_{b\to 1^+} \left(\sec^{-1} 2 - \sec^{-1} b\right) + \lim_{c\to\infty} \left(\sec^{-1} c - \sec^{-1} 2\right)$

$= \left(\frac{\pi}{3} - 0\right) + \left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{\pi}{2}$

19. $\int_1^2 \frac{ds}{s\sqrt{s^2-1}} = \lim_{b\to 1^+} \left[\sec^{-1} s\right]_b^2 = \sec^{-1} 2 - \lim_{b\to 1^+} \sec^{-1} b = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

20. $\int_{-1}^{\infty} \frac{d\theta}{\theta^2+5\theta+6} = \lim_{b\to\infty} \left[\ln\left|\frac{\theta+2}{\theta+3}\right|\right]_{-1}^b = \lim_{b\to\infty} \left[\ln\left|\frac{b+2}{b+3}\right|\right] - \ln\left|\frac{-1+2}{-1+3}\right| = 0 - \ln\left(\frac{1}{2}\right) = \ln 2$

21. $\int_2^{\infty} \frac{2\,dv}{v^2-v} = \lim_{b\to\infty} \left[2\ln\left|\frac{v-1}{v}\right|\right]_2^b = \lim_{b\to\infty} \left(2\ln\left|\frac{b-1}{b}\right| - 2\ln\left|\frac{2-1}{2}\right|\right) = 2\ln(1) - 2\ln\left(\frac{1}{2}\right) = 0 + 2\ln 2 = \ln 4$

22. $\int_2^\infty \frac{2\,dt}{t^2-1} = \lim_{b\to\infty} \left[\ln\left|\frac{t-1}{t+1}\right|\right]_2^b = \lim_{b\to\infty} \left(\ln\left|\frac{b-1}{b+1}\right| - \ln\left|\frac{2-1}{2+1}\right|\right) = \ln(1) - \ln\left(\frac{1}{3}\right) = 0 + \ln 3 = \ln 3$

23. $\int_0^2 \frac{ds}{\sqrt{4-s^2}} = \lim_{b\to 2^-} \left[\sin^{-1}\frac{s}{2}\right]_0^b = \lim_{b\to 2^-} \left(\sin^{-1}\frac{b}{2}\right) - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

24. $\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}} = \lim_{b\to 1^-} \left[2\sin^{-1}(r^2)\right]_0^b = \lim_{b\to 1^-} \left[2\sin^{-1}(b^2)\right] - 2\sin^{-1} 0 = 2\cdot\frac{\pi}{2} - 0 = \pi$

25. $\int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1} v)} = \lim_{b\to\infty} \left[\ln\left|1+\tan^{-1} v\right|\right]_0^b = \lim_{b\to\infty} \left[\ln\left|1+\tan^{-1} b\right|\right] - \ln\left|1+\tan^{-1} 0\right|$

$= \ln\left(1+\frac{\pi}{2}\right) - \ln(1+0) = \ln\left(1+\frac{\pi}{2}\right)$

26. $\int_0^\infty \frac{16\tan^{-1} x}{1+x^2}\,dx = \lim_{b\to\infty} \left[8\left(\tan^{-1} x\right)^2\right]_0^b = \lim_{b\to\infty} \left[8\left(\tan^{-1} b\right)^2\right] - 8\left(\tan^{-1} 0\right)^2 = 8\left(\frac{\pi}{2}\right)^2 - 8(0) = 2\pi^2$

27. $\int_{-1}^4 \frac{dx}{\sqrt{|x|}} = \lim_{b\to 0^-} \int_{-1}^b \frac{dx}{\sqrt{-x}} + \lim_{c\to 0^+} \int_c^4 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^-} \left[-2\sqrt{-x}\right]_{-1}^b + \lim_{c\to 0^+} \left[2\sqrt{x}\right]_c^4$

$= \lim_{b\to 0^-} \left(-2\sqrt{-b}\right) - \left(-2\sqrt{-(-1)}\right) + 2\sqrt{4} - 6 \lim_{c\to 0^+} 2\sqrt{c} = 0 + 2 + 2\cdot 2 - 0 = 6$

28. $\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{b\to 1^-} \left[-2\sqrt{1-x}\right]_0^b + \lim_{c\to 1^+} \left[2\sqrt{x-1}\right]_c^2$

$= \lim_{b\to 1^-} \left(-2\sqrt{1-b}\right) - \left(-2\sqrt{1-0}\right) + 2\sqrt{2-1} - \lim_{c\to 1^+} \left(2\sqrt{c-1}\right) = 0 + 2 + 2 - 0 = 4$

29. $\int_{-\infty}^0 \theta e^\theta\,d\theta = \lim_{b\to -\infty} \left[\theta e^\theta - e^\theta\right]_b^0 = \left(0\cdot e^0 - e^0\right) - \lim_{b\to -\infty} \left[be^b - e^b\right] = -1 - \lim_{b\to -\infty} \left(\frac{b-1}{e^{-b}}\right)$

$= -1 - \lim_{b\to -\infty} \left(\frac{1}{-e^{-b}}\right)$ (l'Hôpital's rule for $\frac{\infty}{\infty}$ form)

$= -1 - 0 = -1$

30. $\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta = \lim_{b\to\infty} \int_0^b 2e^{-\theta}\sin\theta\,d\theta$

$= \lim_{b\to\infty} 2\left[\frac{e^{-\theta}}{1+1}(-\sin\theta - \cos\theta)\right]_0^b$ (FORMULA 107 with $a=-1$, $b=1$)

$$= \lim_{b\to\infty} \frac{-2(\sin b + \cos b)}{2e^b} + \frac{2(\sin 0 + \cos 0)}{2e^0} = 0 + \frac{2(0+1)}{2} = 1$$

31. $\int_{-\infty}^{0} e^{-|x|}\,dx = \int_{-\infty}^{0} e^x\,dx = \lim_{b\to-\infty} \left[e^x\right]_b^0 = \lim_{b\to-\infty} \left(1 - e^b\right) = (1-0) = 1$

32. $\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx = \int_{-\infty}^{0} 2xe^{-x^2}\,dx + \int_{0}^{\infty} 2xe^{-x^2}\,dx = \lim_{b\to\infty} \left[-e^{-x^2}\right]_b^0 + \lim_{c\to\infty} \left[-e^{-x^2}\right]_0^c$

$= \lim_{b\to-\infty} \left[-1 - \left(-e^{-b^2}\right)\right] + \lim_{c\to\infty} \left[-e^{-c^2} - (-1)\right] = (-1-0) + (0+1) = 0$

33. $\int_{0}^{1} x \ln x\,dx = \lim_{b\to0^+} \left[\frac{x^2}{2} \ln x - \frac{x^2}{4}\right]_b^1 = \left(\frac{1}{2} \ln 1 - \frac{1}{4}\right) - \lim_{b\to0^+} \left(\frac{b^2}{2} \ln b - \frac{b^2}{4}\right) = -\frac{1}{4} - \lim_{b\to0^+} \frac{\ln b}{\left(\frac{2}{b^2}\right)} + 0$

$= -\frac{1}{4} - \lim_{b\to0^+} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{4}{b^3}\right)} = -\frac{1}{4} + \lim_{b\to0^+} \left(\frac{b^2}{4}\right) = -\frac{1}{4} + 0 = -\frac{1}{4}$

34. $\int_{0}^{1} (-\ln x)\,dx = \lim_{b\to0^+} [x - x\ln x]_b^1 = [1 - 1\ln 1] - \lim_{b\to0^+} [b - b\ln b] = 1 - 0 + \lim_{b\to0^+} \frac{\ln b}{\left(\frac{1}{b}\right)} = 1 - \lim_{b\to0^+} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)}$

$= 1 + \lim_{b\to0^+} b = 1 + 0 = 1$

35. $\int_{0}^{\pi/2} \tan\theta\,d\theta = \lim_{b\to\frac{\pi}{2}^-} \left[-\ln|\cos\theta|\right]_0^b = \lim_{b\to\frac{\pi}{2}^-} \left[-\ln|\cos b|\right] + \ln 1 = \lim_{b\to\frac{\pi}{2}^-} \left[-\ln|\cos b|\right] = -\infty,$

the integral diverges

36. $\int_{0}^{\pi/2} \cot\theta\,d\theta = \lim_{b\to0^+} \left[\ln|\sin\theta|\right]_b^{\pi/2} = \ln 1 - \lim_{b\to0^+} \left[\ln|\sin b|\right] = -\lim_{b\to0^+} \left[\ln|\sin b|\right] = -\infty,$

the integral diverges

37. $\int_{0}^{\pi} \frac{\sin\theta\,d\theta}{\sqrt{\pi-\theta}}$; $[\pi - \theta = x] \to -\int_{\pi}^{0} \frac{\sin x\,dx}{\sqrt{x}} = \int_{0}^{\pi} \frac{\sin x\,dx}{\sqrt{x}}$. Since $0 \le \frac{\sin x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$ for all $0 \le x \le \pi$ and $\int_{0}^{\pi} \frac{dx}{\sqrt{x}}$

converges, then $\int_{0}^{\pi} \frac{\sin x}{\sqrt{x}}\,dx$ converges by the Direct Comparison Test.

38. $\int_{-\pi/2}^{\pi/2} \frac{\cos\theta\,d\theta}{(\pi-2\theta)^{1/3}}$; $\begin{bmatrix} x = \pi - 2\theta \\ \theta = \frac{\pi}{2} - \frac{x}{2} \\ d\theta = -\frac{dx}{2} \end{bmatrix} \to \int_{2\pi}^{0} \frac{-\cos\left(\frac{\pi}{2} - \frac{x}{2}\right)dx}{2x^{1/3}} = \int_{0}^{2\pi} \frac{\sin\left(\frac{x}{2}\right)dx}{2x^{1/3}}$. Since $0 \le \frac{\sin\frac{x}{2}}{x^{1/3}} \le \frac{1}{x^{1/3}}$ for all

$0 \le x \le 2\pi$ and $\int_0^{2\pi} \frac{dx}{2x^{1/3}}$ converges, then $\int_0^{2\pi} \frac{\sin \frac{x}{2}\, dx}{2x^{1/3}}$ converges by the Direct Comparison Test.

39. $\int_0^{\ln 2} x^{-2}e^{-1/x}\, dx; \left[\frac{1}{x} = y\right] \to \int_{\infty}^{1/\ln 2} \frac{y^2e^{-y}\, dy}{-y^2} = \int_{1/\ln 2}^{\infty} e^{-y}\, dy = \lim_{b\to\infty} \left[-e^{-y}\right]_{1/\ln 2}^{b} = \lim_{b\to\infty} \left[-e^{-b}\right] - \left[-e^{-1/\ln 2}\right]$
$= 0 + e^{-1/\ln 2} = e^{-1/\ln 2}$, so the integral converges.

40. $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx; \left[y = \sqrt{x}\right] \to 2\int_0^1 e^{-y}\, dy = 2 - 2e$, so the integral converges.

41. $\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$. Since for $0 \le t \le \pi$, $0 \le \frac{1}{\sqrt{t} + \sin t} \le \frac{1}{\sqrt{t}}$ and $\int_0^{\pi} \frac{dt}{\sqrt{t}}$ converges, then the original integral converges as well by the Direct Comparison Test.

42. $\int_0^1 \frac{dt}{t - \sin t}$; let $f(t) = \frac{1}{t - \sin t}$ and $g(t) = \frac{1}{t^3}$, then $\lim_{t\to 0} \frac{f(t)}{g(t)} = \lim_{t\to 0} \frac{t^3}{t - \sin t} = \lim_{t\to 0} \frac{3t^2}{1 - \cos t} = \lim_{t\to 0} \frac{6t}{\sin t}$
$= \lim_{t\to 0} \frac{6}{\cos t} = 6$. Now, $\int_0^1 \frac{dt}{t^3} = \lim_{b\to 0^+} \left[-\frac{1}{2t^2}\right]_b^1 = -\frac{1}{2} - \lim_{b\to 0^+} \left[-\frac{1}{2b^2}\right] = +\infty$, which diverges $\Rightarrow \int_0^1 \frac{dt}{t - \sin t}$ diverges by the Limit Comparison Test.

43. $\int_0^2 \frac{dx}{1 - x^2} = \int_0^1 \frac{dx}{1 - x^2} + \int_1^2 \frac{dx}{1 - x^2}$ and $\int_0^1 \frac{dx}{1 - x^2} = \lim_{b\to 1^-} \left[\frac{1}{2}\ln\left|\frac{1 + x}{1 - x}\right|\right]_0^b = \lim_{b\to 1^-} \left[\frac{1}{2}\ln\left|\frac{1 + b}{1 - b}\right|\right] - 0 = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1 - x^2}$ diverges as well.

44. $\int_0^2 \frac{dx}{1 - x} = \int_0^1 \frac{dx}{1 - x} + \int_1^2 \frac{dx}{1 - x}$ and $\int_0^1 \frac{dx}{1 - x} = \lim_{b\to 1^-} \left[-\ln(1 - x)\right]_0^b = \lim_{b\to 1^-} \left[-\ln(1 - b)\right] - 0 = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1 - x}$ diverges as well.

45. $\int_{-1}^1 \ln|x|\, dx = \int_{-1}^0 \ln(-x)\, dx + \int_0^1 \ln x\, dx;\ \int_0^1 \ln x\, dx = \lim_{b\to 0^+} \left[x\ln x - x\right]_b^1 = [1 \cdot 0 - 1] - \lim_{b\to 0^+} [b\ln b - b]$
$= -1 - 0 = -1;\ \int_{-1}^0 \ln(-x)\, dx = -1 \Rightarrow \int_{-1}^1 \ln|x|\, dx = -2$ converges.

46. $\int_{-1}^{1} (-x \ln|x|)\, dx = \int_{-1}^{0} [-x \ln(-x)]\, dx + \int_{0}^{1} (-x \ln x)\, dx = \lim_{b\to 0^+} \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_b^1 - \lim_{c\to 0^+} \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_c^1$

$= \left[\frac{1}{2}\ln 1 - \frac{1}{4}\right] - \lim_{b\to 0^+} \left[\frac{b^2}{2}\ln b - \frac{b^2}{4}\right] - \left[\frac{1}{2}\ln 1 - \frac{1}{4}\right] + \lim_{c\to 0^+} \left[\frac{c^2}{2}\ln c - \frac{c^2}{4}\right] = -\frac{1}{4} - 0 + \frac{1}{4} + 0 = 0 \Rightarrow$ the integral converges (see Exercise 33 for the limit calculations).

47. $\int_{1}^{\infty} \frac{dx}{1+x^3}$; $0 \le \frac{1}{x^3+1} \le \frac{1}{x^3}$ for $1 \le x < \infty$ and $\int_{1}^{\infty} \frac{dx}{x^3}$ converges $\Rightarrow \int_{1}^{\infty} \frac{dx}{1+x^3}$ converges by the Direct Comparison Test.

48. $\int_{4}^{\infty} \frac{dx}{\sqrt{x}-1}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x}-1}\right)}{\left(\frac{1}{\sqrt{x}}\right)} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x}-1} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{\sqrt{x}}} = \frac{1}{1-0} = 1$ and $\int_{4}^{\infty} \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \left[2\sqrt{x}\right]_4^b = \infty$,

which diverges $\Rightarrow \int_{4}^{\infty} \frac{dx}{\sqrt{x}-1}$ diverges by the Limit Comparison Test.

49. $\int_{2}^{\infty} \frac{dv}{\sqrt{v-1}}$; $\lim_{v\to\infty} \frac{\left(\frac{1}{\sqrt{v-1}}\right)}{\left(\frac{1}{\sqrt{v}}\right)} = \lim_{v\to\infty} \frac{\sqrt{v}}{\sqrt{v-1}} = \lim_{v\to\infty} \frac{1}{\sqrt{1-\frac{1}{v}}} = \frac{1}{\sqrt{1-0}} = 1$ and $\int_{2}^{\infty} \frac{dv}{\sqrt{v}} = \lim_{b\to\infty} \left[2\sqrt{v}\right]_2^b = \infty$,

which diverges $\Rightarrow \int_{2}^{\infty} \frac{dv}{\sqrt{v-1}}$ diverges by the Limit Comparison Test.

50. $\int_{4}^{\infty} \frac{2\, dt}{t^{3/2}+1}$; $\lim_{t\to\infty} \frac{t^{3/2}}{t^{3/2}+1} = 1$ and $\int_{4}^{\infty} \frac{2\, dt}{t^{3/2}} = \lim_{b\to\infty} \left[-4t^{-1/2}\right]_4^b = \lim_{b\to\infty} \left(\frac{-4}{\sqrt{b}} + 2\right) = 2 \Rightarrow \int_{4}^{\infty} \frac{2\, dt}{t^{3/2}}$ converges

$\Rightarrow \int_{4}^{\infty} \frac{2\, dt}{t^{3/2}+1}$ converges by the Direct Comparison Test.

51. $\int_{0}^{\infty} \frac{dx}{\sqrt{x^6+1}} = \int_{0}^{1} \frac{dx}{\sqrt{x^6+1}} + \int_{1}^{\infty} \frac{dx}{\sqrt{x^6+1}} < \int_{0}^{1} \frac{dx}{\sqrt{x^6+1}} + \int_{1}^{\infty} \frac{dx}{x^3}$ and $\int_{1}^{\infty} \frac{dx}{x^3} = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_1^b$

$= \lim_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{2}\right) = \frac{1}{2} \Rightarrow \int_{0}^{\infty} \frac{dx}{\sqrt{x^6+1}}$ converges by the Direct Comparison Test.

52. $\int_{2}^{\infty} \frac{dx}{\sqrt{x^2-1}}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x^2-1}}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1$; $\int_{2}^{\infty} \frac{1}{x}\, dx = \lim_{b\to\infty} [\ln b]_2^b = \infty$,

which diverges $\Rightarrow \int_2^{\infty} \frac{dx}{\sqrt{x^2-1}}$ diverges by the Limit Comparison Test.

53. $\int_1^{\infty} \frac{\sqrt{x+1}}{x^2}\,dx$; $\lim_{x\to\infty} \frac{\left(\frac{\sqrt{x}}{x^2}\right)}{\left(\frac{\sqrt{x+1}}{x^2}\right)} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}} = 1$; $\int_1^{\infty} \frac{\sqrt{x}}{x^2}\,dx = \int_1^{\infty} \frac{dx}{x^{3/2}}$

$= \lim_{b\to\infty} \left[-2x^{-1/2}\right]_1^b = \lim_{b\to\infty} \left(\frac{-2}{\sqrt{x}}+2\right) = 2 \Rightarrow \int_1^{\infty} \frac{\sqrt{x+1}}{x^2}\,dx$ converges by the Limit Comparison Test.

54. $\int_2^{\infty} \frac{x\,dx}{\sqrt{x^4-1}}$; $\lim_{x\to\infty} \frac{\left(\frac{x}{\sqrt{x^4-1}}\right)}{\left(\frac{x}{\sqrt{x^4}}\right)} = \lim_{x\to\infty} \frac{\sqrt{x^4}}{\sqrt{x^4-1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^4}}} = 1$; $\int_2^{\infty} \frac{x\,dx}{\sqrt{x^4}} = \int_2^{\infty} \frac{dx}{x} = \lim_{b\to\infty} [\ln x]_2^b = \infty$,

which diverges $\Rightarrow \int_2^{\infty} \frac{x\,dx}{\sqrt{x^4-1}}$ diverges by the Limit Comparison Test.

55. $\int_\pi^{\infty} \frac{2+\cos x}{x}\,dx$; $0 < \frac{1}{x} \le \frac{2+\cos x}{x}$ for $x \ge \pi$ and $\int_\pi^{\infty} \frac{dx}{x} = \lim_{b\to\infty} [\ln x]_\pi^b = \infty$, which diverges

$\Rightarrow \int_\pi^{\infty} \frac{2+\cos x}{x}\,dx$ diverges by the Direct Comparison Test.

56. $\int_\pi^{\infty} \frac{1+\sin x}{x^2}\,dx$; $0 \le \frac{1+\sin x}{x^2} \le \frac{2}{x^2}$ for $x \ge \pi$ and $\int_\pi^{\infty} \frac{2}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{2}{x}\right]_\pi^b = \lim_{b\to\infty} \left(-\frac{2}{b}+\frac{2}{\pi}\right) = \frac{2}{\pi}$

$\Rightarrow \int_\pi^{\infty} \frac{2\,dx}{x^2}$ converges $\Rightarrow \int_\pi^{\infty} \frac{1+\sin x}{x^2}\,dx$ converges by the Direct Comparison Test.

57. $\int_0^{\infty} \frac{d\theta}{1+e^\theta}$; $0 \le \frac{1}{1+e^\theta} \le \frac{1}{e^\theta}$ for $0 \le \theta < \infty$ and $\int_0^{\infty} \frac{d\theta}{e^\theta} = \lim_{b\to 0} \left[-e^{-\theta}\right]_0^b = \lim_{b\to\infty} \left(-e^{-b}+1\right) = 1$

$\Rightarrow \int_0^{\infty} \frac{d\theta}{e^\theta}$ converges $\Rightarrow \int_0^{\infty} \frac{d\theta}{1+e^\theta}$ converges by the Direct Comparison Test.

58. $\int_2^{\infty} \frac{dx}{\ln x}$; $0 < \frac{1}{x} < \frac{1}{\ln x}$ for $x > 2$ and $\int_2^{\infty} \frac{dx}{x}$ diverges $\Rightarrow \int_2^{\infty} \frac{dx}{\ln x}$ diverges by the Direct Comparison Test.

59. $\int_1^{\infty} \frac{e^x}{x}\,dx$; $0 < \frac{1}{x} < \frac{e^x}{x}$ for $x > 1$ and $\int_1^{\infty} \frac{dx}{x}$ diverges $\Rightarrow \int_1^{\infty} \frac{e^x\,dx}{x}$ diverges by the Direct Comparison Test.

60. $\int_{e^e}^{\infty} \ln(\ln x)\,dx$; $[x = e^y] \to \int_{e}^{\infty} (\ln y)\,e^y\,dy$; $0 < \ln y < (\ln y)\,e^y$ for $y \ge e$ and $\int_{e}^{\infty} \ln y\,dy = \lim_{b\to\infty} [y \ln y - y]_e^b$ $= \infty$, which diverges $\Rightarrow \int_{e}^{\infty} \ln e^y\,dy$ diverges $\Rightarrow \int_{e^e}^{\infty} \ln(\ln x)\,dx$ diverges by the Direct Comparison Test.

61. $\int_{1}^{\infty} \frac{dx}{\sqrt{e^x - x}}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{e^x-x}}\right)}{\left(\frac{1}{\sqrt{e^x}}\right)} = \lim_{x\to\infty} \frac{\sqrt{e^x}}{\sqrt{e^x - x}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{x}{e^x}}} = \frac{1}{\sqrt{1-0}} = 1$; $\int_{1}^{\infty} \frac{dx}{\sqrt{e^x}} = \int_{1}^{\infty} e^{-x/2}\,dx$ $= \lim_{b\to\infty} \left[-2e^{-x/2}\right]_1^b = \lim_{b\to\infty} \left(-2e^{-b/2} + 2e^{-1/2}\right) = \frac{2}{\sqrt{e}} \Rightarrow \int_{1}^{\infty} e^{-x/2}\,dx$ converges $\Rightarrow \int_{1}^{\infty} \frac{dx}{\sqrt{e^x - x}}$ converges by the Limit Comparison Test.

62. $\int_{1}^{\infty} \frac{dx}{e^x - 2^x}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{e^x - 2^x}\right)}{\left(\frac{1}{e^x}\right)} = \lim_{x\to\infty} \frac{e^x}{e^x - 2^x} = \lim_{x\to\infty} \frac{1}{1-\left(\frac{2}{e}\right)^x} = \frac{1}{1-0} = 1$ and $\int_{1}^{\infty} \frac{dx}{e^x} = \lim_{b\to\infty} \left[-e^{-x}\right]_1^b$ $= \lim_{b\to\infty} \left(-e^{-b} + e^{-1}\right) = \frac{1}{e} \Rightarrow \int_{1}^{\infty} \frac{dx}{e^x}$ converges $\Rightarrow \int_{1}^{\infty} \frac{dx}{e^x - 2^x}$ converges by the Limit Comparison Test.

63. $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}} = 2\int_{0}^{\infty} \frac{dx}{\sqrt{x^4+1}}$; $\lim_{x\to\infty} \frac{x^2}{\sqrt{x^4+1}} = 1$; $\int_{0}^{\infty} \frac{dx}{\sqrt{x^4+1}} = \int_{0}^{1} \frac{dx}{\sqrt{x^4+1}} + \int_{1}^{\infty} \frac{dx}{\sqrt{x^4+1}}$ $< \int_{0}^{1} \frac{dx}{\sqrt{x^4+1}} + \int_{1}^{\infty} \frac{dx}{x^2}$ and $\int_{1}^{\infty} \frac{dx}{x^2} = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}}$ converges by the Direct Comparison Test.

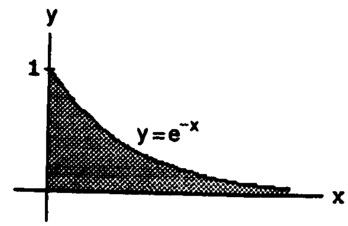
64. $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = 2\int_{0}^{\infty} \frac{dx}{e^x + e^{-x}}$; $0 < \frac{1}{e^x + e^{-x}} < \frac{1}{e^x}$ for $x > 0$; $\int_{0}^{\infty} \frac{dx}{e^x}$ converges $\Rightarrow 2\int_{0}^{\infty} \frac{dx}{e^x + e^{-x}}$ converges by the Direct Comparison Test.

65. (a) $\int_{1}^{2} \frac{dx}{x(\ln x)^p}$; $[t = \ln x] \to \int_{0}^{\ln 2} \frac{dt}{t^p} = \lim_{b\to 0^+} \left[\frac{1}{-p+1} t^{1-p}\right]_b^{\ln 2} = \lim_{b\to 0^+} \frac{b^{1-p}}{p-1} + \frac{1}{1-p}(\ln 2)^{1-p}$
$\Rightarrow$ the integral converges for $p < 1$ and diverges for $p \ge 1$

(b) $\int_{2}^{\infty} \frac{dx}{x(\ln x)^p}$; $[t = \ln x] \to \int_{\ln 2}^{\infty} \frac{dt}{t^p}$ and this integral is essentially the same as in Exercise 67(a): it converges for $p > 1$ and diverges for $p \le 1$

66. $\int_0^\infty \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty} \left[\ln(x^2+1)\right]_0^b = \lim_{b\to\infty} \left[\ln(b^2+1)\right] - 0 = \lim_{b\to\infty} \ln(b^2+1) = \infty \Rightarrow$ the integral $\int_{-\infty}^\infty \frac{2x}{x^2+1}\,dx$ diverges. But $\lim_{b\to\infty} \int_{-b}^b \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty} \left[\ln(x^2+1)\right]_{-b}^b = \lim_{b\to\infty} \left[\ln(b^2+1) - \ln(b^2+1)\right] = \lim_{b\to\infty} \ln\left(\frac{b^2+1}{b^2+1}\right)$ $= \lim_{b\to\infty} (\ln 1) = 0$

67. $A = \int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]_0^b = \lim_{b\to\infty} \left(-e^{-b}\right) - \left(-e^{-0}\right)$ $= 0 + 1 = 1$



68. $V = \int_0^\infty 2\pi x e^{-x}\,dx = 2\pi \int_0^\infty x e^{-x}\,dx = 2\pi \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_0^b = 2\pi\left[\lim_{b\to\infty} \left(-be^{-b} - e^{-b}\right) - 1\right] = 2\pi$

69. $V = \int_0^\infty \pi\left(e^{-x}\right)^2 dx = \pi \int_0^\infty e^{-2x}\,dx = \pi \lim_{b\to\infty} \left[-\frac{1}{2}e^{-2x}\right]_0^b = \pi \lim_{b\to\infty} \left(-\frac{1}{2}e^{-2b} + \frac{1}{2}\right) = \frac{\pi}{2}$

70. $A = \int_0^{\pi/2} (\sec x - \tan x)\,dx = \lim_{b\to\frac{\pi}{2}^-} \left[\ln|\sec x + \tan x| - \ln|\sec x|\right]_0^b = \lim_{b\to\frac{\pi}{2}^-} \left(\ln\left|1 + \frac{\tan b}{\sec b}\right| - \ln|1+0|\right)$ $= \lim_{b\to\frac{\pi}{2}^-} \ln|1 + \sin b| = \ln 2$

71. $\int_3^\infty \left(\frac{1}{x-2} - \frac{1}{x}\right) dx \neq \int_3^\infty \frac{dx}{x-2} - \int_3^\infty \frac{dx}{x}$, since the left hand integral converges but both of the right hand integrals diverge.

72. (a) The statement is true since $\int_{-\infty}^b f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^b f(x)\,dx$, $\int_b^\infty f(x)\,dx = \int_a^\infty f(x)\,dx - \int_a^b f(x)\,dx$ and $\int_a^b f(x)\,dx$ exists since f(x) is integrable on every interval [a, b].

(b) $\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^b f(x)\,dx - \int_a^b f(x)\,dx + \int_a^\infty f(x)\,dx$

$= \int_{-\infty}^b f(x)\,dx + \int_b^a f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx$

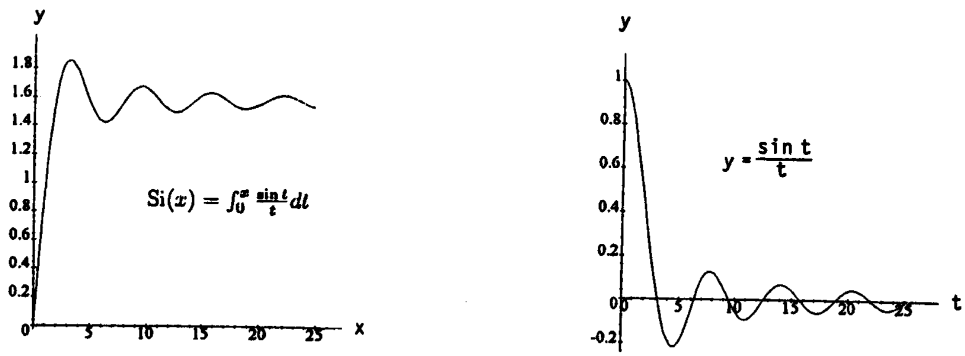
73. (a) $\int_1^{\infty} e^{-3x}\,dx = \lim\limits_{b\to\infty}\left[-\frac{1}{3}e^{-3x}\right]_3^b = \lim\limits_{b\to\infty}\left(-\frac{1}{3}e^{-3b}\right)-\left(-\frac{1}{3}e^{-3\cdot 3}\right) = 0+\frac{1}{3}\cdot e^{-9} = \frac{1}{3}e^{-9}$

$\approx 0.0000411 < 0.000042$. Since $e^{-x^2} \le e^{-3x}$ for $x > 3$, then $\int_3^{\infty} e^{-x^2}\,dx < 0.000042$ and therefore

$\int_0^{\infty} e^{-x^2}\,dx$ can be replaced by $\int_0^3 e^{-x^2}\,dx$ without introducing an error greater than 0.000042.

(b) $\int_0^3 e^{-x^2}\,dx \cong 0.88621$

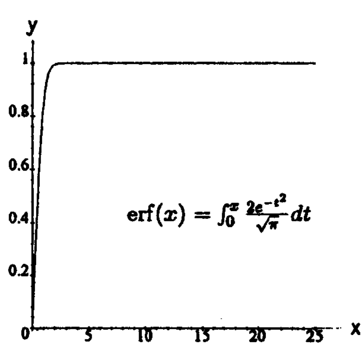
74. (a)



(b) Maple command:

> int((sin(t))/t, t=0..infinity); $\left(\text{answer is } \frac{\pi}{2}\right)$

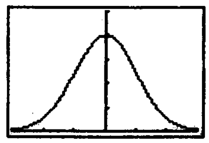
75. (a)



(b) Maple commands:

> f:= 2*exp(−t^2)/sqrt(Pi);
> int(f, t=0..infinity); (answer is 1)

76. (a) $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$



f is increasing on $(-\infty, 0]$. f is decreasing on $[0, \infty)$. f has a local maximum at $(0, f(0)) = \left(0, \frac{1}{\sqrt{2\pi}}\right)$

(b) Maple commands:

>f:=exp(−x^2/2)(sqrt(2*pi);
>int(f,x=−1..1); ≈ 0.683
>int(f,x=−2..2); ≈ 0.954
>int(f,x=−3..3); ≈ 0.997

(c) Part (b) suggests that as b increases, the integral approaches 1. We can make $\int_{-b}^{b} f(x)\,dx$ as close to 1 as we want by choosing b > 1 large enough. Also, we can make $\int_{b}^{\infty} f(x)\,dx$ and $\int_{-\infty}^{-b} f(x)\,dx$ as small as we want by choosing b large enough enough. This is because $0 < f(x) < e^{-x/2}$ for $x > 1$. (Likewise, $0 < f(x) < e^{x/2}$ for $x < -1$.) Thus, $\int_{b}^{\infty} f(x)\,dx < \int_{b}^{\infty} e^{-x/2}\,dx$.

$$\int_{b}^{\infty} e^{-x/2}\,dx = \lim_{c\to\infty} \int_{b}^{c} e^{-x/2}\,dx = \lim_{c\to\infty} \left[-2e^{-x/2}\right]_b^c = \lim_{c\to\infty} \left[-2e^{-c/2} + 2e^{-b/2}\right] = 2e^{-b/2}$$

As $b \to \infty$, $2e^{-b/2} \to 0$, for for large enough b, $\int_{b}^{\infty} f(x)\,dx$ is as small as we want. Likewise, for large enough b, $\int_{-\infty}^{-b} f(x)\,dx$ is as small as we want.

77-80. Use the MAPLE or MATHEMATICA integration commands, as discussed in the text.

CHAPTER 7 PRACTICE EXERCISES

1. $\int x\sqrt{4x^2-9}\,dx;\ \begin{bmatrix} u=4x^2-9 \\ du=8x\,dx \end{bmatrix} \to \frac{1}{8}\int \sqrt{u}\,du = \frac{1}{8}\cdot\frac{2}{3}u^{3/2}+C = \frac{1}{12}\left(4x^2-9\right)^{3/2}+C$

2. $\int x(2x+1)^{1/2}\,dx;\ \begin{bmatrix} u=2x+1 \\ du=2\,dx \end{bmatrix} \to \frac{1}{2}\int \left(\frac{u-1}{2}\right)\sqrt{u}\,du = \frac{1}{4}\left(\int u^{3/2}\,du - \int u^{1/2}\,du\right) = \frac{1}{4}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)+C$

$= \frac{(2x+1)^{5/2}}{10} - \frac{(2x+1)^{3/2}}{6}+C$

3. $\int \frac{x\,dx}{\sqrt{8x^2+1}};\ \begin{bmatrix} u=8x^2+1 \\ du=16x\,dx \end{bmatrix} \to \frac{1}{16}\int \frac{du}{\sqrt{u}} = \frac{1}{16}\cdot 2u^{1/2}+C = \frac{\sqrt{8x^2+1}}{8}+C$

4. $\int \frac{y\,dy}{25+y^2};\ \begin{bmatrix} u=25+y^2 \\ du=2y\,dy \end{bmatrix} \to \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln\left(25+y^2\right)+C$

5. $\int \frac{t^3\,dt}{\sqrt{9-4t^4}};\ \begin{bmatrix} u=9-4t^4 \\ du=-16t^3\,dt \end{bmatrix} \to -\frac{1}{16}\int \frac{du}{\sqrt{u}} = -\frac{1}{16}\cdot 2u^{1/2}+C = -\frac{\sqrt{9-4t^4}}{8}+C$

6. $\int z^{2/3}\left(z^{5/3}+1\right)^{2/3}\,dz;\ \begin{bmatrix} u=z^{5/3}+1 \\ du=\frac{5}{3}z^{2/3}\,dz \end{bmatrix} \to \frac{3}{5}\int u^{2/3}\,du = \frac{3}{5}\cdot\frac{3}{5}u^{5/3}+C = \frac{9}{25}\left(z^{5/3}+1\right)^{5/3}+C$

7. $\int \frac{\sin 2\theta\,d\theta}{(1-\cos 2\theta)^2};\ \begin{bmatrix} u=1-\cos 2\theta \\ du=2\sin 2\theta\,d\theta \end{bmatrix} \to \frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2u}+C = -\frac{1}{2(1-\cos 2\theta)}+C$

8. $\int \frac{\cos 2t\,dt}{1+\sin 2t};\ \begin{bmatrix} u=1+\sin 2t \\ du=2\cos 2t\,dt \end{bmatrix} \to \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln|1+\sin 2t|+C$

9. $\int (\sin 2x)\,e^{\cos 2x}\,dx;\ \begin{bmatrix} u=\cos 2x \\ du=-2\sin 2x\,dx \end{bmatrix} \to -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u+C = -\frac{1}{2}e^{\cos 2x}+C$

10. $\int e^\theta \sec^2\left(e^\theta\right)\,d\theta;\ \begin{bmatrix} u=e^\theta \\ du=e^\theta\,d\theta \end{bmatrix} \to \int \sec^2 u\,du = \tan u + C = \tan\left(e^\theta\right)+C$

11. $\int 2^{x-1}\,dx = \frac{2^{x-1}}{\ln 2}+C$

12. $\int \frac{dv}{v\ln v};\ \begin{bmatrix} u=\ln v \\ du=\frac{1}{v}\,dv \end{bmatrix} \to \int \frac{du}{u} = \ln|u|+C = \ln|\ln v|+C$

13. $\int \frac{dx}{(x^2+1)(2+\tan^{-1}x)}$; $\begin{bmatrix} u = 2+\tan^{-1}x \\ du = \frac{dx}{x^2+1} \end{bmatrix} \to \int \frac{du}{u} = \ln|u| + C = \ln|2+\tan^{-1}x| + C$

14. $\int \frac{2\,dx}{\sqrt{1-4x^2}}$; $\begin{bmatrix} u = 2x \\ du = 2\,dx \end{bmatrix} \to \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}(2x) + C$

15. $\int \frac{dt}{\sqrt{16-9t^2}} = \frac{1}{4}\int \frac{dt}{\sqrt{1-\left(\frac{3t}{4}\right)^2}}$; $\begin{bmatrix} u = \frac{3}{4}t \\ du = \frac{3}{4}\,dt \end{bmatrix} \to \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{3}\sin^{-1}u + C = \frac{1}{3}\sin^{-1}\left(\frac{3t}{4}\right) + C$

16. $\int \frac{dt}{9+t^2} = \frac{1}{9}\int \frac{dt}{1+\left(\frac{t}{3}\right)^2}$; $\begin{bmatrix} u = \frac{1}{3}t \\ du = \frac{1}{3}dt \end{bmatrix} \to \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1}u + C = \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + C$

17. $\int \frac{4\,dx}{5x\sqrt{25x^2-16}} = \frac{4}{25}\int \frac{dx}{x\sqrt{x^2-\frac{16}{25}}} = \frac{1}{5}\sec^{-1}\left|\frac{5x}{4}\right| + C$

18. $\int \frac{dx}{\sqrt{4x-x^2-3}} = \int \frac{d(x-2)}{\sqrt{1-(x-2)^2}} = \sin^{-1}(x-2) + C$

19. $\int \frac{dy}{y^2-4y+8} = \int \frac{d(y-2)}{(y-2)^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{y-2}{2}\right) + C$

20. $\int \frac{dv}{(v+1)\sqrt{v^2+2v}} = \int \frac{d(v+1)}{(v+1)\sqrt{(v+1)^2-1}} = \sec^{-1}|v+1| + C$

21. $\int \cos^2 3x\,dx = \int \frac{1+\cos 6x}{2}\,dx = \frac{x}{2} + \frac{\sin 6x}{12} + C$

22. $\int \sin^3\frac{\theta}{2}\,d\theta = \int \left(1-\cos^2\frac{\theta}{2}\right)\left(\sin\frac{\theta}{2}\right)d\theta$; $\begin{bmatrix} u = \cos\frac{\theta}{2} \\ du = -\frac{1}{2}\sin\frac{\theta}{2}\,d\theta \end{bmatrix} \to -2\int \left(1-u^2\right)du = \frac{2u^3}{3} - 2u + C$

$= \frac{2}{3}\cos^3\frac{\theta}{2} - 2\cos\frac{\theta}{2} + C$

23. $\int \tan^3 2t\, dt = \int (\tan 2t)(\sec^2 2t - 1)\, dt = \int \tan 2t \sec^2 2t\, dt - \int \tan 2t\, dt; \begin{bmatrix} u = 2t \\ du = 2\, dt \end{bmatrix}$

$\to \frac{1}{2}\int \tan u \sec^2 u\, du - \frac{1}{2}\int \tan u\, du = \frac{1}{4}\tan^2 u + \frac{1}{2}\ln|\cos u| + C = \frac{1}{4}\tan^2 2t + \frac{1}{2}\ln|\cos 2t| + C$

$= \frac{1}{4}\tan^2 2t - \frac{1}{2}\ln|\sec 2t| + C$

24. $\int \frac{dx}{2\sin x \cos x} = \int \frac{dx}{\sin 2x} = \int \csc 2x\, dx = -\frac{1}{2}\ln|\csc 2x + \cot 2x| + C$

25. $\int \frac{2\, dx}{\cos^2 x - \sin^2 x} = \int \frac{2\, dx}{\cos 2x}; \begin{bmatrix} u = 2x \\ du = 2\, dx \end{bmatrix} \to \int \frac{du}{\cos u} = \int \sec u\, du = \ln|\sec u + \tan u| + C$

$= \ln|\sec 2x + \tan 2x| + C$

26. $\int_{\pi/4}^{\pi/2} \sqrt{\csc^2 y - 1}\, dy = \int_{\pi/4}^{\pi/2} \cot y\, dy = [\ln|\sin y|]_{\pi/4}^{\pi/2} = \ln 1 - \ln \frac{1}{\sqrt{2}} = \ln \sqrt{2}$

27. $\int_{\pi/4}^{3\pi/4} \sqrt{\cot^2 t + 1}\, dt = \int_{\pi/4}^{3\pi/4} \csc t\, dt = [-\ln|\csc t + \cot t|]_{\pi/4}^{3\pi/4} = -\ln\left|\csc\frac{3\pi}{4} + \cot\frac{3\pi}{4}\right| + \ln\left|\csc\frac{\pi}{4} + \cot\frac{\pi}{4}\right|$

$= -\ln|\sqrt{2} - 1| + \ln|\sqrt{2} + 1| = \ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| = \ln\left|\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{2-1}\right| = \ln(3 + 2\sqrt{2})$

28. $\int_0^{2\pi} \sqrt{1 - \sin^2 \frac{x}{2}}\, dx = \int_0^{2\pi} \left|\cos\frac{x}{2}\right| dx = \int_0^{\pi} \cos\frac{x}{2}\, dx - \int_{\pi}^{2\pi} \cos\frac{x}{2}\, dx = \left[2\sin\frac{x}{2}\right]_0^{\pi} - \left[2\sin\frac{x}{2}\right]_{\pi}^{2\pi}$

$= (2 - 0) - (0 - 2) = 4$

29. $\int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos 2t}\, dt = \sqrt{2}\int_{-\pi/2}^{\pi/2} |\sin t|\, dt = 2\sqrt{2}\int_0^{\pi/2} \sin t\, dt = [-2\sqrt{2}\cos t]_0^{\pi/2} = 2\sqrt{2}\,[0 - (-1)] = 2\sqrt{2}$

30. $\int_{\pi}^{2\pi} \sqrt{1 + \cos 2t}\, dt = \sqrt{2}\int_{\pi}^{2\pi} |\cos t|\, dt = -\sqrt{2}\int_{\pi}^{3\pi/2} \cos t\, dt + \sqrt{2}\int_{3\pi/2}^{2\pi} \cos t\, dt$

$= -\sqrt{2}\,[\sin t]_{\pi}^{3\pi/2} + \sqrt{2}\,[\sin t]_{3\pi/2}^{2\pi} = -\sqrt{2}(-1 - 0) + \sqrt{2}\,[0 - (-1)] = 2\sqrt{2}$

31. $\int \frac{x^2\, dx}{x^2 + 4} = x - \int \frac{4\, dx}{x^2 + 4} = x - 2\tan^{-1}\left(\frac{x}{2}\right) + C$

32. $\int \frac{x^3\, dx}{9 + x^2} = \int \left[\frac{x(x^2 + 9) - 9x}{x^2 + 9}\right] dx = \int \left(x - \frac{9x}{x^2 + 9}\right) dx = \frac{x^2}{2} - \frac{9}{2}\ln(9 + x^2) + C$

33. $\int \frac{2y-1}{y^2+4}\,dy = \int \frac{2y\,dy}{y^2+4} - \int \frac{dy}{y^2+4} = \ln(y^2+4) - \frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) + C$

34. $\int \frac{y+4}{y^2+1}\,dy = \int \frac{y\,dy}{y^2+1} + 4\int \frac{dy}{y^2+1} = \frac{1}{2}\ln(y^2+1) + 4\tan^{-1}y + C$

35. $\int \frac{t+2}{\sqrt{4-t^2}}\,dt = \int \frac{t\,dt}{\sqrt{4-t^2}} + 2\int \frac{dt}{\sqrt{4-t^2}} = -\sqrt{4-t^2} + 2\sin^{-1}\left(\frac{t}{2}\right) + C$

36. $\int \frac{2t^2+\sqrt{1-t^2}}{t\sqrt{1-t^2}}\,dt = \int \frac{2t\,dt}{\sqrt{1-t^2}} + \int \frac{dt}{t} = -2\sqrt{1-t^2} + \ln|t| + C$

37. $\int \frac{\tan x\,dx}{\tan x + \sec x} = \int \frac{\sin x\,dx}{\sin x + 1} = \int \frac{(\sin x)(1-\sin x)}{1-\sin^2 x}\,dx = \int \frac{\sin x - 1 + \cos^2 x}{\cos^2 x}\,dx$

$= -\int \frac{d(\cos x)}{\cos^2 x} - \int \frac{dx}{\cos^2 x} + \int dx = \frac{1}{\cos x} - \tan x + x + C = x - \tan x + \sec x + C$

38. $\int x\csc(x^2+3)\,dx = \frac{1}{2}\int \csc(x^2+3)\,d(x^2+3) = -\frac{1}{2}\ln\left|\csc(x^2+3) + \cot(x^2+3)\right| + C$

39. $\int \cot\left(\frac{x}{4}\right)dx = 4\int \cot\left(\frac{x}{4}\right)d\left(\frac{x}{4}\right) = 4\ln\left|\sin\left(\frac{x}{4}\right)\right| + C$

40. $\int x\sqrt{1-x}\;\;dx;\left[\begin{array}{c} u = 1-x \\ du = -dx \end{array}\right] \to -\int (1-u)\sqrt{u}\,du = \int \left(u^{3/2} - u^{1/2}\right)du = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + C = -2\left[\frac{\left(\sqrt{1-x}\right)^3}{3} - \frac{\left(\sqrt{1-x}\right)^5}{5}\right] + C$

41. $\int \left(16+z^2\right)^{-3/2}dz;\left[\begin{array}{c} z = 4\tan\theta \\ dz = 4\sec^2\theta\,d\theta \end{array}\right] \to \int \frac{4\sec^2\theta\,d\theta}{64\sec^3\theta\,d\theta} = \frac{1}{16}\int \cos\theta\,d\theta = \frac{1}{16}\sin\theta + C = \frac{z}{16\sqrt{16+z^2}} + C$

$= \frac{z}{16\left(16+z^2\right)^{1/2}} + C$

42. $\int \frac{dy}{\sqrt{25+y^2}} = \frac{1}{5}\int \frac{dy}{\sqrt{1+\left(\frac{y}{5}\right)^2}} = \int \frac{du}{\sqrt{1+u^2}}, \left[u = \frac{y}{5}\right];\left[\begin{array}{c} u = \tan\theta \\ du = \sec^2\theta\,d\theta \end{array}\right] \to \int \frac{\sec^2\theta\,d\theta}{\sqrt{1+\tan^2\theta}} = \int \sec\theta\,d\theta$

$= \ln|\sec\theta + \tan\theta| + C_1 = \ln\left|\sqrt{1+u^2} + u\right| + C_1 = \ln\left|\sqrt{1+\left(\frac{y}{5}\right)^2} + \frac{y}{5}\right| + C_1 = \ln\left|\frac{\sqrt{25+y^2}+y}{5}\right| + C_1$

$= \ln\left|y + \sqrt{25+y^2}\right| + C$

43. $\int \frac{dx}{x^2\sqrt{1-x^2}}; \begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{bmatrix} \to \int \frac{\cos\theta\, d\theta}{\sin^2\theta\cos\theta} = \int \csc^2\theta\, d\theta = -\cot\theta + C = -\frac{\cos\theta}{\sin\theta} + C = \frac{-\sqrt{1-x^2}}{x} + C$

44. $\int \frac{x^2\, dx}{\sqrt{1-x^2}}; \begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{bmatrix} \to \int \frac{\sin^2\theta\cos\theta\, d\theta}{\cos\theta} = \int \sin^2\theta\, d\theta = \int \frac{1-\cos 2\theta}{2}\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta$

$= \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta = \frac{\sin^{-1}x}{2} - \frac{x\sqrt{1-x^2}}{2} + C$

45. $\int \frac{dx}{\sqrt{x^2-9}}; \begin{bmatrix} x = 3\sec\theta \\ dx = 3\sec\theta\tan\theta\, d\theta \end{bmatrix} \to \int \frac{3\sec\theta\tan\theta\, d\theta}{\sqrt{9\sec^2\theta - 9}} = \int \frac{3\sec\theta\tan\theta\, d\theta}{3\tan\theta} = \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + C_1 = \ln\left|\frac{x}{3} + \sqrt{\left(\frac{x}{3}\right)^2 - 1}\right| + C_1 = \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| + C_1 = \ln\left|x + \sqrt{x^2-9}\right| + C$

46. $\int \frac{12\, dx}{(x^2-1)^{3/2}}; \begin{bmatrix} x = \sec\theta \\ dx = \sec\theta\tan\theta\, d\theta \end{bmatrix} \to \int \frac{12\sec\theta\tan\theta\, d\theta}{\tan^3\theta} = \int \frac{12\cos\theta\, d\theta}{\sin^2\theta}; \begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \to \int \frac{12\, du}{u^2}$

$= -\frac{12}{u} + C = -\frac{12}{\sin\theta} + C = -\frac{12\sec\theta}{\tan\theta} + C = -\frac{12x}{\sqrt{x^2-1}} + C$

47. $u = \ln(x+1)$, $du = \frac{dx}{x+1}$; $dv = dx$, $v = x$;

$\int \ln(x+1)\, dx = x\ln(x+1) - \int \frac{x}{x+1}\, dx = x\ln(x+1) - \int dx + \int \frac{dx}{x+1} = x\ln(x+1) - x + \ln(x+1) + C_1$

$= (x+1)\ln(x+1) - x + C_1 = (x+1)\ln(x+1) - (x+1) + C$, where $C = C_1 + 1$

48. $u = \ln x$, $du = \frac{dx}{x}$; $dv = x^2\, dx$, $v = \frac{1}{3}x^3$;

$\int x^2\ln x\, dx = \frac{1}{3}x^3\ln x - \int \frac{1}{3}x^3\left(\frac{1}{x}\right) dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$

49. $u = \tan^{-1}3x$, $du = \frac{3\, dx}{1+9x^2}$; $dv = dx$, $v = x$;

$\int \tan^{-1}3x\, dx = x\tan^{-1}3x - \int \frac{3x\, dx}{1+9x^2}; \begin{bmatrix} y = 1+9x^2 \\ dy = 18x\, dx \end{bmatrix} \to x\tan^{-1}3x - \frac{1}{6}\int \frac{dy}{y}$

$= x\tan^{-1}(3x) - \frac{1}{6}\ln(1+9x^2) + C$

50. $u = \cos^{-1}\left(\frac{x}{2}\right)$, $du = \frac{-dx}{\sqrt{4-x^2}}$; $dv = dx$, $v = x$;

$\int \cos^{-1}\left(\frac{x}{2}\right) dx = x\cos^{-1}\left(\frac{x}{2}\right) + \int \frac{x\, dx}{\sqrt{4-x^2}}; \begin{bmatrix} y = 4-x^2 \\ dy = -2x\, dx \end{bmatrix} \to x\cos^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\int \frac{dy}{\sqrt{y}}$

$= x\cos^{-1}\left(\frac{x}{2}\right) - \sqrt{4-x^2} + C = x\cos^{-1}\left(\frac{x}{2}\right) - 2\sqrt{1-\left(\frac{x}{2}\right)^2} + C$

51. e^x

$(x+1)^2 \xrightarrow{(+)} e^x$

$2(x+1) \xrightarrow{(-)} e^x$

$2 \xrightarrow{(+)} e^x$

$0 \qquad \Rightarrow \int (x+1)^2 e^x\,dx = \left[(x+1)^2 - 2(x+1) + 2\right]e^x + C$

52. $\sin(1-x)$

$x^2 \xrightarrow{(+)} \cos(1-x)$

$2x \xrightarrow{(-)} -\sin(1-x)$

$2 \xrightarrow{(+)} -\cos(1-x)$

$0 \qquad \Rightarrow \int x^2 \sin(1-x)\,dx = x^2\cos(1-x) + 2x\sin(1-x) - 2\cos(1-x) + C$

53. $u = \cos 2x,\ du = -2\sin 2x\,dx;\ dv = e^x\,dx,\ v = e^x;$

$I = \int e^x \cos 2x\,dx = e^x \cos 2x + 2\int e^x \sin 2x\,dx;$

$u = \sin 2x,\ du = 2\cos 2x\,dx;\ dv = e^x\,dx,\ v = e^x;$

$I = e^x\cos 2x + 2\left[e^x \sin 2x - 2\int e^x \cos 2x\,dx\right] = e^x\cos 2x + 2e^x \sin 2x - 4I \Rightarrow I = \frac{e^x\cos 2x}{5} + \frac{2e^x \sin 2x}{5} + C$

54. $u = \sin 3x,\ du = 3\cos 3x\,dx;\ dv = e^{-2x}\,dx,\ v = -\frac{1}{2}e^{-2x};$

$I = \int e^{-2x}\sin 3x\,dx = -\frac{1}{2}e^{-2x}\sin 3x + \frac{3}{2}\int e^{-2x}\cos 3x\,dx;$

$u = \cos 3x,\ du = -3\sin 3x\,dx;\ dv = e^{-2x}\,dx,\ v = -\frac{1}{2}e^{-2x};$

$I = -\frac{1}{2}e^{-2x}\sin 3x + \frac{3}{2}\left[-\frac{1}{2}e^{-2x}\cos 3x - \frac{3}{2}\int e^{-2x}\sin 3x\,dx\right] = -\frac{1}{2}e^{-2x}\sin 3x - \frac{3}{4}e^{-2x}\cos 3x - \frac{9}{4}I$

$\Rightarrow I = \frac{4}{13}\left(-\frac{1}{2}e^{-2x}\sin 3x - \frac{3}{4}e^{-2x}\cos 3x\right) + C = -\frac{2}{13}e^{-2x}\sin 3x - \frac{3}{13}e^{-2x}\cos 3x + C$

55. $\int \frac{x\,dx}{x^2 - 3x + 2} = \int \frac{2\,dx}{x-2} - \int \frac{dx}{x-1} = 2\ln|x-2| - \ln|x-1| + C$

56. $\int \frac{dx}{x(x+1)^2} = \int \left(\frac{1}{x} - \frac{2}{x+1} + \frac{x}{(x+1)^2}\right)dx = \ln|x| - 2\ln|x+1| + \left(\ln|x+1| + \frac{1}{x+1}\right) + C$

$= \ln|x| - \ln|x+1| + \frac{1}{x+1} + C$

57. $\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2};\ [\cos\theta = y] \to -\int \frac{dy}{y^2 + y - 2} = -\frac{1}{3}\int \frac{dy}{y-1} + \frac{1}{3}\int \frac{dy}{y+2} = \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C$

$= \frac{1}{3}\ln\left|\frac{\cos\theta + 2}{\cos\theta - 1}\right| + C = -\frac{1}{3}\ln\left|\frac{\cos\theta - 1}{\cos\theta + 2}\right| + C$

58. $\int \frac{3x^2+4x+4}{x^3+x}\,dx = \int \frac{4}{x}\,dx - \int \frac{x-4}{x^2+1}\,dx = 4\ln|x| - \frac{1}{2}\ln(x^2+1) + 4\tan^{-1}x + C$

59. $\int \frac{(v+3)\,dv}{2v^3-8v} = \frac{1}{2}\int\left(-\frac{3}{4v} + \frac{5}{8(v-2)} + \frac{1}{8(v+2)}\right)dv = -\frac{3}{8}\ln|v| + \frac{5}{16}\ln|v-2| + \frac{1}{16}\ln|v+2| + C$

$= \frac{1}{16}\ln\left|\frac{(v-2)^5(v+2)}{v^6}\right| + C$

60. $\int \frac{dt}{t^4+4t^2+3} = \frac{1}{2}\int \frac{dt}{t^2+1} - \frac{1}{2}\int \frac{dt}{t^2+3} = \frac{1}{2}\tan^{-1}t - \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C = \frac{1}{2}\tan^{-1}t - \frac{\sqrt{3}}{6}\tan^{-1}\frac{t}{\sqrt{3}} + C$

61. $\int \frac{x^3+x^2}{x^2+x-2}\,dx = \int\left(x + \frac{2x}{x^2+x-2}\right)dx = \int x\,dx + \frac{2}{3}\int \frac{dx}{x-1} + \frac{4}{3}\int \frac{dx}{x+2}$

$= \frac{x^2}{2} + \frac{4}{3}\ln|x+2| + \frac{2}{3}\ln|x-1| + C$

62. $\int \frac{x^3+4x^2}{x^2+4x+3}\,dx = \int\left(x - \frac{3x}{x^2+4x+3}\right)dx = \int x\,dx + \frac{3}{2}\int \frac{dx}{x+1} - \frac{9}{2}\int \frac{dx}{x+3}$

$= \frac{x^2}{2} - \frac{9}{2}\ln|x+3| + \frac{3}{2}\ln|x+1| + C$

63. $\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx = \int\left[(2x-3) + \frac{x}{x^2+2x-8}\right]dx = \int (2x-3)\,dx + \frac{1}{3}\int \frac{dx}{x-2} + \frac{2}{3}\int \frac{dx}{x+4}$

$= x^2 - 3x + \frac{2}{3}\ln|x+4| + \frac{1}{3}\ln|x-2| + C$

64. $\int \frac{dx}{x(3\sqrt{x+1})};\ \begin{bmatrix} u = \sqrt{x+1} \\ du = \frac{dx}{2\sqrt{x+1}} \\ dx = 2u\,du \end{bmatrix} \rightarrow \frac{2}{3}\int \frac{u\,du}{(u^2-1)u} = \frac{1}{3}\int \frac{du}{u-1} - \frac{1}{3}\int \frac{du}{u+1} = \frac{1}{3}\ln|u-1| - \frac{1}{3}\ln|u+1| + C$

$= \frac{1}{3}\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$

65. $\int \frac{ds}{e^s-1};\ \begin{bmatrix} u = e^s - 1 \\ du = e^s\,ds \\ ds = \frac{du}{u+1} \end{bmatrix} \rightarrow \int \frac{du}{u(u-1)} = \int \frac{du}{u-1} - \int \frac{du}{u} = \ln\left|\frac{u-1}{u}\right| + C = \ln\left|\frac{e^s-1}{e^s}\right| + C = \ln|1-e^{-s}| + C$

66. $\int \frac{ds}{\sqrt{e^s+1}}; \begin{bmatrix} u = \sqrt{e^s+1} \\ du = \frac{e^s\,ds}{2\sqrt{e^s+1}} \\ ds = \frac{2u\,du}{u^2-1} \end{bmatrix} \rightarrow \int \frac{2u\,du}{u(u^2-1)} = 2\int \frac{du}{(u+1)(u-1)} = \int \frac{du}{u-1} - \int \frac{du}{u+1} = \ln\left|\frac{u-1}{u+1}\right| + C$

$= \ln\left|\frac{\sqrt{e^s+1}-1}{\sqrt{e^s+1}+1}\right| + C$

67. (a) $\int \frac{y\,dy}{\sqrt{16-y^2}} = -\frac{1}{2}\int \frac{d(16-y^2)}{\sqrt{16-y^2}} = -\sqrt{16-y^2} + C$

(b) $\int \frac{y\,dy}{\sqrt{16-y^2}}; [y = 4\sin x] \rightarrow 4\int \frac{\sin x \cos x\,dx}{\cos x} = -4\cos x + C = -\frac{4\sqrt{16-y^2}}{4} + C = -\sqrt{16-y^2} + C$

68. (a) $\int \frac{x\,dx}{\sqrt{4+x^2}} = \frac{1}{2}\int \frac{d(4+x^2)}{\sqrt{4+x^2}} = \sqrt{4+x^2} + C$

(b) $\int \frac{x\,dx}{\sqrt{4+x^2}}; [x = 2\tan y] \rightarrow \int \frac{2\tan y \cdot 2\sec^2 y\,dy}{2\sec y} = 2\int \sec y \tan y\,dy = 2\sec y + C = \sqrt{4+x^2} + C$

69. (a) $\int \frac{x\,dx}{4-x^2} = -\frac{1}{2}\int \frac{d(4-x^2)}{4-x^2} = -\frac{1}{2}\ln|4-x^2| + C$

(b) $\int \frac{x\,dx}{4-x^2}; [x = 2\sin\theta] \rightarrow \int \frac{2\sin\theta \cdot 2\cos\theta\,d\theta}{4\cos^2\theta} = \int \tan\theta\,d\theta = -\ln|\cos\theta| + C = -\ln\sqrt{4-x^2} + C$

$= -\frac{1}{2}\ln|4-x^2| + C$

70. (a) $\int \frac{t\,dt}{\sqrt{4t^2-1}} = \frac{1}{8}\int \frac{d(4t^2-1)}{\sqrt{4t^2-1}} = \frac{1}{4}\sqrt{4t^2-1} + C$

(b) $\int \frac{t\,dt}{\sqrt{4t^2-1}}; \left[t = \frac{1}{2}\sec\theta\right] \rightarrow \int \frac{\frac{1}{2}\sec\theta\tan\theta \cdot \frac{1}{2}\sec\theta\,d\theta}{\tan\theta} = \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{\tan\theta}{4} + C = \frac{\sqrt{4t^2-1}}{4} + C$

71. $\int \frac{x\,dx}{9-x^2}; \begin{bmatrix} u = 9-x^2 \\ du = -2x\,dx \end{bmatrix} \rightarrow -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| + C = \ln\frac{1}{\sqrt{u}} + C = \ln\frac{1}{\sqrt{9-x^2}} + C$

72. $\int \frac{dx}{x(9-x^2)} = \frac{1}{9}\int \frac{dx}{x} + \frac{1}{18}\int \frac{dx}{3-x} - \frac{1}{18}\int \frac{dx}{3+x} = \frac{1}{9}\ln|x| - \frac{1}{18}\ln|3-x| - \frac{1}{18}\ln|3+x| + C$

$= \frac{1}{9}\ln|x| - \frac{1}{18}\ln|9-x^2| + C$

73. $\int \frac{dx}{9-x^2} = \frac{1}{6}\int \frac{dx}{3-x} + \frac{1}{6}\int \frac{dx}{3+x} = -\frac{1}{6}\ln|3-x| + \frac{1}{6}\ln|3+x| + C = \frac{1}{6}\ln\left|\frac{x+3}{x-3}\right| + C$

74. $\int \frac{dx}{\sqrt{9-x^2}}; \begin{bmatrix} x = 3\sin\theta \\ dx = 3\cos\theta\, d\theta \end{bmatrix} \to \int \frac{3\cos\theta}{3\cos\theta}\, d\theta = \int d\theta = \theta + C = \sin^{-1}\frac{x}{3} + C$

75. $\int \frac{x\,dx}{1+\sqrt{x}}; \begin{bmatrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int \frac{u^2 \cdot 2u\,du}{1+u} = \int \left(2u^2 - 2u + 2 - \frac{2}{1+u}\right) du = \frac{2}{3}u^3 - u^2 + 2u - 2\ln|1+u| + C$

$= \frac{2x^{3/2}}{3} - x + 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$

76. $\int \frac{dx}{x\left(x^2+1\right)^2}; \begin{bmatrix} x = \tan\theta \\ dx = \sec^2\theta\, d\theta \end{bmatrix} \to \int \frac{\sec^2\theta\, d\theta}{\tan\theta \sec^4\theta} = \int \frac{\cos^3\theta\, d\theta}{\sin\theta} = \int \left(\frac{1-\sin^2\theta}{\sin\theta}\right) d(\sin\theta)$

$= \ln|\sin\theta| - \frac{1}{2}\sin^2\theta + C = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| - \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}}\right)^2 + C$

77. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx; \begin{bmatrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int \frac{\cos u \cdot 2u\,du}{u} = 2\int \cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C$

78. $\int \frac{dx}{\sqrt{-2x-x^2}} = \int \frac{d(x+1)}{\sqrt{1-(x+1)^2}} = \sin^{-1}(x+1) + C$

79. $\int \frac{du}{\sqrt{1+u^2}}; [u = \tan\theta] \to \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \ln|\sec\theta + \tan\theta| + C = \ln\left|\sqrt{1+u^2} + u\right| + C$

80. $\int \frac{2 - \cos x + \sin x}{\sin^2 x}\, dx = \int 2\csc^2 x\, dx - \int \frac{\cos x\, dx}{\sin^2 x} + \int \csc x\, dx = -2\cot x + \frac{1}{\sin x} - \ln|\csc x + \cot x| + C$

$= -2\cot x + \csc x - \ln|\csc x + \cot x| + C$

81. $\int \frac{9\,dv}{81-v^4} = \frac{1}{2}\int \frac{dv}{v^2+9} + \frac{1}{12}\int \frac{dv}{3-v} + \frac{1}{12}\int \frac{dv}{3+v} = \frac{1}{12}\ln\left|\frac{3+v}{3-v}\right| + \frac{1}{6}\tan^{-1}\frac{v}{3} + C$

82.

$\cos(2\theta+1)$

$\theta \xrightarrow{(+)} \frac{1}{2}\sin(2\theta+1)$

$1 \xrightarrow{(-)} -\frac{1}{4}\cos(2\theta+1)$

0

$\int \theta\cos(2\theta+1)\, d\theta = \frac{\theta}{2}\sin(2\theta+1) + \frac{1}{4}\cos(2\theta+1) + C$

83. $\int \frac{x^3\,dx}{x^2-2x+1} = \int \left(x+2+\frac{3x+2}{x^2-2x+1}\right) dx = \int (x+2)\,dx + 3\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}$

$= \frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1} + C$

84. $\int \frac{d\theta}{\sqrt{1+\sqrt{\theta}}};\ \begin{bmatrix} x = 1+\sqrt{\theta} \\ dx = \frac{d\theta}{2\sqrt{\theta}} \\ d\theta = 2(x-1)\,dx \end{bmatrix} \to \int \frac{2(x-1)\,dx}{\sqrt{x}} = 2\int \sqrt{x}\,dx - 2\int \frac{dx}{\sqrt{x}} = \frac{4}{3}x^{3/2} - 4x^{1/2} + C$

$= \frac{4}{3}\left(1+\sqrt{\theta}\right)^{3/2} - 4\left(1+\sqrt{\theta}\right)^{1/2} + C = 4\left[\frac{\left(\sqrt{1+\sqrt{\theta}}\right)^3}{3} - \sqrt{1+\sqrt{\theta}}\right] + C$

85. $\int \frac{2\sin\sqrt{x}\,dx}{\sqrt{x}\sec\sqrt{x}};\ \begin{bmatrix} y = \sqrt{x} \\ dy = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int \frac{2\sin y \cdot 2y\,dy}{y\sec y} = \int 2\sin 2y\,dy = -\cos(2y) + C = -\cos\left(2\sqrt{x}\right) + C$

86. $\int \frac{x^5\,dx}{x^4-16} = \int \left(x + \frac{16x}{x^4-16}\right) dx = \frac{x^2}{2} + \int \left(\frac{2x}{x^2-4} - \frac{2x}{x^2+4}\right) dx = \frac{x^2}{2} + \ln\left|\frac{x^2-4}{x^2+4}\right| + C$

87. $\int \frac{d\theta}{\theta^2-2\theta+4} = \int \frac{d\theta}{(\theta-1)^2+3} = \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{\theta-1}{\sqrt{3}}\right) + C$

88. $\int \frac{dr}{(r+1)\sqrt{r^2+2r}} = \int \frac{d(r+1)}{(r+1)\sqrt{(r+1)^2-1}} = \sec^{-1}|r+1| + C$

89. $\int \frac{\sin 2\theta\,d\theta}{(1+\cos 2\theta)^2} = -\frac{1}{2}\int \frac{d(1+\cos 2\theta)}{(1+\cos 2\theta)^2} = \frac{1}{2(1+\cos 2\theta)} + C = \frac{1}{4}\sec^2\theta + C$

90. $\int \frac{dx}{\left(x^2-1\right)^2} = \int \frac{dx}{\left(1-x^2\right)^2} = \frac{x}{2\left(1-x^2\right)} + \frac{1}{2}\int \frac{dx}{1-x^2}$ (FORMULA 19)

$= \frac{x}{2\left(1-x^2\right)} + \frac{1}{4}\int \frac{dx}{1-x} + \frac{1}{4}\int \frac{dx}{1+x} = \frac{x}{2\left(1-x^2\right)} - \frac{1}{4}\ln|1-x| + \frac{1}{4}\ln|1+x| + C = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2\left(x^2-1\right)} + C$

91. $\int \frac{x\,dx}{\sqrt{2-x}};\ \begin{bmatrix} y = 2-x \\ dy = -dx \end{bmatrix} \to -\int \frac{(2-y)\,dy}{\sqrt{y}} = \frac{2}{3}y^{3/2} - 4y^{1/2} + C = \frac{2}{3}(2-x)^{3/2} - 4(2-x)^{1/2} + C$

$= 2\left[\frac{\left(\sqrt{2-x}\right)^3}{3} - 2\sqrt{2-x}\right] + C$

92. $\int \frac{dy}{y^2-2y+2} = \int \frac{d(y-1)}{(y-1)^2+1} = \tan^{-1}(y-1) + C$

93. $\int \ln \sqrt{x-1}\, dx; \left[\begin{array}{c} y=\sqrt{x-1} \\ dy=\frac{dx}{2\sqrt{x-1}} \end{array}\right] \rightarrow \int \ln y \cdot 2y\, dy;\ u=\ln y,\ du=\frac{dy}{y};\ dv=2y\, dy,\ v=y^2$

$\Rightarrow \int 2y \ln y\, dy = y^2 \ln y - \int y\, dy = y^2 \ln y - \frac{1}{2}y^2 + C = (x-1)\ln\sqrt{x-1} - \frac{1}{2}(x-1) + C_1$

$= \frac{1}{2}[(x-1)\ln|x-1| - x] + \left(C_1 + \frac{1}{2}\right) = \frac{1}{2}[x \ln|x-1| - x - \ln|x-1|] + C$

94. $\int \frac{x\, dx}{\sqrt{8-2x^2-x^4}} = \frac{1}{2}\int \frac{d(x^2+1)}{\sqrt{9-(x^2+1)^2}} = \frac{1}{2}\sin^{-1}\left(\frac{x^2+1}{3}\right) + C$

95. $\int \frac{z+1}{z^2(z^2+4)}\, dz = \frac{1}{4}\int \left(\frac{1}{z} + \frac{1}{z^2} - \frac{z+1}{z^2+4}\right) dz = \frac{1}{4}\ln|z| - \frac{1}{4z} - \frac{1}{8}\ln(z^2+4) - \frac{1}{8}\tan^{-1}\frac{z}{2} + C$

96. $\int x^3 e^{x^2}\, dx = \frac{1}{2}\int x^2 e^{x^2}\, d(x^2) = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C = \frac{(x^2-1)e^{x^2}}{2} + C$

97. $u=\tan^{-1}x,\ du=\frac{dx}{1+x^2};\ dv=\frac{dx}{x^2},\ v=-\frac{1}{x};$

$\int \frac{\tan^{-1}x\, dx}{x^2} = -\frac{1}{x}\tan^{-1}x + \int \frac{dx}{x(1+x^2)} = -\frac{1}{x}\tan^{-1}x + \int \frac{dx}{x} - \int \frac{x\, dx}{1+x^2}$

$= -\frac{1}{x}\tan^{-1}x + \ln|x| - \frac{1}{2}\ln(1+x^2) + C = -\frac{\tan^{-1}x}{x} + \ln|x| - \ln\sqrt{1+x^2} + C$

98. $\int \frac{e^t\, dt}{e^{2t}+3e^t+2};\ [e^t = x] \rightarrow \int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \ln|x+1| - \ln|x+2| + C$

$= \ln\left|\frac{x+1}{x+2}\right| + C = \ln\left(\frac{e^t+1}{e^t+2}\right) + C$

99. $\int \frac{1-\cos 2x}{1+\cos 2x}\, dx = \int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x + C$

100. $\int \frac{\cos(\sin^{-1}x)\, dx}{\sqrt{1-x^2}}; \left[\begin{array}{c} u=\sin^{-1}x \\ du=\frac{dx}{\sqrt{1-x^2}} \end{array}\right] \rightarrow \int \cos u\, du = \sin u + C = \sin(\sin^{-1}x) + C = x + C$

101. $\int \frac{\cos x\, dx}{\sin^3 x - \sin x} = -\int \frac{\cos x\, dx}{(\sin x)(1-\sin^2 x)} = -\int \frac{\cos x\, dx}{(\sin x)(\cos^2 x)} = -\int \frac{2\, dx}{\sin 2x} = -2\int \csc 2x\, dx$

$= \ln|\csc(2x) + \cot(2x)| + C$

102. $\int \frac{e^t\, dt}{1+e^t} = \ln(1+e^t) + C$

103. $\int_1^{\infty} \frac{\ln y\, dy}{y^3}; \begin{bmatrix} x = \ln y \\ dx = \frac{dy}{y} \\ dy = e^x\, dx \end{bmatrix} \to \int_0^{\infty} \frac{x \cdot e^x}{e^{3x}}\, dx = \int_0^{\infty} xe^{-2x}\, dx = \lim_{b\to\infty} \left[-\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\right]_0^b$

$= \lim_{b\to\infty} \left(\frac{-b}{2e^{2b}} - \frac{1}{4e^{2b}}\right) - \left(0 - \frac{1}{4}\right) = \frac{1}{4}$

104. $\int \frac{\cot v\, dv}{\ln(\sin v)} = \int \frac{\cos v\, dv}{(\sin v)\ln(\sin v)}; \begin{bmatrix} u = \ln(\sin v) \\ du = \frac{\cos v\, dv}{\sin v} \end{bmatrix} \to \int \frac{du}{u} = \ln|u| + C = \ln|\ln(\sin v)| + C$

105. $\int \frac{dx}{(2x-1)\sqrt{x^2 - x}} = \int \frac{2\, dx}{(2x-1)\sqrt{4x^2 - 4x}} = \int \frac{2\, dx}{(2x-1)\sqrt{(2x-1)^2 - 1}}; \begin{bmatrix} u = 2x - 1 \\ du = 2\, dx \end{bmatrix} \to \int \frac{du}{u\sqrt{u^2 - 1}}$

$= \sec^{-1}|u| + C = \sec^{-1}|2x - 1| + C$

106. $\int e^{\ln\sqrt{x}}\, dx = \int \sqrt{x}\, dx = \frac{2}{3}x^{3/2} + C$

107. $\int e^{\theta}\sqrt{3 + 4e^{\theta}}\, d\theta; \begin{bmatrix} u = 4e^{\theta} \\ du = 4e^{\theta}\, d\theta \end{bmatrix} \to \frac{1}{4}\int \sqrt{3+u}\, du = \frac{1}{4}\cdot\frac{2}{3}(3+u)^{3/2} + C = \frac{1}{6}\left(3 + 4e^{\theta}\right)^{3/2} + C$

108. $\int \frac{dv}{\sqrt{e^{2v} - 1}}; \begin{bmatrix} x = e^v \\ dx = e^v\, dv \end{bmatrix} \to \int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x + C = \sec^{-1}(e^v) + C$

109. $\int (27)^{3\theta+1}\, d\theta = \frac{1}{3}\int (27)^{3\theta+1}\, d(3\theta+1) = \frac{1}{3\ln 27}(27)^{3\theta+1} + C = \frac{1}{3}\left(\frac{27^{3\theta+1}}{\ln 27}\right) + C$

110.

		$\sin x$
x^5	$\xrightarrow{(+)}$	$-\cos x$
$5x^4$	$\xrightarrow{(-)}$	$-\sin x$
$20x^3$	$\xrightarrow{(+)}$	$\cos x$
$60x^2$	$\xrightarrow{(-)}$	$\sin x$
$120x$	$\xrightarrow{(+)}$	$-\cos x$
120	$\xrightarrow{(-)}$	$-\sin x$
0		

$\int x^5 \sin x\, dx = -x^5 \cos x + 5x^4 \sin x + 20x^3 \cos x - 60x^2 \sin x - 120x \cos x + 120 \sin x + C$

111. $\int \frac{dr}{1+\sqrt{r}}; \begin{bmatrix} u = \sqrt{r} \\ du = \frac{dr}{2\sqrt{r}} \end{bmatrix} \to \int \frac{2u\, du}{1+u} = \int \left(2 - \frac{2}{1+u}\right) du = 2u - 2\ln|1+u| + C = 2\sqrt{r} - 2\ln\left(1 + \sqrt{r}\right) + C$

112. $\int \frac{8\,dy}{y^3(y+2)} = \int \frac{dy}{y} - \int \frac{2\,dy}{y^2} + \int \frac{4\,dy}{y^3} - \int \frac{dy}{(y+2)} = \ln\left|\frac{y}{y+2}\right| + \frac{2}{y} - \frac{2}{y^2} + C$

113. $\int \frac{8\,dm}{m\sqrt{49m^2-4}} = \frac{8}{7}\int \frac{dm}{m\sqrt{m^2-\left(\frac{2}{7}\right)^2}} = 4\sec^{-1}\left(\frac{7m}{2}\right) + C$

114. $\int \frac{dt}{t(1+\ln t)\sqrt{(\ln t)(2+\ln t)}}; \begin{bmatrix} u = \ln t \\ du = \frac{dt}{t} \end{bmatrix} \to \int \frac{du}{(1+u)\sqrt{u(2+u)}} = \int \frac{du}{(u+1)\sqrt{(u+1)^2-1}}$

$= \sec^{-1}|u+1| + C = \sec^{-1}|\ln t + 1| + C$

115. $\lim_{t\to 0} \frac{t - \ln(1+2t)}{t^2} = \lim_{t\to 0} \frac{1 - \frac{2}{1+2t}}{2t} = \infty$ for $t \to 0^-$ and $-\infty$ for $t \to 0^+$

The limit does not exist.

116. $\lim_{t\to 0} \frac{\tan 3t}{\tan 5t} = \lim_{t\to 0} \frac{3\sec^2 3t}{5\sec^2 5t} = \frac{3}{5}$

117. $\lim_{x\to 0} \frac{x\sin x}{1-\cos x} = \lim_{x\to 0} \frac{x\cos x + \sin x}{\sin x} = \lim_{x\to 0} \frac{-x\sin x + \cos x + \cos x}{\cos x} = 2$

118. The limit leads to the indeterminate form 1^∞. $f(x) = x^{1/(1-x)} \Rightarrow \ln f(x) = \frac{\ln x}{1-x}$

$\Rightarrow \lim_{x\to 1} \frac{\ln x}{1-x} = \lim_{x\to 1} \frac{1/x}{-1} = -1 \Rightarrow \lim_{x\to 1} x^{1/(1-x)} = \lim_{x\to 1} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$

119. The limit leads to the indeterminate form ∞^0. $f(x) = x^{1/x} \Rightarrow \ln f(x) = \frac{\ln x}{x} \Rightarrow \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$

$\Rightarrow \lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1$

120. The limit leads to the indeterminate form 1^∞. $f(x) = \left(1+\frac{3}{x}\right)^x \Rightarrow \ln f(x) = x\ln\left(1+\frac{3}{x}\right) = \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}$

$\Rightarrow \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{-3/x^2}{1+3/x}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{3x}{x+3} = 3 \Rightarrow \lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x = \lim_{x\to\infty} e^{\ln f(x)} = e^3$

121. $\lim_{r\to\infty} \frac{\cos r}{\ln r} = 0$ since $|\cos r| \le 1$ and $\ln r \to \infty$ as $r \to \infty$.

122. $\lim_{\theta\to\pi/2} \left(\theta - \frac{\pi}{2}\right)\sec\theta = \lim_{\theta\to\pi/2} \frac{\theta - \frac{\pi}{2}}{\cos\theta} = \lim_{\theta\to\pi/2} \frac{1}{-\sin\theta} = -1$

123. $\lim\limits_{x\to 1}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)=\lim\limits_{x\to 1}\left[\frac{\ln x-x+1}{(x-1)\ln x}\right]=\lim\limits_{x\to 1}\frac{\frac{1}{x}-1}{\frac{x-1}{x}+\ln x}=\lim\limits_{x\to 1}\frac{1-x}{x-1+x\ln x}=\lim\limits_{x\to 1}\frac{-1}{1+x/x+\ln x}=-\frac{1}{2}$

124. The limit leads to the indeterminate form ∞^0. $f(x)=\left(1+\frac{1}{x}\right)^x \Rightarrow \ln f(x)=x\ln\left(1+\frac{1}{x}\right)=\frac{\ln(1+1/x)}{1/x}$

$\Rightarrow \lim\limits_{x\to 0^+}\frac{\ln(1+1/x)}{1/x}=\lim\limits_{x\to 0^+}\frac{\frac{-1/x^2}{1+1/x}}{-1/x^2}=\lim\limits_{x\to 0^+}\frac{x}{x+1}=0 \Rightarrow \lim\limits_{x\to 0^+}\left(1+\frac{1}{x}\right)^x=\lim\limits_{x\to 0^+}e^{\ln f(x)}=e^0=1$

125. The limit leads to the indeterminate form 0^0. $f(\theta)=(\tan\theta)^\theta \Rightarrow \ln f(\theta)=\theta\ln(\tan\theta)=\frac{\ln(\tan\theta)}{1/\theta}$

$\Rightarrow \lim\limits_{x\to 0^+}\frac{\ln(\tan\theta)}{1/\theta}=\lim\limits_{x\to 0^+}\frac{\frac{\sec^2\theta}{\tan\theta}}{-\frac{1}{\theta^2}}=\lim\limits_{x\to 0^+}-\frac{\theta^2}{\sin\theta\cos\theta}=\lim\limits_{x\to 0^+}\frac{-2\theta}{-\sin^2\theta+\cos^2\theta}=0$

$\Rightarrow \lim\limits_{x\to 0^+}(\tan\theta)^\theta=\lim\limits_{x\to 0^+}e^{\ln f(\theta)}=e^0=1$

126. $\lim\limits_{\theta\to\infty}\theta^2\sin\left(\frac{1}{\theta}\right)=\lim\limits_{t\to 0^+}\frac{\sin t}{t^2}=\lim\limits_{t\to 0^+}\frac{\cos t}{2t}=\infty$

127. $\lim\limits_{x\to\infty}\frac{x^3-3x^2+1}{2x^2+x-3}=\lim\limits_{x\to\infty}\frac{3x^2-6x}{4x+1}=\lim\limits_{x\to\infty}\frac{6x-6}{4}=\infty$

128. $\lim\limits_{x\to\infty}\frac{3x^2-x+1}{x^4-x^3+2}=\lim\limits_{x\to\infty}\frac{6x-1}{4^3-3x^2}=\lim\limits_{x\to\infty}\frac{6}{12x^2-6x}=0$

129. $\int_0^3\frac{dx}{\sqrt{9-x^2}}=\lim\limits_{b\to 3^-}\int_0^b\frac{dx}{\sqrt{9-x^2}}=\lim\limits_{b\to 3^-}\left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^b=\lim\limits_{b\to 3^-}\sin^{-1}\left(\frac{b}{3}\right)-\sin^{-1}\left(\frac{0}{3}\right)=\frac{\pi}{2}-0=\frac{\pi}{2}$

130. $\int_0^1\ln x\,dx=\lim\limits_{b\to 0^+}[x\ln x-x]_b^1=(1\cdot\ln 1-1)-\lim\limits_{b\to 0^+}[b\ln b-b]=-1-\lim\limits_{b\to 0^+}\frac{\ln b}{\left(\frac{1}{b}\right)}=-1-\lim\limits_{b\to 0^+}\frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)}$

$=-1+0=-1$

131. $\int_{-1}^1\frac{dy}{y^{2/3}}=\int_{-1}^0\frac{dy}{y^{2/3}}+\int_0^1\frac{dy}{y^{2/3}}=2\int_0^1\frac{dy}{y^{2/3}}=2\cdot 3\lim\limits_{b\to 0^+}\left[y^{1/3}\right]_b^1=6\left(1-\lim\limits_{b\to 0^+}b^{1/3}\right)=6$

132. $\int_{-2}^0\frac{d\theta}{(\theta+1)^{3/5}}=\int_{-2}^{-1}\frac{d\theta}{(\theta+1)^{3/5}}+\int_{-1}^0\frac{d\theta}{(\theta+1)^{3/5}}=\lim\limits_{b\to -1^-}\left[\frac{5}{2}(\theta+1)^{2/5}\right]_{-2}^b+\lim\limits_{b\to -1^+}\left[\frac{5}{2}(\theta+1)^{2/5}\right]_b^0$

$=\lim\limits_{b\to -1^-}\left(\frac{5}{2}(b+1)^{2/5}-\frac{5}{2}\right)+\lim\limits_{b\to -1^+}\left(\frac{5}{2}-\frac{5}{2}(b+1)^{2/5}\right)=-\frac{5}{2}+\frac{5}{2}=0$

133. $\int_3^\infty \frac{2\,du}{u^2-2u} = \int_3^\infty \frac{du}{u-2} - \int_3^\infty \frac{du}{u} = \lim_{b\to\infty} \left[\ln\left|\frac{u-2}{u}\right|\right]_3^b = \lim_{b\to\infty} \left[\ln\left|\frac{b-2}{b}\right|\right] - \ln\left|\frac{3-2}{3}\right| = 0 - \ln\left(\frac{1}{3}\right) = \ln 3$

134. $\int_1^\infty \frac{3v-1}{4v^3-v^2}\,dv = \int_1^\infty \left(\frac{1}{v} + \frac{1}{v^2} - \frac{4}{4v-1}\right) dv = \lim_{b\to\infty} \left[\ln v - \frac{1}{v} - \ln(4v-1)\right]_1^b$

$= \lim_{b\to\infty} \left[\ln\left(\frac{b}{4b-1}\right) - \frac{1}{b}\right] - (\ln 1 - 1 - \ln 3) = \ln\frac{1}{4} + 1 + \ln 3 = 1 + \ln\frac{3}{4}$

135. $\int_0^\infty x^2 e^{-x}\,dx = \lim_{b\to\infty} \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^b = \lim_{b\to\infty} \left(-b^2e^{-b} - 2be^{-b} - 2e^{-b}\right) - (-2) = 0 + 2 = 2$

136. $\int_{-\infty}^0 xe^{3x}\,dx = \lim_{b\to-\infty} \left[\frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}\right]_b^0 = -\frac{1}{9} - \lim_{b\to-\infty} \left(\frac{b}{3}e^{3b} - \frac{1}{9}e^{3b}\right) = -\frac{1}{9} - 0 = -\frac{1}{9}$

137. $\int_{-\infty}^\infty \frac{dx}{4x^2+9} = 2\int_0^\infty \frac{dx}{4x^2+9} = \frac{1}{2}\int_0^\infty \frac{dx}{x^2+\frac{9}{4}} = \frac{1}{2}\lim_{b\to\infty} \left[\frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right)\right]_0^b = \frac{1}{2}\lim_{b\to\infty} \left[\frac{2}{3}\tan^{-1}\left(\frac{2b}{3}\right)\right] - \frac{1}{3}\tan^{-1}(0)$

$= \frac{1}{2}\left(\frac{2}{3}\cdot\frac{\pi}{2}\right) - 0 = \frac{\pi}{6}$

138. $\int_{-\infty}^\infty \frac{4\,dx}{x^2+16} = 2\int_0^\infty \frac{4\,dx}{x^2+16} = 2\lim_{b\to\infty} \left[\tan^{-1}\left(\frac{x}{4}\right)\right]_0^b = 2\lim_{b\to\infty} \left[\tan^{-1}\left(\frac{b}{4}\right)\right] - \tan^{-1}(0) = 2\left(\frac{\pi}{2}\right) - 0 = \pi$

139. $\lim_{\theta\to\infty} \frac{\theta}{\sqrt{\theta^2+1}} = 1$ and $\int_6^\infty \frac{d\theta}{\theta}$ diverges $\Rightarrow \int_6^\infty \frac{d\theta}{\sqrt{\theta^2+1}}$ diverges

140. $I = \int_0^\infty e^{-u}\cos u\,du = \lim_{b\to\infty} \left[-e^{-u}\cos u\right]_0^b - \int_0^\infty e^{-u}\sin u\,du = 1 - \lim_{b\to\infty} \left[e^{-u}\sin u\right]_0^b + \int_0^\infty \left(-e^{-u}\right)\cos u\,du$

$\Rightarrow I = 1 + 0 + I \Rightarrow 2I = 1 \Rightarrow I = \frac{1}{2}$ converges

141. $\int_1^\infty \frac{\ln z}{z}\,dz = \int_1^e \frac{\ln z}{z}\,dz + \int_e^\infty \frac{\ln z}{z}\,dz = \left[(\ln z)^2\right]_1^e + \lim_{b\to\infty} \left[(\ln z)^2\right]_e^b = \left(1^2 - 0\right) + \lim_{b\to\infty} \left[(\ln b)^2 - 1\right]$

$= \infty \Rightarrow$ diverges

142. $0 < \frac{e^{-t}}{\sqrt{t}} \le e^{-t}$ for $t \ge 1$ and $\int_1^\infty e^{-t}\,dt$ converges $\Rightarrow \int_1^\infty \frac{e^{-t}}{\sqrt{t}}\,dt$ converges

143. $0 < \frac{e^{-x}}{3+e^{-2x}} = \frac{1}{3e^x + e^{-x}} < \frac{1}{e^x + e^{-x}}$ and $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = 2\int_0^{\infty} \frac{dx}{e^x + e^{-x}} < \int_0^{\infty} \frac{2\,dx}{e^x}$ converges

$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{-x}}{3+e^{-2x}}\,dx$ converges

144. $\int_{-\infty}^{\infty} \frac{dx}{x^2(1+e^x)} = \int_{-\infty}^{-1} \frac{dx}{x^2(1+e^x)} + \int_{-1}^{0} \frac{dx}{x^2(1+e^x)} + \int_{0}^{1} \frac{dx}{x^2(1+e^x)} + \int_{1}^{\infty} \frac{dx}{x^2(1+e^x)}$;

$\lim_{x\to 0} \frac{\left(\frac{1}{x^2}\right)}{\left[\frac{1}{x^2(1+e^x)}\right]} = \lim_{x\to 0} \frac{x^2(1+e^x)}{x^2} = \lim_{x\to 0} (1+e^x) = 2$ and $\int_0^1 \frac{dx}{x^2}$ diverges $\Rightarrow \int_0^1 \frac{dx}{x^2(1+e^x)}$ diverges

$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2(1+e^x)}$ diverges

145. $\frac{1}{y^2-y}\,dy = e^x\,dx \Rightarrow \int \frac{1}{y(y-1)}\,dy = \int e^x\,dx = e^x + C$; $\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1} \Rightarrow 1 = A(y-1) + B(y)$
$= (A+B)y - A$

Equating coefficients of like terms gives $A + B = 0$ and $-A = 1$. Solving the system simultaneously yields $A = -1$, $B = 1$.

$\int \frac{1}{y(y-1)}\,dy = \int -\frac{1}{y}\,dy + \int \frac{1}{y-1}\,dy = -\ln|y| + \ln|y-1| + C_2 \Rightarrow -\ln|y| + \ln|y-1| = e^x + C$

Substitute $x = 0$, $y = 2 \Rightarrow -\ln 2 + 0 = 1 + C$ or $C = -1 - \ln 2$.

The solution to the initial value problem is $-\ln|y| + \ln|y-1| = e^x - 1 - \ln 2$.

146. $\frac{1}{(y+1)^2}\,dy = \sin\theta\,d\theta$; $\int \frac{1}{(y+1)^2}\,dy = \int \sin\theta\,d\theta \Rightarrow -\frac{1}{y+1} = -\cos\theta + C$

Substitute $x = \frac{\pi}{2}$, $y = 0 \Rightarrow -1 = 0 + C$ or $C = -1$.

The solution to the initial value problem is $-\frac{1}{y+1} = -\cos\theta - 1 \Rightarrow y + 1 = \frac{1}{\cos\theta + 1} \Rightarrow y = \frac{1}{\cos\theta + 1} - 1$

147. $dy = \frac{dx}{x^2 - 3x + 2}$; $x^2 - 3x + 2 = (x-2)(x-1) \Rightarrow \frac{1}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x-2)$
$\Rightarrow 1 = (A+B)x - A - 2B$

Equating coefficients of like terms gives $A + B = 0$, $-A - 2B = 1$. Solving the system simultaneously yields $A = 1$, $B = -1$.

$\int dy = \int \frac{dx}{x^2-3x+2} = \int \frac{dx}{x-2} - \int \frac{dx}{x-1} \Rightarrow y = \ln|x-2| - \ln|x-1| + C$

Substitute $x = 3$, $y = 0 \Rightarrow 0 = 0 - \ln 2 + C$ or $C = \ln 2$.

The solution to the initial value problem is $y = \ln|x-2| - \ln|x-1| + \ln 2$.

148. $\frac{ds}{2x+2} = \frac{dt}{t^2+2t}$; $\int \frac{ds}{2x+2} = \frac{1}{2}\int \frac{ds}{s+1} = \frac{1}{2}\ln|s+1| + C_1$; $t^2+2t = t(t+2) \Rightarrow \frac{1}{t^2+2t} = \frac{A}{t} + \frac{B}{t+2}$

$\Rightarrow 1 = A(t+2) + Bt \Rightarrow 1 = (A+B)t + 2A$.

Equating coefficients of like terms gives $A + B = 0$ and $2A = 1$. Solving the system simultaneously yields $A = \frac{1}{2}$, $B = -\frac{1}{2}$.

$$\int \frac{dt}{t^2+2t} = \int \frac{1/2}{t}\,dt - \int \frac{1/2}{t+2}\,dt = \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t+2| + C_2 \Rightarrow \frac{1}{2}\ln|s+1| = \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t+2| + C_3$$

$\Rightarrow \ln|s+1| = \ln|t| - \ln|t+2| + C$

Substitute $t = 1$, $x = 1 \Rightarrow \ln 2 = 0 - \ln 3 + C$ or $C = \ln 2 + \ln 3 = \ln 6$.

The solution to the initial value problem is $\ln|s+1| = \ln|t| - \ln|t+2| + \ln 6 \Rightarrow \ln|s+1| = \ln\left|\frac{6t}{t+2}\right|$

$\Rightarrow |s+1| = \left|\frac{6t}{t+2}\right|$

CHAPTER 7 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. $u = (\sin^{-1} x)^2$, $du = \frac{2\sin^{-1} x\,dx}{\sqrt{1-x^2}}$; $dv = dx$, $v = x$;

$$\int (\sin^{-1} x)^2\,dx = x(\sin^{-1} x)^2 - \int \frac{2x\sin^{-1} x\,dx}{\sqrt{1-x^2}};$$

$u = \sin^{-1} x$, $du = \frac{dx}{\sqrt{1-x^2}}$; $dv = -\frac{2x\,dx}{\sqrt{1-x^2}}$, $v = 2\sqrt{1-x^2}$;

$\int \frac{2x\sin^{-1} x\,dx}{\sqrt{1-x^2}} = 2(\sin^{-1} x)\sqrt{1-x^2} - \int 2\,dx = 2(\sin^{-1} x)\sqrt{1-x^2} - 2x + C$; therefore

$$\int (\sin^{-1} x)^2\,dx = x(\sin^{-1} x)^2 + 2(\sin^{-1} x)\sqrt{1-x^2} - 2x + C$$

2. $\frac{1}{x} = \frac{1}{x}$,

$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$,

$\frac{1}{x(x+1)(x+2)} = \frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)}$,

$\frac{1}{x(x+1)(x+2)(x+3)} = \frac{1}{6x} - \frac{1}{2(x+1)} + \frac{1}{2(x+2)} - \frac{1}{6(x+3)}$,

$\frac{1}{x(x+1)(x+2)(x+3)(x+4)} = \frac{1}{24x} - \frac{1}{6(x+1)} + \frac{1}{4(x+2)} - \frac{1}{6(x+3)} + \frac{1}{24(x+4)} \Rightarrow$ the following pattern:

$\frac{1}{x(x+1)(x+2)\cdots(x+m)} = \sum_{k=0}^{m} \frac{(-1)^k}{(k!)(m-k)!(x+k)}$; therefore $\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)}$

$$= \sum_{k=0}^{m} \left[\frac{(-1)^k}{(k!)(m-k)!} \ln|x+k| \right] + C$$

3. $u = \sin^{-1} x,\ du = \frac{dx}{\sqrt{1-x^2}};\ dv = x\,dx,\ v = \frac{x^2}{2};$

$$\int x \sin^{-1} x\,dx = \frac{x^2}{2}\sin^{-1} x - \int \frac{x^2\,dx}{2\sqrt{1-x^2}};\ \left[\begin{array}{c} x = \sin\theta \\ dx = \cos\theta\,d\theta \end{array}\right] \to \int x \sin^{-1} x\,dx = \frac{x^2}{2}\sin^{-1} x - \int \frac{\sin^2\theta\cos\theta\,d\theta}{2\cos\theta}$$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \sin^2\theta\,d\theta = \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + C = \frac{x^2}{2}\sin^{-1} x + \frac{\sin\theta\cos\theta - \theta}{4} + C$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{x\sqrt{1-x^2} - \sin^{-1} x}{4} + C$$

4. $\int \sin^{-1}\sqrt{y}\,dy;\ \left[\begin{array}{c} z = \sqrt{y} \\ dz = \frac{dy}{2\sqrt{y}} \end{array}\right] \to \int 2z \sin^{-1} z\,dz$; from Exercise 3, $\int z \sin^{-1} z\,dz$

$$= \frac{z^2 \sin^{-1} z}{2} + \frac{z\sqrt{1-z^2} - \sin^{-1} z}{4} + C \Rightarrow \int \sin^{-1}\sqrt{y}\,dy = y \sin^{-1}\sqrt{y} + \frac{\sqrt{y}\sqrt{1-y} - \sin^{-1}\sqrt{y}}{2} + C$$

$$= y \sin^{-1}\sqrt{y} + \frac{\sqrt{y-y^2}}{2} - \frac{\sin^{-1}\sqrt{y}}{2} + C$$

5. $\int \frac{d\theta}{1-\tan^2\theta} = \int \frac{\cos^2\theta}{\cos^2\theta - \sin^2\theta}\,d\theta = \int \frac{1+\cos 2\theta}{2\cos 2\theta}\,d\theta = \frac{1}{2}\int (\sec 2\theta + 1)\,d\theta = \frac{\ln|\sec 2\theta + \tan 2\theta| + 2\theta}{4} + C$

6. $u = \ln\left(\sqrt{x} + \sqrt{1+x}\right),\ du = \left(\frac{dx}{\sqrt{x}+\sqrt{1+x}}\right)\left(\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{1+x}}\right) = \frac{dx}{2\sqrt{x}\sqrt{1+x}};\ dv = dx,\ v = x;$

$$\int \ln\left(\sqrt{x}+\sqrt{1+x}\right)dx = x\ln\left(\sqrt{x}+\sqrt{1+x}\right) - \frac{1}{2}\int \frac{x\,dx}{\sqrt{x}\sqrt{1+x}} = \frac{1}{2}\int \frac{x\,dx}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}};$$

$$\left[\begin{array}{c} x + \frac{1}{2} = \frac{1}{2}\sec\theta \\ dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta \end{array}\right] \to \frac{1}{4}\int \frac{(\sec\theta - 1)\cdot\sec\theta\tan\theta\,d\theta}{\left(\frac{1}{2}\tan\theta\right)} = \frac{1}{2}\int \left(\sec^2\theta - \sec\theta\right)d\theta$$

$$= \frac{\tan\theta - \ln|\sec\theta + \tan\theta|}{2} + C = \frac{2\sqrt{x^2+x} - \ln\left|2x+1+2\sqrt{x^2+x}\right|}{2} + C$$

$$\Rightarrow \int \ln\left(\sqrt{x}+\sqrt{1+x}\right)dx = x\ln\left(\sqrt{x}+\sqrt{1+x}\right) - \frac{2\sqrt{x^2+x} - \ln\left|2x+1+2\sqrt{x^2+x}\right|}{4} + C$$

7. $\int \frac{dt}{t-\sqrt{1-t^2}}$; $\left[\begin{array}{c} t=\sin\theta \\ dt=\cos\theta\, d\theta \end{array}\right] \to \int \frac{\cos\theta\, d\theta}{\sin\theta-\cos\theta} = \int \frac{d\theta}{\tan\theta-1}$; $\left[\begin{array}{c} u=\tan\theta \\ du=\sec^2\theta\, d\theta \\ d\theta=\frac{du}{u^2+1} \end{array}\right] \to \int \frac{du}{(u-1)(u^2+1)}$

$= \frac{1}{2}\int \frac{du}{u-1} - \frac{1}{2}\int \frac{du}{u^2+1} - \frac{1}{2}\int \frac{u\,du}{u^2+1} = \frac{1}{2}\ln\left|\frac{u-1}{\sqrt{u^2+1}}\right| - \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\ln\left|\frac{\tan\theta-1}{\sec\theta}\right| - \frac{1}{2}\theta + C$

$= \frac{1}{2}\ln\left(t-\sqrt{1-t^2}\right) - \frac{1}{2}\sin^{-1}t + C$

8. $\int \frac{(2e^{2x}-e^x)\,dx}{\sqrt{3e^{2x}-6e^x-1}}$; $\left[\begin{array}{c} u=e^x \\ du=e^x\, dx \end{array}\right] \to \int \frac{(2u-1)\,du}{\sqrt{3u^2-6u-1}} = \frac{1}{\sqrt{3}}\int \frac{(2u-1)\,du}{\sqrt{(u-1)^2-\frac{4}{3}}}$;

$\left[\begin{array}{c} u-1=\frac{2}{\sqrt{3}}\sec\theta \\ du=\frac{2}{\sqrt{3}}\sec\theta\tan\theta\, d\theta \end{array}\right] \to \frac{1}{\sqrt{3}}\int \left(\frac{4}{\sqrt{3}}\sec\theta+1\right)(\sec\theta)\, d\theta = \frac{4}{3}\int \sec^2\theta\, d\theta + \frac{1}{\sqrt{3}}\int \sec\theta\, d\theta$

$= \frac{4}{3}\tan\theta + \frac{1}{\sqrt{3}}\ln|\sec\theta+\tan\theta| + C_1 = \frac{4}{3}\cdot\sqrt{\frac{3}{4}(u-1)^2-1} + \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}}{2}(u-1)+\sqrt{\frac{3}{4}(u-1)^2-1}\right| + C_1$

$= \frac{2}{3}\sqrt{3u^2-6u-1} + \frac{1}{\sqrt{3}}\ln\left|u-1+\sqrt{(u-1)^2-\frac{4}{3}}\right| + \left(C_1+\frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{\sqrt{3}}\left[2\sqrt{e^{2x}-2e^x-\frac{1}{3}} + \ln\left|e^x-1+\sqrt{e^{2x}-2e^x-\frac{1}{3}}\right|\right] + C$

9. $\int \frac{1}{x^4+4}\,dx = \int \frac{1}{(x^2+2)^2-4x^2}\,dx = \int \frac{1}{(x^2+2x+2)(x^2-2x+2)}\,dx$

$= \frac{1}{16}\int \left[\frac{2x+2}{x^2+2x+2} + \frac{2}{(x+1)^2+1} - \frac{2x-2}{x^2-2x+2} + \frac{2}{(x-1)^2+1}\right]dx$

$= \frac{1}{16}\ln\left|\frac{x^2+2x+2}{x^2-2x+2}\right| + \frac{1}{8}\left[\tan^{-1}(x+1)+\tan^{-1}(x-1)\right] + C$

10. $\int \frac{1}{x^6-1}\,dx = \frac{1}{6}\int \left(\frac{1}{x-1} - \frac{1}{x+1} + \frac{x-2}{x^2-x+1} - \frac{x+2}{x^2+x+1}\right)dx$

$= \frac{1}{6}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{12}\int \left[\frac{2x-1}{x^2-x+1} - \frac{3}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{2x+1}{x^2+x+1} - \frac{3}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\right]dx$

$= \frac{1}{6}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{12}\left[\ln\left|\frac{x^2-x+1}{x^2+x+1}\right| - 2\sqrt{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) - 2\sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right] + C$

11. $\lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-} [\sin^{-1} x]_0^b = \lim_{b\to 1^-} (\sin^{-1} b - \sin^{-1} 0) = \lim_{b\to 1^-} (\sin^{-1} b - 0) = \lim_{b\to 1^-} \sin^{-1} b = \frac{\pi}{2}$

12. $\lim_{x\to\infty} \frac{1}{x} \int_0^x \tan^{-1} t\,dt = \lim_{x\to\infty} \frac{\int_0^x \tan^{-1} t\,dt}{x}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$= \lim_{x\to\infty} \frac{\tan^{-1} x}{1} = \frac{\pi}{2}$

13. $y = (\cos\sqrt{x})^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln(\cos\sqrt{x})$ and $\lim_{x\to 0^+} \frac{\ln(\cos\sqrt{x})}{x} = \lim_{x\to 0^+} \frac{-\sin\sqrt{x}}{2\sqrt{x}\cos\sqrt{x}} = \frac{-1}{2}\lim_{x\to 0^+} \frac{\tan\sqrt{x}}{\sqrt{x}}$

$= -\frac{1}{2}\lim_{x\to 0^+} \frac{\frac{1}{2}x^{-1/2}\sec^2\sqrt{x}}{\frac{1}{2}x^{-1/2}} = -\frac{1}{2} \Rightarrow \lim_{x\to 0^+} (\cos\sqrt{x})^{1/x} = e^{-1/2} = \frac{1}{\sqrt{e}}$

14. $y = (x+e^x)^{2/x} \Rightarrow \ln y = \frac{2\ln(x+e^x)}{x} \Rightarrow \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{2(1+e^x)}{x+e^x} = \lim_{x\to\infty} \frac{2e^x}{1+e^x} = \lim_{x\to\infty} \frac{2e^x}{e^x} = 2$

$\Rightarrow \lim_{x\to\infty} (x+e^x)^{2/x} = \lim_{x\to\infty} e^y = e^2$

15. $\lim_{x\to\infty} \int_{-x}^x \sin t\,dt = \lim_{x\to\infty} [-\cos t]_{-x}^x = \lim_{x\to\infty} [-\cos x + \cos(-x)] = \lim_{x\to\infty} (-\cos x + \cos x) = \lim_{x\to\infty} 0 = 0$

16. $\lim_{x\to 0^+} \int_x^1 \frac{\cos t}{t^2}\,dt$; $\lim_{t\to 0^+} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{\cos t}{t^2}\right)} - \lim_{t\to 0^+} \frac{1}{\cos t} = 1 \Rightarrow \lim_{x\to 0^+} \int_x^1 \frac{\cos t}{t^2}\,dt$ diverges since $\int_0^1 \frac{dt}{t^2}$ diverges; thus

$\lim_{x\to 0^+} x\int_x^1 \frac{\cos t}{t^2}\,dt$ is an indeterminate $0\cdot\infty$ form and we apply l'Hôpital's rule:

$$\lim_{x\to 0^+} x\int_x^1 \frac{\cos t}{t^2}\,dt = \lim_{x\to 0^+} \frac{-\int_1^x \frac{\cos t}{t^2}\,dt}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{-\left(\frac{\cos x}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+} \cos x = 1$$

17. $\frac{dy}{dx} = \sqrt{\cos 2x} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cos 2x = 2\cos^2 x$; $L = \int_0^{\pi/4} \sqrt{1 + \left(\sqrt{\cos 2t}\right)^2}\,dt = \sqrt{2}\int_0^{\pi/4} \sqrt{\cos^2 t}\,dt$

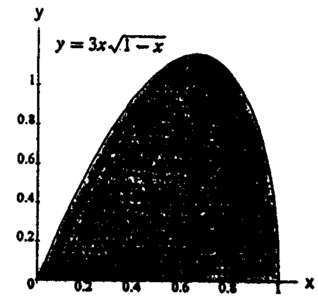
$= \sqrt{2}\,[\sin t]_0^{\pi/4} = 1$

18. $\frac{dy}{dx} = \frac{-2x}{1-x^2} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{(1-x^2)^2 + 4x^2}{(1-x^2)^2} = \frac{1+2x^2+x^4}{(1-x^2)^2} = \left(\frac{1+x^2}{1-x^2}\right)^2$; $L = \int_0^{1/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$$= \int_0^{1/2} \left(\frac{1+x^2}{1-x^2}\right) dx = \int_0^{1/2} \left(-1 + \frac{2}{1-x^2}\right) dx = \int_0^{1/2} \left(-1 + \frac{1}{1+x} + \frac{1}{1-x}\right) dx = \left[-x + \ln\left|\frac{1+x}{1-x}\right|\right]_0^{1/2}$$

$$= \left(-\tfrac{1}{2} + \ln 3\right) - (0 + \ln 1) = \ln 3 - \tfrac{1}{2}$$

19. $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^1 2\pi xy\, dx$

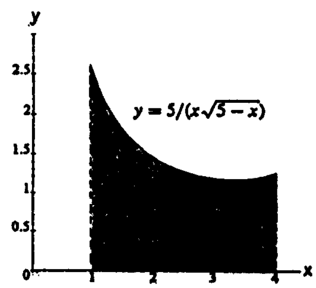


$$= 6\pi \int_0^1 x^2\sqrt{1-x}\, dx; \begin{bmatrix} u = 1-x \\ du = -dx \\ x^2 = (1-u)^2 \end{bmatrix}$$

$$\to -6\pi \int_1^0 (1-u)^2\sqrt{u}\, du = -6\pi \int_1^0 \left(u^{1/2} - 2u^{3/2} + u^{5/2}\right) du$$

$$= -6\pi\left[\tfrac{2}{3}u^{3/2} - \tfrac{4}{5}u^{5/2} + \tfrac{2}{7}u^{7/2}\right]_1^0 = 6\pi\left(\tfrac{2}{3} - \tfrac{4}{5} + \tfrac{2}{7}\right) = 6\pi\left(\frac{70-84+30}{105}\right) = 6\pi\left(\frac{16}{105}\right) = \frac{32\pi}{35}$$

20. $V = \int_a^b \pi y^2\, dx = \pi \int_1^4 \frac{25\, dx}{x^2(5-x)} = \pi \int_1^4 \left(\frac{dx}{x} + \frac{5\, dx}{x^2} + \frac{dx}{5-x}\right)$

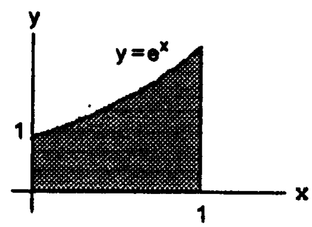


$$= \pi\left[\ln\left|\frac{x}{5-x}\right| - \frac{5}{x}\right]_1^4 = \pi\left(\ln 4 - \tfrac{5}{4}\right) - \pi\left(\ln \tfrac{1}{4} - 5\right)$$

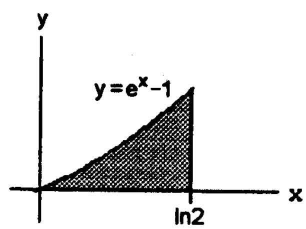
$$= \frac{15\pi}{4} + 2\pi \ln 4$$

21. $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^1 2\pi x e^x\, dx$

$$= 2\pi\left[xe^x - e^x\right]_0^1 = 2\pi$$



22. $V = \int_0^{\ln 2} 2\pi(\ln 2 - x)(e^x - 1)\, dx$

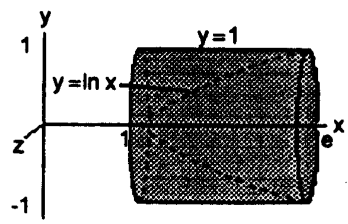


$$= 2\pi \int_0^{\ln 2} \left[(\ln 2)e^x - \ln 2 - xe^x + x\right] dx$$

$$= 2\pi\left[(\ln 2)e^x - (\ln 2)x - xe^x + e^x + \frac{x^2}{2}\right]_0^{\ln 2}$$

$$= 2\pi\left[2\ln 2 - (\ln 2)^2 - 2\ln 2 + 2 + \frac{(\ln 2)^2}{2}\right] - 2\pi(\ln 2 + 1) = 2\pi\left[-\frac{(\ln 2)^2}{2} - \ln 2 + 1\right]$$

23. (a) $V = \int_1^e \pi\left[1 - (\ln x)^2\right] dx$

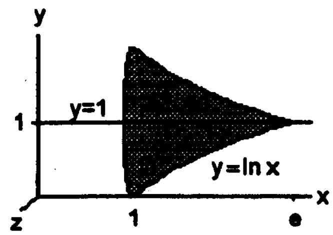


$= \pi\left[x - x(\ln x)^2\right]_1^e - 2\pi \int_1^e \ln x \, dx$ (FORMULA 110)

$= \pi\left[x - x(\ln x)^2 + 2(x \ln x - x)\right]_1^e$

$= \pi\left[-x - x(\ln x)^2 + 2x \ln x\right]_1^e = \pi\left[-e - e + 2e - (-1)\right] = \pi$

(b) $V = \int_1^e \pi(1 - \ln x)^2 \, dx = \pi \int_1^e \left[1 - 2\ln x + (\ln x)^2\right] dx$



$= \pi\left[x - 2(x \ln x - x) + x(\ln x)^2\right]_1^e - 2\pi \int_1^e \ln x \, dx$

$= \pi\left[x - 2(x \ln x - x) + x(\ln x)^2 - 2(x \ln x - x)\right]_1^e$

$= \pi\left[5x - 4x \ln x + x(\ln x)^2\right]_1^e = \pi\left[(5e - 4e + e) - (5)\right]$

$= \pi(2e - 5)$

24. (a) $V = \pi \int_0^1 \left[(e^y)^2 - 1\right] dy = \pi \int_0^1 \left(e^{2y} - 1\right) dy = \pi\left[\frac{e^{2y}}{2} - y\right]_0^1 = \pi\left[\frac{e^2}{2} - 1 - \left(\frac{1}{2}\right)\right] = \frac{\pi\left(e^2 - 3\right)}{2}$

(b) $V = \pi \int_0^1 \left(e^y - 1\right)^2 dy = \pi \int_0^1 \left(e^{2y} - 2e^y + 1\right) dy = \pi\left[\frac{e^{2y}}{2} - 2e^y + y\right]_0^1 = \pi\left[\left(\frac{e^2}{2} - 2e + 1\right) - \left(\frac{1}{2} - 2\right)\right]$

$= \pi\left(\frac{e^2}{2} - 2e + \frac{5}{2}\right) = \frac{\pi\left(e^2 - 4e + 5\right)}{2}$

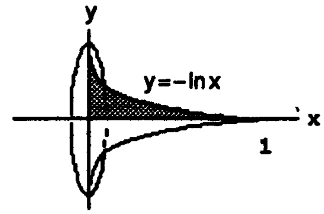
25. (a) $\lim_{x \to 0^+} x \ln x = 0 \Rightarrow \lim_{x \to 0^+} f(x) = 0 = f(0) \Rightarrow$ f is continuous

(b) $V = \int_0^2 \pi x^2 (\ln x)^2 \, dx; \left[\begin{array}{c} u = (\ln x)^2 \\ du = (2 \ln x) \frac{dx}{x} \\ dv = x^2 \\ v = \frac{x^3}{3} \end{array}\right] \to \pi\left(\lim_{b \to 0^+} \left[\frac{x^3}{3}(\ln x)^2\right]_b^2 - \int_0^2 \left(\frac{x^3}{3}\right)(2 \ln x) \frac{dx}{x}\right)$

$= \pi\left[\left(\frac{8}{3}\right)(\ln 2)^2 - \left(\frac{2}{3}\right) \lim_{b \to 0^+} \left[\frac{x^3}{3} \ln x - \frac{x^3}{9}\right]_b^2\right] = \pi\left[\frac{8(\ln 2)^2}{3} - \frac{16(\ln 2)}{9} + \frac{16}{27}\right]$

26. $V = \int_0^1 \pi(-\ln x)^2\,dx = \pi\left(\lim_{b\to 0}\left[x(\ln x)^2\right]_b^1 - 2\int_0^1 \ln x\,dx\right)$

$= -2\pi \lim_{b\to 0}\left[x\ln x - x\right]_b^1 = 2\pi$



27. $u = \frac{1}{1+y}$, $du = -\frac{dy}{(1+y)^2}$; $dv = ny^{n-1}\,dy$, $v = y^n$;

$\lim_{n\to\infty}\int_0^1 \frac{ny^{n-1}}{1+y}\,dy = \lim_{n\to\infty}\left(\left[\frac{y^n}{1+y}\right]_0^1 + \int_0^1 \frac{y^n}{1+y^2}\,dy\right) = \frac{1}{2} + \lim_{n\to\infty}\int_0^1 \frac{y^n}{1+y^2}\,dy$. Now, $0 \le \frac{y^n}{1+y^2} \le y^n$

$\Rightarrow 0 \le \lim_{n\to\infty}\int_0^1 \frac{y^n}{1+y^2}\,dy \le \lim_{n\to\infty}\int_0^1 y^n\,dy = \lim_{n\to\infty}\left[\frac{y^{n+1}}{n+1}\right]_0^1 = \lim_{n\to\infty}\frac{1}{n+1} = 0 \Rightarrow \lim_{n\to\infty}\int_0^1 \frac{ny^{n-1}}{1+y}\,dy$

$= \frac{1}{2} + 0 = \frac{1}{2}$

28. $u = x^2 - a^2 \Rightarrow du = 2x\,dx$;

$\int x\left(\sqrt{x^2-a^2}\right)^n dx = \frac{1}{2}\int\left(\sqrt{u}\right)^n du = \frac{1}{2}\int u^{n/2}\,du = \frac{1}{2}\left(\frac{u^{n/2+1}}{\frac{n}{2}+1}\right) + C,\ n \ne -2$

$= \frac{u^{(n+2)/2}}{n+2} + C = \frac{\left(\sqrt{u}\right)^{n+2}}{n+2} + C = \frac{\left(\sqrt{x^2-a^2}\right)^{n+2}}{n+2} + C$

29. $\frac{\pi}{6} = \sin^{-1}\frac{1}{2} = \left[\sin^{-1}\frac{x}{2}\right]_0^1 = \int_0^1 \frac{dx}{\sqrt{4-x^2}} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \int_0^1 \frac{dx}{\sqrt{4-2x^2}} = \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}} \frac{du}{\sqrt{4-u^2}}$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{u}{2}\right]_0^{\sqrt{2}} = \frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right) = \frac{\pi\sqrt{2}}{8}$

30. $\int_1^\infty \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right)dx = \lim_{b\to\infty}\int_1^b \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right)dx = \lim_{b\to\infty}\left[\frac{a}{2}\ln(x^2+1) - \frac{1}{2}\ln x\right]_1^b = \lim_{b\to\infty}\left[\frac{1}{2}\ln\frac{(x^2+1)^a}{x}\right]_1^b$

$= \lim_{b\to\infty}\frac{1}{2}\left[\ln\frac{(b^2+1)^a}{b} - \ln 2^a\right]$; $\lim_{b\to\infty}\frac{(b^2+1)^a}{b} > \lim_{b\to\infty}\frac{b^{2a}}{b} = \lim_{b\to\infty} b^{2\left(a-\frac{1}{2}\right)} = \infty$ if $a > \frac{1}{2} \Rightarrow$ the improper

integral diverges if $a > \frac{1}{2}$; for $a = \frac{1}{2}$: $\lim_{b\to\infty}\frac{\sqrt{b^2+1}}{b} = \lim_{b\to\infty}\sqrt{1+\frac{1}{b^2}} = 1 \Rightarrow \lim_{b\to\infty}\frac{1}{2}\left[\ln\frac{(b^2+1)^{1/2}}{b} - \ln 2^{1/2}\right]$

$= \frac{1}{2}\left(\ln 1 - \frac{1}{2}\ln 2\right) = -\frac{\ln 2}{4}$; if $a < \frac{1}{2}$: $0 \le \lim_{b\to\infty}\frac{(b^2+1)^a}{b} < \lim_{b\to\infty}\frac{(b+1)^{2a}}{b+1} = \lim_{b\to\infty}(b+1)^{2a-1} = 0$

$\Rightarrow \lim_{b\to\infty}\ln\frac{(b^2+1)^a}{b} = -\infty \Rightarrow$ the improper integral diverges if $a < \frac{1}{2}$; in summary, the improper integral

$\int_1^\infty \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right)dx$ converges only when $a = \frac{1}{2}$ and has the value $-\frac{\ln 2}{4}$

31. Let $u = f(x) \Rightarrow du = f'(x)\,dx$ and $dv = dx \Rightarrow v = x$;

$$\int_{\pi/2}^{3\pi/2} f(x)\,dx = \left[x\,f(x)\right]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} xf'(x)\,dx = \left[\frac{3\pi}{2} f\left(\frac{3\pi}{2}\right) - \frac{\pi}{2} f\left(\frac{\pi}{2}\right)\right] - \int_{\pi/2}^{3\pi/2} \cos x\,dx$$

$$= \frac{3\pi}{2}b - \frac{\pi}{2}a - [\sin x]_{\pi/2}^{3\pi/2} = \frac{\pi}{2}(3b - a) - [-1 - 1] = \frac{\pi}{2}(3b - a) + 2$$

32. $\int_0^a \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^a = \tan^{-1} a$; $\int_a^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \left[\tan^{-1} x\right]_a^b = \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} a\right) = \frac{\pi}{2} - \tan^{-1} a$;

therefore, $\tan^{-1} a = \frac{\pi}{2} - \tan^{-1} a \Rightarrow \tan^{-1} a = \frac{\pi}{4} \Rightarrow a = 1$ for $a > 0$.

33. $L = 4\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dy$; $x^{2/3} + y^{2/3} = 1 \Rightarrow y = \left(1 - x^{2/3}\right)^{3/2} \Rightarrow \frac{dy}{dx} = -\frac{3}{2}\left(1 - x^{2/3}\right)^{1/2}\left(x^{-1/3}\right)\left(\frac{2}{3}\right)$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1 - x^{2/3}}{x^{2/3}} \Rightarrow L = 4\int_0^1 \sqrt{1 + \left(\frac{1 - x^{2/3}}{x^{2/3}}\right)}\,dx = 4\int_0^1 \frac{dx}{x^{1/3}} = 6\left[x^{2/3}\right]_0^1 = 6$$

34. $\left(\frac{dy}{dx}\right)^2 = \frac{1}{4x} \Rightarrow \frac{dy}{dx} = \frac{\pm 1}{2\sqrt{x}} \Rightarrow y = \pm\sqrt{x},\ 0 \le x \le 4$

35. $P(x) = ax^2 + bx + c$, $P(0) = c = 1$ and $P'(0) = 0 \Rightarrow b = 0 \Rightarrow P(x) = ax^2 + 1$. Next,

$\frac{ax^2+1}{x^3(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2}$; for the integral to be a rational function, we must have $A = 0$ and $D = 0$. Thus, $ax^2 + 1 = Bx(x-1)^2 + C(x-1)^2 + Ex^3 = (B+E)x^3 + (C - 2B)x^2 + (B - 2C)x + C$

$\Rightarrow C = 1$; B-2C $= 0 \Rightarrow B = 2$; $C - 2B = a \Rightarrow a = -3$; therefore, $P(x) = -3x^2 + 1$

36. The integral $\int_{-1}^{1} \sqrt{1-x^2}\,dx$ is the area enclosed by the x-axis and the semicircle $y = \sqrt{1-x^2}$. This area is half the circle's area, or $\frac{\pi}{2}$ and multiplying by 2 gives π. The length of the circular arc $y = \sqrt{1-x^2}$ from $x = -1$ to $x = 1$ is $L = \int_{-1}^{1} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{-1}^{1} \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \frac{1}{2}(2\pi) = \pi$ since L is half the circle's circumference. In conclusion, $2\int_{-1}^{1} \sqrt{1-x^2}\,dx = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$.

37. $A = \int_1^{\infty} \frac{dx}{x^p}$ converges if $p > 1$ and diverges if $p \le 1$ (Exercise 67 in Section 7.6). Thus, $p \le 1$ for infinite area.

The volume of the solid of revolution about the x-axis is $V = \int_1^{\infty} \pi\left(\frac{1}{x^p}\right)^2 dx = \pi \int_1^{\infty} \frac{dx}{x^{2p}}$ which converges if $2p > 1$ and diverges if $2p \le 1$. Thus we want $p > \frac{1}{2}$ for finite volume. In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $\frac{1}{2} < p \le 1$.

38. The area is given by the integral $A = \int_0^1 \frac{dx}{x^p}$;

$p = 1$: $A = \lim_{b \to 0^+} [\ln x]_b^1 = -\lim_{b \to 0^+} \ln b = \infty$, diverges;

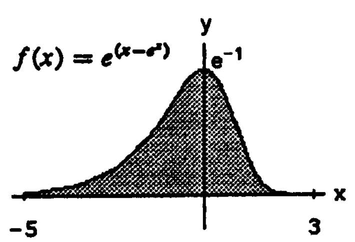
$p > 1$: $A = \lim_{b \to 0^+} \left[x^{1-p}\right]_b^1 = 1 - \lim_{b \to 0^+} b^{1-p} = -\infty$, diverges;

$p < 1$: $A = \lim_{b \to 0^+} \left[x^{1-p}\right]_b^1 = 1 - \lim_{b \to 0^+} b^{1-p} = 1 - 0$, converges; thus, $p \ge 1$ for infinite area.

The volume of the solid of revolution about the x-axis is $V_x = \pi \int_0^1 \frac{dx}{x^{2p}}$ which converges if $2p < 1$ or $p < \frac{1}{2}$, and diverges if $p \ge \frac{1}{2}$. Thus, V_x is infinite whenever the area is infinite ($p \ge 1$).

The volume of the solid of revolution about the y-axis is $V_y = \pi \int_1^{\infty} [R(y)]^2 \, dy = \pi \int_1^{\infty} \frac{dy}{y^{2/p}}$ which converges if $\frac{2}{p} > 1 \Leftrightarrow p < 2$ (see Exercise 39). In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $1 \le p < 2$, as described above.

39. (a)



(b) $\int_{-\infty}^{\infty} e^{(x-e^x)} dx = \int_{-\infty}^{\infty} e^{(-e^x)} e^x \, dx$

$$= \lim_{a\to-\infty} \int_a^0 e^{(-e^x)} e^x \, dx + \lim_{b\to+\infty} \int_0^b e^{(-e^x)} e^x \, dx;$$

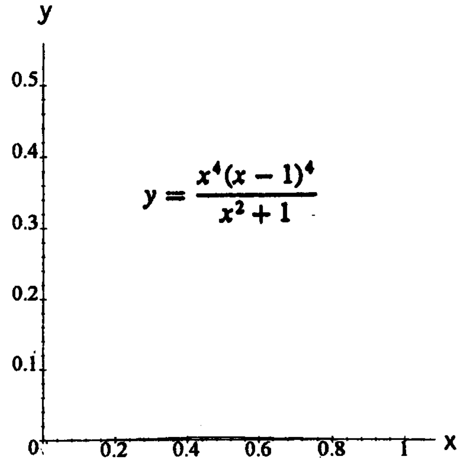
$$\begin{bmatrix} u = e^x \\ du = e^x \, dx \end{bmatrix} \to \lim_{a\to-\infty} \int_{e^a}^1 e^{-u} \, du + \lim_{b\to+\infty} \int_1^{e^b} e^{-u} \, du$$

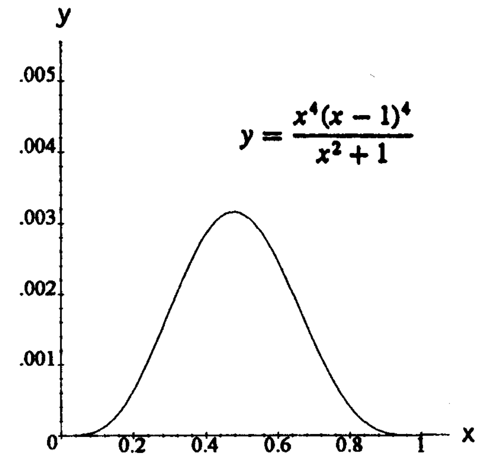
$$= \lim_{a\to-\infty} \left[-e^{-u}\right]_{e^a}^1 + \lim_{b\to-\infty} \left[-e^{-u}\right]_1^{e^b} = \lim_{a\to-\infty} \left[-\tfrac{1}{e} + e^{-(e^a)}\right] + \lim_{b\to+\infty} \left[-e^{-(e^b)} + \tfrac{1}{e}\right] = \left(-\tfrac{1}{e} + e^0\right) + \left(0 + \tfrac{1}{e}\right) = 1$$

40. (a) $\int_0^1 \frac{x^4(x-1)^4}{x^2+1} \, dx = \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}\right) dx = \frac{22}{7} - \pi$

(b) $\frac{\frac{22}{7} - \pi}{\pi} \cdot 100\% \cong 0.04\%$

(c) The area is less than 0.003





41. e^{2x} (+) $\cos 3x$

$2e^{2x}$ (−) $\frac{1}{3} \sin 3x$

$4e^{2x}$ $\underline{\quad (+) \quad}$ $-\frac{1}{9} \cos 3x$

$I = \frac{e^{2x}}{3} \sin 3x + \frac{2e^{2x}}{9} \cos 3x - \frac{4}{9} I \Rightarrow \frac{13}{9} I = \frac{e^{2x}}{9}(3 \sin 3x + 2 \cos 3x) \Rightarrow I = \frac{e^{2x}}{13}(3 \sin 3x + 2 \cos 3x) + C$

42. e^{3x} (+) $\sin 4x$

$3e^{3x}$ (−) $-\frac{1}{4} \cos 4x$

$9e^{3x}$ $\underline{\quad (+) \quad}$ $-\frac{1}{16} \sin 4x$

$I = -\frac{e^{3x}}{4} \cos 4x + \frac{3e^{3x}}{16} \sin 4x - \frac{9}{16} I \Rightarrow \frac{25}{16} I = \frac{e^{3x}}{16}(3 \sin 4x - 4 \cos 4x) \Rightarrow I = \frac{e^{3x}}{25}(3 \sin 4x - 4 \cos 4x) + C$

43. $\sin 3x \quad (+) \quad \sin x$

$3\cos 3x \quad (-) \quad -\cos x$

$-9\sin 3x \xrightarrow{(+)} -\sin x$

$I = -\sin 3x \cos x + 3\cos 3x \sin x + 9I \Rightarrow -8I = -\sin 3x \cos x + 3\cos 3x \sin x$

$\Rightarrow I = \frac{\sin 3x \cos x - 3\cos 3x \sin x}{8} + C$

44. $\cos 5x \quad (+) \quad \sin 4x$

$-5\sin 5x \quad (-) \quad -\frac{1}{4}\cos 4x$

$-25\cos 5x \xrightarrow{(+)} -\frac{1}{16}\sin 4x$

$I = -\frac{1}{4}\cos 5x \cos 4x - \frac{5}{16}\sin 5x \sin 4x + \frac{25}{16}I \Rightarrow -\frac{9}{16}I = -\frac{1}{4}\cos 5x \cos 4x - \frac{5}{16}\sin 5x \sin 4x$

$\Rightarrow I = \frac{1}{9}(4\cos 5x \cos 4x) \Rightarrow I = \frac{1}{9}(4\cos 5x \cos 4x + 5\sin 5x \sin 4x) + C$

45. $e^{ax} \quad (+) \quad \sin bx$

$ae^{ax} \quad (-) \quad -\frac{1}{b}\cos bx$

$a^2e^{ax} \xrightarrow{(+)} -\frac{1}{b^2}\sin bx$

$I = -\frac{e^{ax}}{b}\cos bx + \frac{ae^{ax}}{b^2}\sin bx - \frac{a^2}{b^2}I \Rightarrow \left(\frac{a^2+b^2}{b^2}\right)I = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx)$

$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

46. $e^{ax} \quad (+) \quad \cos bx$

$ae^{ax} \quad (-) \quad \frac{1}{b}\sin bx$

$a^2e^{ax} \xrightarrow{(+)} -\frac{1}{b^2}\cos bx$

$I = -\frac{e^{ax}}{b}\sin bx + \frac{ae^{ax}}{b^2}\cos bx - \frac{a^2}{b^2}I \Rightarrow \left(\frac{a^2+b^2}{b^2}\right)I = \frac{e^{ax}}{b^2}(a\cos bx + b\sin bx)$

$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

47. $\ln(ax) \quad (+) \quad 1$

$\frac{1}{x} \xrightarrow{(-)} x$

$I = x\ln(ax) - \int \left(\frac{1}{x}\right)x\,dx = x\ln(ax) - x + C$

48. $\ln(ax)$ $(+)$ x^2

$\frac{1}{x}$ $\underline{\quad(-)\quad}$ $\frac{1}{3}x^3$

$I = \frac{1}{3}x^3 \ln(ax) - \int \left(\frac{1}{x}\right)\left(\frac{x^3}{3}\right) dx = \frac{1}{3}x^3 \ln(ax) - \frac{1}{9}x^3 + C$

49. (a) $\Gamma(1) = \int_0^\infty e^{-t}\,dt = \lim_{b\to\infty} \int_0^b e^{-t}\,dt = \lim_{b\to\infty} \left[-e^{-t}\right]_0^b = \lim_{b\to\infty} \left[-\frac{1}{e^b} - (-1)\right] = 0 + 1 = 1$

(b) $u = t^x$, $du = xt^{x-1}\,dt$; $dv = e^{-t}\,dt$, $v = -e^{-t}$; x = fixed positive real

$$\Rightarrow \Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt = \lim_{b\to\infty} \left[-t^x e^{-t}\right]_0^b + x\int_0^\infty t^{x-1}e^{-t}\,dt = \lim_{b\to\infty} \left(-\frac{b^x}{e^b} + 0^x e^0\right) + x\Gamma(x) = x\Gamma(x)$$

(c) $\Gamma(n+1) = n\Gamma(n) = n!$:

$n = 0$: $\Gamma(0+1) = \Gamma(1) = 0!$;

$n = k$: Assume $\Gamma(k+1) = k!$ for some $k > 0$;

$n = k+1$: $\Gamma(k+1+1) = (k+1)\,\Gamma(k+1)$ from part (b)

$= (k+1)k!$ induction hypothesis

$= (k+1)!$ definition of factorial

Thus, $\Gamma(n+1) = n\Gamma(n) = n!$ for every positive integer n.

50. (a) $\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}$ and $n\Gamma(n) = n! \Rightarrow n! \approx n\left(\frac{n}{e}\right)^n \sqrt{\frac{2\pi}{n}} = \left(\frac{n}{e}\right)^n \sqrt{2n\pi}$

(b)

n	$\left(\frac{n}{e}\right)^n\sqrt{2n\pi}$	calculator
10	3598695.619	3628800
20	2.4227868×10^{18}	2.432902×10^{18}
30	2.6451710×10^{32}	2.652528×10^{32}
40	8.1421726×10^{47}	8.1591528×10^{47}
50	3.0363446×10^{64}	3.0414093×10^{64}
60	8.3094383×10^{81}	8.3209871×10^{81}

(c)

n	$\left(\frac{n}{e}\right)^n\sqrt{2n\pi}$	$\left(\frac{n}{e}\right)^n\sqrt{2n\pi}\,e^{1/12n}$	calculator
10	3598695.619	3628810.051	3628800

NOTES: