CHAPTER 1 LIMITS AND CONTINUITY

1.1 RATE OF CHANGE AND LIMITS

1. (a) $\frac{\Delta f}{\Delta x} = \frac{f(3) - f(2)}{3 - 2} = \frac{28 - 9}{1} = 19$ (b) $\frac{\Delta f}{\Delta x} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2 - 0}{2} = 1$

2. $\frac{\Delta R}{\Delta \theta} = \frac{R(2) - R(0)}{2 - 0} = \frac{\sqrt{8+1} - \sqrt{1}}{2} = \frac{3 - 1}{2} = 1$

3. (a) $\frac{\Delta h}{\Delta t} = \frac{h\left(\frac{3\pi}{4}\right) - h\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4} - \frac{\pi}{4}} = \frac{-1 - 1}{\frac{\pi}{2}} = -\frac{4}{\pi}$ (b) $\frac{\Delta h}{\Delta t} = \frac{h\left(\frac{\pi}{2}\right) - h\left(\frac{\pi}{6}\right)}{\frac{\pi}{2} - \frac{\pi}{6}} = \frac{0 - \sqrt{3}}{\frac{\pi}{3}} = \frac{-3\sqrt{3}}{\pi}$

4. (a) $\frac{\Delta g}{\Delta t} = \frac{g(\pi) - g(0)}{\pi - 0} = \frac{(2 - 1) - (2 + 1)}{\pi - 0} = -\frac{2}{\pi}$ (b) $\frac{\Delta g}{\Delta t} = \frac{g(\pi) - g(-\pi)}{\pi - (-\pi)} = \frac{(2 - 1) - (2 - 1)}{2\pi} = 0$

5. (a)

Q	Slope of PQ $= \frac{\Delta p}{\Delta t}$
$Q_1(10, 225)$	$\frac{650 - 225}{20 - 10} = 42.5$ m/sec
$Q_2(14, 375)$	$\frac{650 - 375}{20 - 14} = 45.83$ m/sec
$Q_3(16.5, 475)$	$\frac{650 - 475}{20 - 16.5} = 50.00$ m/sec
$Q_4(18, 550)$	$\frac{650 - 550}{20 - 18} = 50.00$ m/sec

(b) At $t = 20$, the Cobra was traveling approximately 50 m/sec or 180 km/h.

6. (a)

Q	Slope of PQ $= \frac{\Delta p}{\Delta t}$
$Q_1(5, 20)$	$\frac{80 - 20}{10 - 5} = 12$ m/sec
$Q_2(7, 39)$	$\frac{80 - 39}{10 - 7} = 13.7$ m/sec
$Q_3(8.5, 58)$	$\frac{80 - 58}{10 - 8.5} = 14.7$ m/sec
$Q_4(9.5, 72)$	$\frac{80 - 72}{10 - 9.5} = 16$ m/sec

(b) Approximately 16 m/sec

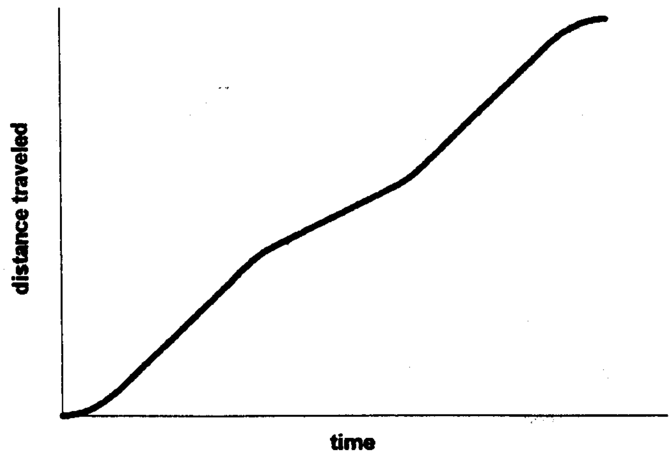
7. A plot of the data shows that the slope of the secant between $t = 0.8$ sec and $t = 1.0$ sec underestimates the instantaneous velocity (i.e., the slope of the tangent) at $t = 1.0$ sec, whereas the slope of the secant between $t = 1.0$ sec and $t = 1.2$ sec overestimates it.

Lower bound: $a = \frac{13.10 - 8.39}{1.0 - 0.8} = 23.55$ ft/sec

Upper bound: $b = \frac{18.87 - 13.10}{1.2 - 1.0} = 28.85$ ft/sec

$v(1) \approx \frac{a+b}{2} = \frac{23.55 + 28.85}{2} = 26.20$ ft/sec

8. There are many graphs that would be correct. One possible solution looks like this:



9. (a) Does not exist. As x approaches 1 from the right, g(x) approaches 0. As x approaches 1 from the left, g(x) approaches 1. There is no single number L that all the values g(x) get arbitrarily close to as $x \to 1$.
   (b) 1
   (c) 0

10. (a) 0
   (b) -1
   (c) Does not exist. As t approaches 0 from the left, f(t) approaches -1. As t approaches 0 from the right, f(t) approaches 1. There is no single number L that f(t) gets arbitrarily close to as $t \to 0$.

11. (a) True (b) True (c) False
   (d) False (e) False (f) True

12. (a) False (b) False (c) True
   (d) True (e) True

13. $\lim_{x \to 0} \frac{x}{|x|}$ does not exist because $\frac{x}{|x|} = \frac{x}{x} = 1$ if $x > 0$ and $\frac{x}{|x|} = \frac{x}{-x} = -1$ if $x < 0$. As x approaches 0 from the left, $\frac{x}{|x|}$ approaches -1. As x approaches 0 from the right, $\frac{x}{|x|}$ approaches 1. There is no single number L that all the function values get arbitrarily close to as $x \to 0$.

14. As x approaches 1 from the left, the values of $\frac{1}{x-1}$ become increasingly large and negative. As x approaches 1 from the right, the values become increasingly large and positive. There is no one number L that all the function values get arbitrarily close to as $x \to 1$, so $\lim_{x \to 1} \frac{1}{x-1}$ does not exist.

15. Nothing can be said about $\lim_{x \to x_0} f(x)$ because the existence of a limit as $x \to x_0$ does not depend on how the function is defined at x_0. In order for a limit to exist, f(x) must be arbitrarily close to a single real number L when x is close enough to x_0. That is, the existence of a limit depends on the values of f(x) for x <u>near</u> x_0, not on the definition of f(x) at x_0 itself.

16. Nothing can be said. In order for $\lim_{x\to 0} f(x)$ to exist, f(x) must close to a single value for x near 0 regardless of the value f(0) itself.

17. No, the definition does not require that f be defined at x = 1 in order for a limiting value to exist there. If f(1) is defined, it can be any real number, so we can conclude nothing about f(1) from $\lim_{x\to 1} f(x) = 5$.

18. No, because the existence of a limit depends on the values of f(x) when x is near 1, not on f(1) itself. If $\lim_{x\to 1} f(x)$ exists, its value may be some number other than f(1) = 5. We can conclude nothing about $\lim_{x\to 1} f(x)$, whether it exists or what its value is if it does exist, from knowing the value of f(1) alone.

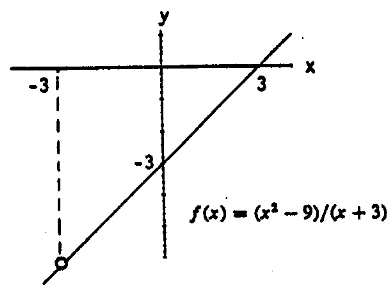
19. (a) $f(x) = (x^2 - 9)/(x + 3)$

x	−3.1	−3.01	−3.001	−3.0001	−3.00001	−3.000001
f(x)	−6.1	−6.01	−6.001	−6.0001	−6.00001	−6.000001

x	−2.9	−2.99	−2.999	−2.9999	−2.99999	−2.999999
f(x)	−5.9	−5.99	−5.999	−5.9999	−5.99999	−5.999999

The estimate is $\lim_{x\to -3} f(x) = -6$.

(b)



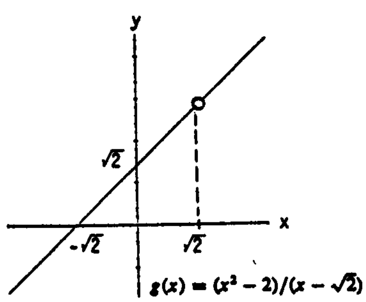
(c) $f(x) = \frac{x^2 - 9}{x + 3} = \frac{(x+3)(x-3)}{x+3} = x - 3$ if $x \neq -3$, and $\lim_{x\to -3} (x - 3) = -3 - 3 = -6$.

20. (a) $g(x) = (x^2 - 2)/(x - \sqrt{2})$

x	1.4	1.41	1.414	1.4142	1.41421	1.414213
g(x)	2.81421	2.82421	2.82821	2.828413	2.828423	2.828426

The estimate is $\lim_{x\to \sqrt{2}} g(x) = 2\sqrt{2}$.

(b)



(c) $g(x) = \frac{x^2-2}{x-\sqrt{2}} = \frac{(x+\sqrt{2})(x-\sqrt{2})}{(x-\sqrt{2})} = x+\sqrt{2}$ if $x \neq \sqrt{2}$, and $\lim_{x\to\sqrt{2}} (x+\sqrt{2}) = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$.

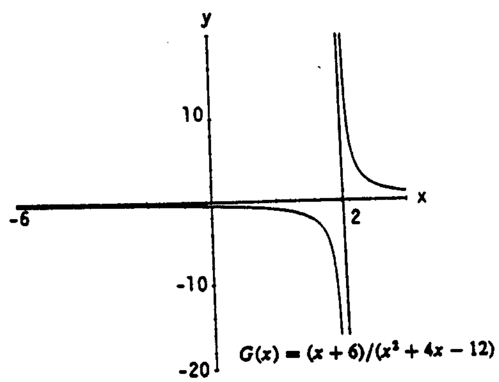
21. (a) $G(x) = (x+6)/(x^2+4x-12)$

x	−5.9	−5.99	−5.999	−5.9999	−5.99999	−5.999999
G(x)	−0.126582	−0.1251564	−0.1250156	−0.1250016	−0.12500016	−0.12500002

x	−6.1	−6.01	−6.001	−6.0001	−6.00001	−6.000001
G(x)	−0.123457	−0.1248439	−0.1249844	−0.1249984	−0.12499984	−0.12499998

The estimate is $\lim_{x\to -6} G(x) = -0.125$.

(b)



(c) $G(x) = \frac{x+6}{(x^2+4x-12)} = \frac{x+6}{(x+6)(x-2)} = \frac{1}{x-2}$ if $x \neq -6$, and $\lim_{x\to -6} \frac{1}{x-2} = \frac{1}{-6-2} = -\frac{1}{8} = -0.125$.

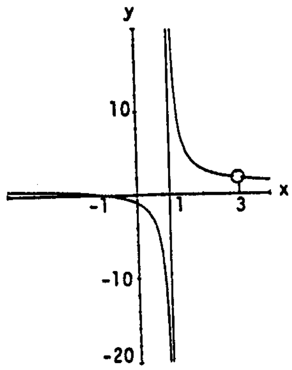
22. (a) $h(x) = (x^2-2x-3)/(x^2-4x+3)$

x	2.9	2.99	2.999	2.9999	2.99999	2.999999
h(x)	2.052631	2.005025	2.000500	2.000050	2.000005	2.0000005

x	3.1	3.01	3.001	3.0001	3.00001	3.000001
h(x)	1.952380	1.995024	1.999500	1.999950	1.999995	1.999999

The estimate is $\lim_{x\to 3} h(x) = 2$.

(b)



$h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$

(c) $h(x) = \frac{x^2 - 2x - 3}{x^2 - 4x + 3} = \frac{(x-3)(x+1)}{(x-3)(x-1)} = \frac{x+1}{x-1}$ if $x \neq 3$, and $\lim_{x \to 3} \frac{x+1}{x-1} = \frac{3+1}{3-1} = \frac{4}{2} = 2$.

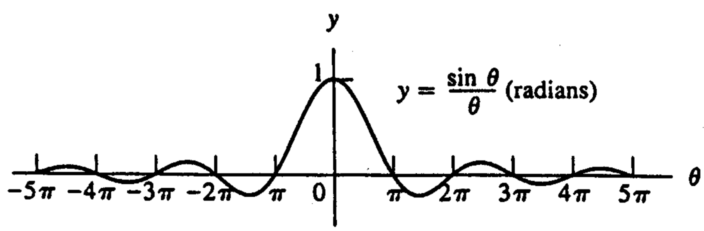
23. (a) $g(\theta) = (\sin\theta)/\theta$

θ	.1	.01	.001	.0001	.00001	.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

θ	−.1	−.01	−.001	−.0001	−.00001	−.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

$\lim_{\theta \to 0} g(\theta) = 1$

(b)



NOT TO SCALE

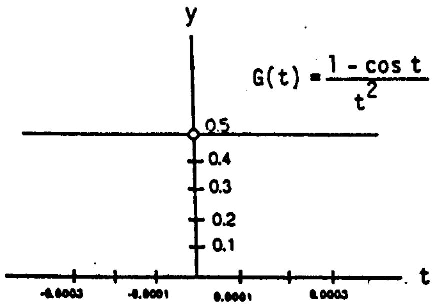
24. (a) $G(t) = (1 - \cos t)/t^2$

t	.1	.01	.001	.0001	.00001	.000001
G(t)	.499583	.499995	.499999	.5	.5	.5

t	−.1	−.01	−.001	−.0001	−.00001	−.000001
G(t)	.499583	.499995	.499999	.5	.5	.5

$\lim_{t \to 0} G(t) = 0.5$

(b)



Graph is NOT TO SCALE

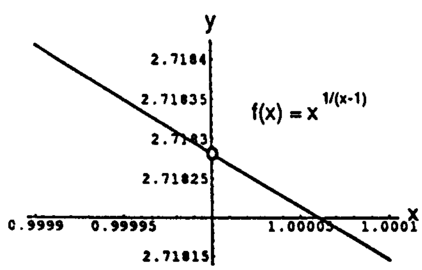
25. (a) $f(x) = x^{1/(1-x)}$

x	.9	.99	.999	.9999	.99999	.999999
f(x)	.348678	.366032	.367695	.367861	.367878	.367879

x	1.1	1.01	1.001	1.0001	1.00001	1.000001
f(x)	.385543	.369711	.368063	.367898	.367881	.367880

$\lim_{x \to 1} f(x) \approx 0.36788$

(b)



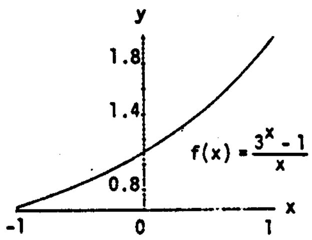
26. (a) $f(x) = (3^x - 1)/x$

x	.1	.01	.001	.0001	.00001	.000001
f(x)	1.161231	1.104669	1.099215	1.098672	1.098618	1.098612

x	−.1	−.01	−.001	−.0001	−.00001	−.000001
f(x)	1.040415	1.092599	1.098009	1.098551	1.098606	1.098611

$\lim_{x \to 0} f(x) \approx 1.0986$

(b)



27. Step 1: $|x-5| < \delta \Rightarrow -\delta < x-5 < \delta \Rightarrow -\delta+5 < x < \delta+5$
    Step 2: From the graph, $-\delta+5 = 4.9 \Rightarrow \delta = 0.1$, or $\delta+5 = 5.1 \Rightarrow \delta = 0.1$; thus $\delta = 0.1$ in either case.

28. Step 1: $|x-(-3)| < \delta \Rightarrow -\delta < x+3 < \delta \Rightarrow -\delta-3 < x < \delta-3$
    Step 2: From the graph, $-\delta-3 = -3.1 \Rightarrow \delta = 0.1$, or $\delta-3 = -2.9 \Rightarrow \delta = 0.1$; thus $\delta = 0.1$.

29. Step 1: $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow -\delta+1 < x < \delta+1$
    Step 2: From the graph, $-\delta+1 = \frac{9}{16} \Rightarrow \delta = \frac{7}{16}$, or $\delta+1 = \frac{25}{16} \Rightarrow \delta = \frac{9}{16}$; thus $\delta = \frac{7}{16}$.

30. Step 1: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta+2 < x < \delta+2$
    Step 2: From the graph, $-\delta+2 = \sqrt{3} \Rightarrow \delta = 2-\sqrt{3} \approx 0.2679$, or $\delta+2 = \sqrt{5} \Rightarrow \delta = \sqrt{5}-2 \approx 0.2361$; thus $\delta = \sqrt{5}-2$.

31. Step 1: $|(x+1)-5| < 0.01 \Rightarrow |x-4| < 0.01 \Rightarrow -0.01 < x-4 < 0.01 \Rightarrow 3.99 < x < 4.01$
    Step 2: $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4 \Rightarrow \delta = 0.01$.

32. Step 1: $|(2x-2)-(-6)| < 0.02 \Rightarrow |2x+4| < 0.02 \Rightarrow -0.02 < 2x+4 < 0.02 \Rightarrow -4.02 < 2x < -3.98$ $\Rightarrow -2.01 < x < -1.99$
    Step 2: $|x-(-2)| < \delta \Rightarrow -\delta < x+2 < \delta \Rightarrow -\delta-2 < x < \delta-2 \Rightarrow \delta = 0.01$.

33. Step 1: $|\sqrt{x+1}-1| < 0.1 \Rightarrow -0.1 < \sqrt{x+1}-1 < 0.1 \Rightarrow 0.9 < \sqrt{x+1} < 1.1 \Rightarrow 0.81 < x+1 < 1.21$ $\Rightarrow -0.19 < x < 0.21$
    Step 2: $|x-0| < \delta \Rightarrow -\delta < x < \delta \Rightarrow \delta = 0.19$.

34. Step 1: $|\sqrt{19-x}-3| < 1 \Rightarrow -1 < \sqrt{19-x}-3 < 1 \Rightarrow 2 < \sqrt{19-x} < 4 \Rightarrow 4 < 19-x < 16$ $\Rightarrow -4 > x-19 > -16 \Rightarrow 15 > x > 3$ or $3 < x < 15$
    Step 2: $|x-10| < \delta \Rightarrow -\delta < x-10 < \delta \Rightarrow -\delta+10 < x < \delta+10$.
    Then $-\delta+10 = 3 \Rightarrow \delta = 7$, or $\delta+10 = 15 \Rightarrow \delta = 5$; thus $\delta = 5$.

35. Step 1: $\left|\frac{1}{x}-\frac{1}{4}\right| < 0.05 \Rightarrow -0.05 < \frac{1}{x}-\frac{1}{4} < 0.05 \Rightarrow 0.2 < \frac{1}{x} < 0.3 \Rightarrow \frac{10}{2} > x > \frac{10}{3}$ or $\frac{10}{3} < x < 5$.
    Step 2: $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4$.
    Then $-\delta+4 = \frac{10}{3}$ or $\delta = \frac{2}{3}$, or $\delta+4 = 5$ or $\delta = 1$; thus $\delta = \frac{2}{3}$.

36. Step 1: $|x^2-3|<0.1 \Rightarrow -0.1<x^2-3<0.1 \Rightarrow 2.9<x^2<3.1 \Rightarrow \sqrt{2.9}<x<\sqrt{3.1}$

Step 2: $|x-\sqrt{3}|<\delta \Rightarrow -\delta<x-\sqrt{3}<\delta \Rightarrow -\delta+\sqrt{3}<x<\delta+\sqrt{3}$.

Then $-\delta+\sqrt{3}=\sqrt{2.9} \Rightarrow \delta=\sqrt{3}-\sqrt{2.9}\approx 0.0291$, or $\delta+\sqrt{3}=\sqrt{3.1} \Rightarrow \delta=\sqrt{3.1}-\sqrt{3}\approx 0.0286$; thus $\delta=0.0286$.

37. $|A-9|\le 0.01 \Rightarrow -0.01\le \pi\left(\frac{x}{2}\right)^2-9\le 0.01 \Rightarrow 8.99\le \frac{\pi x^2}{4}\le 9.01 \Rightarrow \frac{4}{\pi}(8.99)\le x^2\le \frac{4}{\pi}(9.01)$

$\Rightarrow 2\sqrt{\frac{8.99}{\pi}}\le x\le 2\sqrt{\frac{9.01}{\pi}}$ or $3.384\le x\le 3.387$. To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

38. $V=RI \Rightarrow \frac{V}{R}=I \Rightarrow \left|\frac{V}{R}-5\right|\le 0.1 \Rightarrow -0.1\le\frac{120}{R}-5\le 0.1 \Rightarrow 4.9\le\frac{120}{R}\le 5.1 \Rightarrow \frac{10}{49}\ge\frac{R}{120}\ge\frac{10}{51} \Rightarrow$

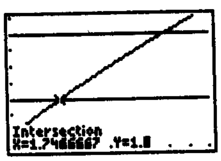
$\frac{(120)(10)}{51}\le R\le\frac{(120)(10)}{49} \Rightarrow 23.53\le R\le 24.49.$

To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

39. (a) The limit can be found by substitution.

$\lim_{x\to 2} f(x)=f(2)=\sqrt{3(2)-2}=\sqrt{4}=2$

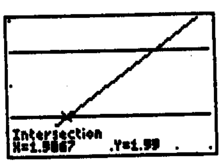
(b) The graphs of $y_1=f(x)$, $y_2=1.8$, and $y_3=2.2$ are shown.



[1.5, 2.5] by [1.5, 2.3]

The intersections of y_1 with y_2 and y_3 are at $x\approx 1.7467$ and $x=2.28$, respectively, so we may choose any value of a in $[1.7467, 2)$ (approximately) and any value of b in $[2, 2.28]$.
One possible answer: $a=1.75$, $b=2.28$.

(c) The graphs of $y_1=f(x)$, $y_2=1.99$, and $y_3=2.01$ are shown.

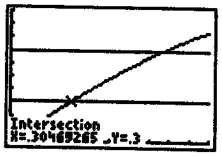


[1.97, 2.03] by [1.98, 2.02]

The intersections of y_1 with y_2 and y_3 are at $x=1.9867$ and $x\approx 2.0134$, respectively, so we may choose any value of a in $[1.9867, 2)$ and any value of b in $[2, 2.0134]$ (approximately).
One possible answer: $a=1.99$, $b=2.01$.

40. (a) $f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

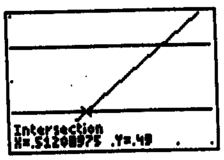
(b) The graphs of $y_1 = f(x)$, $y_2 = 0.3$, and $y_3 = 0.7$ are shown.



[0, 1] by [0, 1]

The intersections of y_1 with y_2 and y_3 are at $x \approx 0.3047$ and $x \approx 0.7754$, respectively, so we may choose any value of a in $\left[0.3047, \frac{\pi}{6}\right)$ and any value of b in $\left(\frac{\pi}{6}, 0.7754\right]$, where the interval endpoints are approximate. One possible answer: $a = 0.305$, $b = 0.775$.

(c) The graphs of $y_1 = f(x)$, $y_2 = 0.49$, and $y_3 = 0.51$ are shown.



[0.49, 0.55] by [0.48, 0.52]

The intersections of y_1 with y_2 and y_3 are at $x \approx 0.5121$ and $x \approx 0.5352$, respectively, so we may choose any value of a in $\left[0.5121, \frac{\pi}{6}\right)$ and any value of b in $\left(\frac{\pi}{6}, 0.5352\right]$, where the interval endpoints are approximate. One possible answer: $a = 0.513$, $b = 5.35$.

41. (a) In three seconds, the ball falls $4.9(3)^2 = 44.1$ m, so its average speed is $\frac{44.1}{3} = 14.7$ m/sec.

(b) The average speed over the interval from time $t = 3$ to time $3 + h$ is

$$\frac{\Delta y}{\Delta t} = \frac{4.9(3+h)^2 - 4.9(3)^2}{(3+h) - 3} = \frac{4.9(6h + h^2)}{h} = 29.4 + 4.9h$$

Since $\lim_{h \to 0} (29.4 + 4.9h) = 29.4$, the instantaneous speed is 29.4 m/sec.

42. (a) $y = gt^2 \rightarrow 20 = g(4^2) \rightarrow g = \frac{20}{16} = \frac{5}{4} = 1.25$

(b) Average speed $= \frac{20}{4} = 5$ m/sec.

(c) If the rock had not been stopped, its average speed over the interval from time $t = 4$ to time $t = 4 + h$ is

$$\frac{\Delta y}{\Delta t} = \frac{1.25(4+h)^2 - 1.25(4)^2}{(4+h) - 4} = \frac{1.25(8h + h^2)}{h} = 10 + 1.25h$$

Since $\lim_{h \to 0} (10 + 1.25h) = 10$, the instantaneous speed is 10 m/sec.

43. (a)

x	−0.1	−0.01	−0.001	−0.0001
f(x)	−0.054402	−0.005064	−0.000827	−0.000031

(b)

x	0.1	0.01	0.001	0.0001
f(x)	−0.054402	−0.005064	−0.000827	−0.000031

The limit appears to be 0.

44. (a)

x	−0.1	−0.01	−0.001	−0.0001
f(x)	0.5440	−0.5064	−0.8269	0.3056

(b)

x	0.1	0.01	0.001	0.0001
f(x)	−0.5440	−0.5064	0.8269	−0.3056

There is no clear indication of a limit.

45. (a)

x	−0.1	−0.01	−0.001	−0.0001
f(x)	2.0567	2.2763	2.2999	2.3023

(b)

x	0.1	0.01	0.001	0.0001
f(x)	2.5893	2.3293	2.3052	2.3029

The limit appears to be approximately 2.3.

46. (a)

x	−0.1	−0.01	−0.001	−0.0001
f(x)	0.074398	−0.009943	0.000585	0.000021

(b)

x	0.1	0.01	0.001	0.0001
f(x)	−0.074398	0.009943	−0.000585	−0.000021

The limit appears to be 0.

47-50. Example CAS commands:

Maple:

```
f:=x -> (x ∧ 4 − 81)/(x − 3);
plot (f(x), x=2.9..3.1);
limit (f(x), x=−1);
```

Mathematica:

```
x0=3; f=(x ∧ 4 − 81)/(x − 3)
Plot [f,{x,x0−0.1,x0 + 0.1}]
Limit [f,x - > x0]
```

51-54. (values of del may vary for a specified eps):

Maple:

```
f:=x -> (x ∧ 4 − 81)/(x − 3);
x0:='x0': eps :='eps':L:='L':del:='del':
```

```
y1:=x -> L - eps: y2:=x -> L + eps:
x0:=3: L=limit(f(x),x=x0);
eps:=0.1: del:= 0.16:
xmin:= x0 - 2*del: xmax :=x0 + 2*del:
ymin:=L - 2*eps: ymax:=L + 2*eps:
plot({f(x),y1(x),y2(x)}, x=x0-del..x0+del,view = [xmin..xmax,ymin..ymax]);
```

Mathematica:

```
Clear [f,x,L,eps,del]
y1 := L - eps; y2 := L + eps;
x0 = 3; f = (x ∧ 4 - 81)/(x-3)
Plot [f, {x,x0 - 0.2,x0 + 0.2}]
L = Limit[f, x -> x0]
eps = 0.1; del = 0.0015;
Plot [{f,y1,y2}, {x,x0 - del,x0 + del},
  PlotRange -> {{x0 - del,x0 + del}, {L - eps,L + eps}}}]
```

1.2 RULES FOR FINDING LIMITS

1. (a) $\lim_{x\to 3^-} f(x) = 3$

   (b) $\lim_{x\to 3^+} f(x) = -2$

   (c) $\lim_{x\to 3} f(x)$ does not exist, because the left- and right-hand limits are not equal.

   (d) $f(3) = 1$

2. (a) $\lim_{t\to -4^-} g(t) = 5$

   (b) $\lim_{t\to -4^+} g(t) = 2$

   (c) $\lim_{t\to -4} g(t)$ does not exist, because the left- and right-hand limits are not equal.

   (d) $g(-4) = 2$

3. (a) $\lim_{h\to 0^-} f(h) = -4$

   (b) $\lim_{h\to 0^+} f(h) = -4$

   (c) $\lim_{h\to 0} f(h) = -4$

   (d) $f(0) = -4$

4. (a) $\lim_{x\to -2^-} p(s) = 3$

   (b) $\lim_{x\to -2^+} p(s) = 3$

   (c) $\lim_{x\to -2} p(s) = 3$

   (d) $p(-2) = 3$

5. (a) $\lim_{x\to 0^-} F(x) = 4$

   (b) $\lim_{x\to 0^+} F(x) = -3$

   (c) $\lim_{x\to 0} F(x) =$ does not exist because the left- and right-hand limits are not equal.

   (d) $F(0) = 4$

6. (a) $\lim_{x\to 2^-} G(x) = 1$

   (b) $\lim_{x\to 2^+} G(x) = 1$

   (c) $\lim_{x\to 2} G(x) = 1$

   (d) $G(2) = 3$

7. (a) quotient rule
   (b) difference and power rules
   (c) sum and constant multiple rules

8. (a) quotient rule
   (b) power and product rules
   (c) difference and constant multiple rules

9. (a) $\lim_{x \to c} f(x)g(x) = [\lim_{x \to c} f(x)][\lim_{x \to c} g(x)] = (5)(-2) = -10$

(b) $\lim_{x \to c} 2f(x)g(x) = 2[\lim_{x \to c} f(x)][\lim_{x \to c} g(x)] = 2(5)(-2) = -20$

(c) $\lim_{x \to c} [f(x) + 3g(x)] = \lim_{x \to c} f(x) + 3 \lim_{x \to c} g(x) = 5 + 3(-2) = -1$

(d) $\lim_{x \to c} \frac{f(x)}{f(x) - g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} f(x) - \lim_{x \to c} g(x)} = \frac{5}{5 - (-2)} = \frac{5}{7}$

10. (a) $\lim_{x \to 4} [g(x) + 3] = \lim_{x \to 4} g(x) + \lim_{x \to 4} 3 = -3 + 3 = 0$

(b) $\lim_{x \to 4} xf(x) = \lim_{x \to 4} x \cdot \lim_{x \to 4} f(x) = (4)(0) = 0$

(c) $\lim_{x \to 4} [g(x)]^2 = \left[\lim_{x \to 4} g(x)\right]^2 = [-3]^2 = 9$

(d) $\lim_{x \to 4} \frac{g(x)}{f(x) - 1} = \frac{\lim_{x \to 4} g(x)}{\lim_{x \to 4} f(x) - \lim_{x \to 4} 1} = \frac{-3}{0 - 1} = 3$

11. (a) $\lim_{x \to -7} (2x + 5) = 2(-7) + 5 = -14 + 5 = -9$

(b) $\lim_{t \to 6} 8(t - 5)(t - 7) = 8(6 - 5)(6 - 7) = -8$

(c) $\lim_{y \to 2} \frac{y + 2}{y^2 + 5y + 6} = \frac{2 + 2}{(2)^2 + 5(2) + 6} = \frac{4}{4 + 10 + 6} = \frac{4}{20} = \frac{1}{5}$

(d) $\lim_{h \to 0} \frac{3}{\sqrt{3h + 1} + 1} = \frac{3}{\sqrt{3(0) + 1} + 1} = \frac{3}{\sqrt{1} + 1} = \frac{3}{2}$

12. (a) $\lim_{r \to -2} \left(r^3 - 2r^2 + 4r + 8\right) = (-2)^3 - 2(-2)^2 + 4(-2) + 8 = -8 - 8 - 8 + 8 = -16$

(b) $\lim_{x \to 2} \frac{x + 3}{x + 6} = \frac{2 + 3}{2 + 6} = \frac{5}{8}$

(c) $\lim_{y \to -3} (5 - y)^{4/3} = [5 - (-3)]^{4/3} = (8)^{4/3} = \left((8)^{1/3}\right)^4 = 2^4 = 16$

(d) $\lim_{\theta \to 5} \frac{\theta - 5}{\theta^2 - 25} = \lim_{\theta \to 5} \frac{\theta - 5}{(\theta + 5)(\theta - 5)} = \lim_{\theta \to 5} \frac{1}{\theta + 5} = \frac{1}{5 + 5} = \frac{1}{10}$

13. (a) $\lim_{t \to -5} \frac{t^2 + 3t - 10}{t + 5} = \lim_{t \to -5} \frac{(t + 5)(t - 2)}{t + 5} = \lim_{t \to -5} (t - 2) = -5 - 2 = -7$

(b) $\lim_{x \to -2} \frac{-2x - 4}{x^3 + 2x^2} = \lim_{x \to -2} \frac{-2(x + 2)}{x^2(x + 2)} = \lim_{x \to -2} \frac{-2}{x^2} = \frac{-2}{4} = -\frac{1}{2}$

(c) $\lim_{y \to 1} \frac{y - 1}{\sqrt{y + 3} - 2} = \lim_{y \to 1} \frac{(y - 1)\left(\sqrt{y + 3} + 2\right)}{\left(\sqrt{y + 3} - 2\right)\left(\sqrt{y + 3} + 2\right)} = \lim_{y \to 1} \frac{(y - 1)\left(\sqrt{y + 3} + 2\right)}{(y + 3) - 4} = \lim_{y \to 1} \left(\sqrt{y + 3} + 2\right)$

$= \sqrt{4} + 2 = 4$

(d) $\lim_{x \to 3} \sin\left(\frac{1}{x} - \frac{1}{2}\right) = \sin\left(\frac{1}{3} - \frac{1}{2}\right) = -\sin\left(\frac{1}{6}\right) \approx -0.1659$

14. (a) $\lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \lim_{x \to -1} \frac{(\sqrt{x^2+8}-3)(\sqrt{x^2+8}+3)}{(x+1)(\sqrt{x^2+8}+3)} = \lim_{x \to -1} \frac{(x^2+8)-9}{(x+1)(\sqrt{x^2+8}+3)}$

$= \lim_{x \to -1} \frac{(x+1)(x-1)}{(x+1)(\sqrt{x^2+8}+3)} = \lim_{x \to -1} \frac{x-1}{\sqrt{x^2+8}+3} = \frac{-2}{3+3} = -\frac{1}{3}$

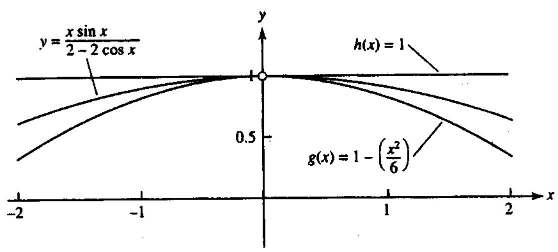
(b) $\lim_{\theta \to 1} \frac{\theta^4-1}{\theta^3-1} = \lim_{\theta \to 1} \frac{(\theta^2+1)(\theta+1)(\theta-1)}{(\theta^2+\theta+1)(\theta-1)} = \lim_{\theta \to 1} \frac{(\theta^2+1)(\theta+1)}{\theta^2+\theta+1} = \frac{(1+1)(1+1)}{1+1+1} = \frac{4}{3}$

(c) $\lim_{t \to 9} \frac{3-\sqrt{t}}{9-t} = \lim_{t \to 9} \frac{\sqrt{t}-3}{(\sqrt{t}-3)(\sqrt{t}+3)} = \lim_{t \to 9} \frac{1}{\sqrt{t}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

(d) Let $\frac{\pi - s}{2} = u$ so that $u \to 0$ as $s \to \pi$, and then rewrite and evaluate the limit as $\lim_{u \to 0} (\pi - 2u)\cos(u) = \lim_{u \to 0} (\pi - 2u) \cdot \lim_{u \to 0} \cos(u) = \pi \cdot 1 = \pi$

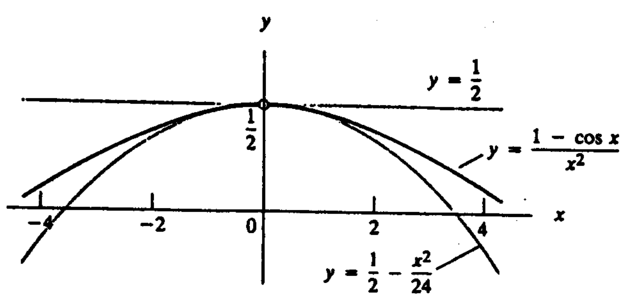
15. (a) $\lim_{x \to 0} \left(1 - \frac{x^2}{6}\right) = 1 - \frac{0}{6} = 1$ and $\lim_{x \to 0} 1 = 1$; by the sandwich theorem, $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x} = 1$

(b) For $x \neq 0$, $y = (x \sin x)/(2 - 2\cos x)$ lies between the other two graphs in the figure, and the graphs converge as $x \to 0$.



16. (a) $\lim_{x \to 0} \left(\frac{1}{2} - \frac{x^2}{24}\right) = \lim_{x \to 0} \frac{1}{2} - \lim_{x \to 0} \frac{x^2}{24} = \frac{1}{2} - 0 = \frac{1}{2}$ and $\lim_{x \to 0} \frac{1}{2} = \frac{1}{2}$; by the sandwich theorem, $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$.

(b) For all $x \neq 0$, the graph of $f(x) = (1 - \cos x)/x^2$ lies between the line $y = \frac{1}{2}$ and the parabola $y = \frac{1}{2} - x^2/24$, and the graphs converge as $x \to 0$.



17. $\lim\limits_{h\to 0} \frac{(1+h)^2 - 1^2}{h} = \lim\limits_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim\limits_{h\to 0} \frac{h(2+h)}{h} = \lim\limits_{h\to 0} (2+h) = 2$

18. $\lim\limits_{h\to 0} \frac{[3(2+h)-4]-[3(2)-4]}{h} = \lim\limits_{h\to 0} \frac{3h}{h} = 3$

19. $\lim\limits_{h\to 0} \frac{\left(\frac{1}{-2+h}\right)-\left(\frac{1}{-2}\right)}{h} = \lim\limits_{h\to 0} \frac{\frac{-2}{-2+h}-1}{-2h} = \lim\limits_{h\to 0} \frac{-2-(-2+h)}{-2h(-2+h)} = \lim\limits_{h\to 0} \frac{-h}{h(4-2h)} = -\frac{1}{4}$

20. $\lim\limits_{h\to 0} \frac{\sqrt{7+h}-\sqrt{7}}{h} = \lim\limits_{h\to 0} \frac{\left(\sqrt{7+h}-\sqrt{7}\right)\left(\sqrt{7+h}+\sqrt{7}\right)}{h\left(\sqrt{7+h}+\sqrt{7}\right)} = \lim\limits_{h\to 0} \frac{(7+h)-7}{h\left(\sqrt{7+h}+\sqrt{7}\right)}$

$= \lim\limits_{h\to 0} \frac{h}{h\left(\sqrt{7+h}+\sqrt{7}\right)} = \lim\limits_{h\to 0} \frac{1}{\sqrt{7+h}+\sqrt{7}} = \frac{1}{2\sqrt{7}}$

21. (a) False (b) True (c) False (d) True
(e) True (f) True (g) False (h) False
(i) False (j) False (k) True (l) False

22. (a) $\lim\limits_{x\to 2^+} f(x) = \frac{2}{2}+1 = 2$, $\lim\limits_{x\to 2^-} f(x) = 3-2 = 1$

(b) No, $\lim\limits_{x\to 2} f(x)$ does not exist because $\lim\limits_{x\to 2^+} f(x) \neq \lim\limits_{x\to 2^-} f(x)$

(c) $\lim\limits_{x\to 4^-} f(x) = \frac{4}{2}+1 = 3$, $\lim\limits_{x\to 4^+} f(x) = \frac{4}{2}+1 = 3$

(d) Yes, $\lim\limits_{x\to 4} f(x) = 3$ because $3 = \lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^+} f(x)$

23. (a) No, $\lim\limits_{x\to 0^+} f(x)$ does not exist since $\sin\left(\frac{1}{x}\right)$ does not approach any single value as x approaches 0

(b) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} 0 = 0$

(c) $\lim\limits_{x\to 0} f(x)$ does not exist because $\lim\limits_{x\to 0^+} f(x)$ does not exist

24. (a) Yes, $\lim\limits_{x\to 0^+} g(x) = 0$ by the sandwich theorem since $-\sqrt{x} \le g(x) \le \sqrt{x}$ when $x > 0$

(b) No, $\lim\limits_{x\to 0^-} g(x)$ does not exist since $\sqrt{x}$ does not exist, and therefore the function is not defined, for $x < 0$

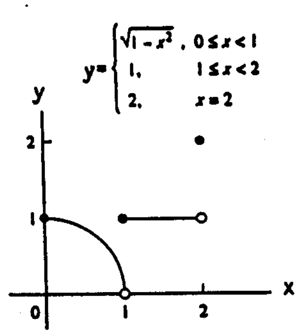
(c) No, $\lim\limits_{x\to 0} g(x)$ does not exist since $\lim\limits_{x\to 0^-} g(x)$ does not exist

25. (a) domain: $0 \le x \le 2$
range: $0 < y \le 1$ and $y = 2$

(b) $\lim\limits_{x\to c} f(x)$ exists for c belonging to
$(0,1) \cup (1,2)$

(c) $x = 2$

(d) $x = 0$

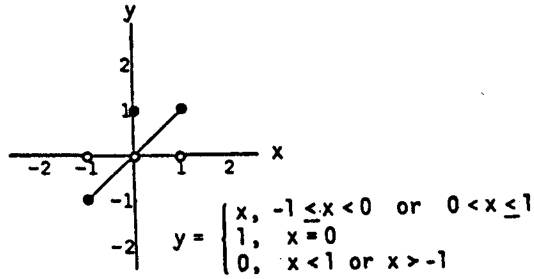


26. (a) domain: $-\infty < x < \infty$
range: $-1 \le y \le 1$

(b) $\lim\limits_{x \to c} f(x)$ exists for c belonging to $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(c) none

(d) none



27. $\lim\limits_{x \to -0.5^-} \sqrt{\frac{x+2}{x+1}} = \sqrt{\frac{-0.5+2}{-0.5+1}} = \sqrt{\frac{3/2}{1/2}} = \sqrt{3}$

28. $\lim\limits_{x \to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right) = \left(\frac{-2}{-2+1}\right)\left(\frac{2(-2)+5}{(-2)^2+(-2)}\right) = (2)\left(\frac{1}{2}\right) = 1$

29. $\lim\limits_{h \to 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h} = \lim\limits_{h \to 0^+} \left(\frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}\right)\left(\frac{\sqrt{h^2+4h+5}+\sqrt{5}}{\sqrt{h^2+4h+5}+\sqrt{5}}\right)$

$= \lim\limits_{h \to 0^+} \frac{(h^2+4h+5)-5}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)} = \lim\limits_{h \to 0^+} \frac{h(h+4)}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)} = \frac{0+4}{\sqrt{5}+\sqrt{5}} = \frac{2}{\sqrt{5}}$

30. $\lim\limits_{h \to 0^-} \frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h} = \lim\limits_{h \to 0^-} \left(\frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}\right)\left(\frac{\sqrt{6}+\sqrt{5h^2+11h+6}}{\sqrt{6}+\sqrt{5h^2+11h+6}}\right)$

$= \lim\limits_{h \to 0^-} \frac{6-(5h^2+11h+6)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)} = \lim\limits_{h \to 0^-} \frac{-h(5h+11)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)} = \frac{-(0+11)}{\sqrt{6}+\sqrt{6}} = -\frac{11}{2\sqrt{6}}$

31. (a) $\lim\limits_{x \to -2^+} (x+3)\frac{|x+2|}{x+2} = \lim\limits_{x \to -2^+} (x+3)\frac{(x+2)}{(x+2)}$ $\quad (|x+2| = x+2 \text{ for } x > -2)$

$= \lim\limits_{x \to -2^+} (x+3) = (-2)+3 = 1$

(b) $\lim\limits_{x \to -2^-} (x+3)\frac{|x+2|}{x+2} = \lim\limits_{x \to -2^-} (x+3)\left[\frac{-(x+2)}{(x+2)}\right]$ $\quad (|x+2| = -(x+2) \text{ for } x < -2)$

$= \lim\limits_{x \to -2^-} (x+3)(-1) = -(-2+3) = -1$

32. (a) $\lim\limits_{x \to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim\limits_{x \to 1^+} \frac{\sqrt{2x}(x-1)}{(x-1)}$ $\quad (|x-1| = x-1 \text{ for } x > 1)$

$= \lim\limits_{x \to 1^+} \sqrt{2x} = \sqrt{2}$

(b) $\lim\limits_{x \to 1^-} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim\limits_{x \to 1^-} \frac{\sqrt{2x}(x-1)}{-(x-1)}$ $\quad (|x-1| = -(x-1) \text{ for } x < 1)$

$= \lim\limits_{x \to 1^-} -\sqrt{2x} = -\sqrt{2}$

33. $\lim_{x\to c} f(x)$ exists at those points c where $\lim_{x\to c} x^4 = \lim_{x\to c} x^2$. Thus, $c^4 = c^2 \Rightarrow c^2(1-c^2) = 0$ $\Rightarrow c = 0,\ 1,$ or -1. Moreover, $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = 0$ and $\lim_{x\to -1} f(x) = \lim_{x\to 1} f(x) = 1$.

34. Nothing can be concluded about the values of f, g, and h at x = 2. Yes, f(2) could be 0. Since the conditions of the sandwich theorem are satisfied, $\lim_{x\to 2} f(x) = -5 \neq 0$.

35. (a) $1 = \lim_{x\to -2} \frac{f(x)}{x^2} = \frac{\lim_{x\to -2} f(x)}{\lim_{x\to -2} x^2} = \frac{\lim_{x\to -2} f(x)}{4} \Rightarrow \lim_{x\to -2} f(x) = 4.$

(b) $1 = \lim_{x\to -2} \frac{f(x)}{x^2} = \left[\lim_{x\to -2} \frac{f(x)}{x}\right]\left[\lim_{x\to -2} \frac{1}{x}\right] = \left[\lim_{x\to -2} \frac{f(x)}{x}\right]\left(\frac{1}{-2}\right) \Rightarrow \lim_{x\to -2} \frac{f(x)}{x} = -2.$

36. (a) $0 = 3\cdot 0 = \left[\lim_{x\to 2} \frac{f(x)-5}{x-2}\right]\left[\lim_{x\to 2}(x-2)\right] = \lim_{x\to 2}\left[\left(\frac{f(x)-5}{x-2}\right)(x-2)\right] = \lim_{x\to 2}[f(x)-5] = \lim_{x\to 2} f(x) - 5$ $\Rightarrow \lim_{x\to 2} f(x) = 5.$

(b) $0 = 4\cdot 0 = \left[\lim_{x\to 2} \frac{f(x)-5}{x-2}\right]\left[\lim_{x\to 2}(x-2)\right] \Rightarrow \lim_{x\to 2} f(x) = 5$ as in part (a).

37. Yes. If $\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$, then $\lim_{x\to a} f(x) = L$. If $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$, then $\lim_{x\to a} f(x)$ does not exist.

38. Since $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$, then $\lim_{x\to c} f(x)$ can be found by calculating $\lim_{x\to c^+} f(x)$.

39. $I = (5, 5+\delta) \Rightarrow 5 < x < 5+\delta$. Also, $\sqrt{x-5} < \epsilon \Rightarrow x-5 < \epsilon^2 \Rightarrow x < 5+\epsilon^2$. Choose $\delta = \epsilon^2$ $\Rightarrow \lim_{x\to 5^+} \sqrt{x-5} = 0$.

40. $I = (4-\delta, 4) \Rightarrow 4-\delta < x < 4$. Also, $\sqrt{4-x} < \epsilon \Rightarrow 4-x < \epsilon^2 \Rightarrow x > 4-\epsilon^2$. Choose $\delta = \epsilon^2$ $\Rightarrow \lim_{x\to 4^-} \sqrt{4-x} = 0$.

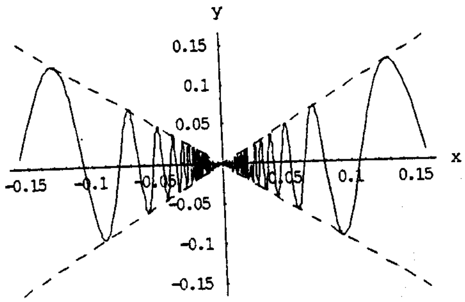
41. If f is an odd function of x, then $f(-x) = -f(x)$. Given $\lim_{x\to 0^+} f(x) = 3$, then $\lim_{x\to 0^-} f(x) = -3$.

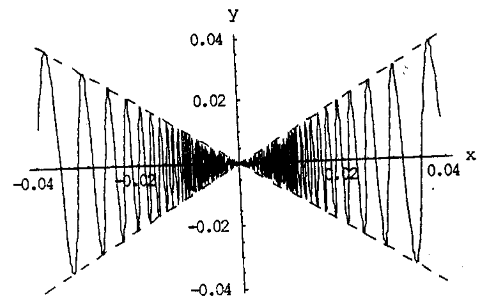
42. If f is an even function of x, then $f(-x) = f(x)$. Given $\lim_{x\to 2^-} f(x) = 7$ then $\lim_{x\to -2^+} f(x) = 7$. However, nothing can be said about $\lim_{x\to -2^-} f(x)$ because we don't know $\lim_{x\to 2^+} f(x)$.

43. (a) $g(x) = x\sin\left(\frac{1}{x}\right)$

$-\frac{\pi}{20} \le x \le \frac{\pi}{20}$

$-\frac{\pi}{80} \le x \le \frac{\pi}{180}$

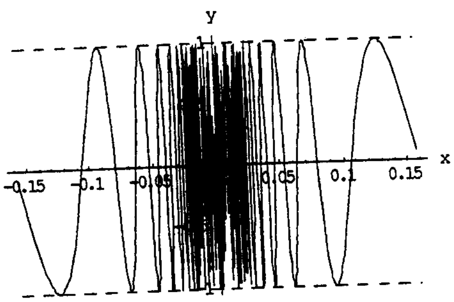




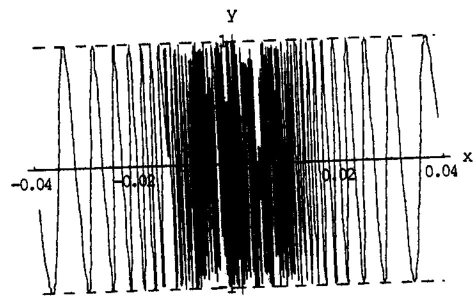
The graphs suggest that $\lim_{x\to 0} g(x) = 0$.

(b) $k(x) = \sin\left(\frac{1}{x}\right)$

$-\frac{\pi}{20} \le x \le \frac{\pi}{20}$



$-\frac{\pi}{80} \le x \le \frac{\pi}{80}$

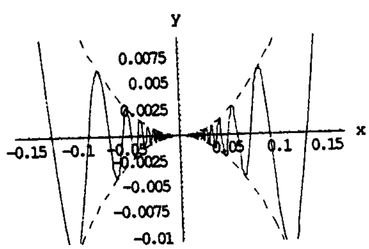


The graphs suggest that $\lim_{x\to 0} k(x)$ does not exist.

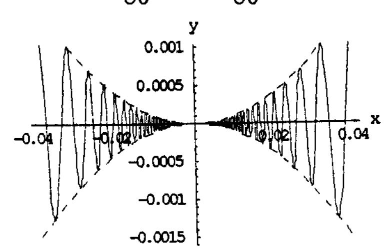
For both g(x) and k(x), the frequency of the oscillations increases without bound as $x \to 0$. For g(x), the sandwich theorem can be applied. If $x > 0$, $-x \le x \sin\left(\frac{1}{x}\right) \le x \Rightarrow \lim_{x\to 0^+} g(x) = 0$ and if $x < 0$, $x \le x \sin\left(\frac{1}{x}\right) \le -x \Rightarrow \lim_{x\to 0^-} g(x) = 0$. Therefore, $\lim_{x\to 0} g(x) = 0$ since the left- and right-hand limits are both 0. For k(x), the amplitude of the oscillations remains equal to one. Therefore, k(x) cannot be kept arbitrarily close to any number by keeping x sufficiently close to 0.

44. (a) $h(x) = x^2 \cos\left(\frac{1}{x}\right)$

$-\frac{\pi}{20} \le x \le \frac{\pi}{20}$



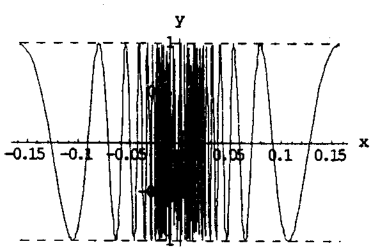
$-\frac{\pi}{80} \le x \le \frac{\pi}{80}$



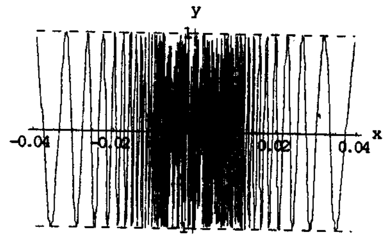
The graphs suggest that $\lim_{x\to 0} h(x) = 0$.

(b) $k(x) = \cos\left(\frac{1}{x}\right)$

$-\frac{\pi}{20} \le x \le \frac{\pi}{20}$



$-\frac{\pi}{80} \le x \le \frac{\pi}{80}$



The graphs suggest that $\lim_{x \to 0} k(x)$ does not exist.

For both h(x) and k(x), the frequency of the oscillations increases without bound as $x \to 0$. For h(x), the sandwich theorem can be applied: $-x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2 \Rightarrow \lim_{x \to 0} g(x) = 0$. For k(x), the amplitude of the oscillations remains equal to one. Therefore, k(x) cannot be kept arbitrarily close to any number by keeping x sufficiently close to 0.

1.3 LIMITS INVOLVING INFINITY

Note: In these exercises we use the result $\lim_{x \to \pm\infty} \frac{1}{x^{m/n}} = 0$ whenever $\frac{m}{n} > 0$. This result follows immediately from Example 1 and the power rule in Theorem 7: $\lim_{x \to \pm\infty} \left(\frac{1}{x^{m/n}}\right) = \lim_{x \to \pm\infty} \left(\frac{1}{x}\right)^{m/n} = \left(\lim_{x \to \pm\infty} \frac{1}{x}\right)^{m/n} = 0^{m/n}$ $= 0$.

1. (a) π (b) π

2. (a) $\frac{1}{2}$ (b) $\frac{1}{2}$

3. (a) $-\frac{5}{3}$ (b) $-\frac{5}{3}$

4. (a) $\frac{3}{4}$ (b) $\frac{3}{4}$

5. $-\frac{1}{x} \le \frac{\sin 2x}{x} \le \frac{1}{x} \Rightarrow \lim_{x \to \infty} \frac{\sin 2x}{x} = 0$ by the Sandwich Theorem

6. $\lim_{t \to -\infty} \frac{2 - t + \sin t}{t + \cos t} = \lim_{t \to -\infty} \frac{\frac{2}{t} - 1 + \left(\frac{\sin t}{t}\right)}{1 + \left(\frac{\cos t}{t}\right)} = \frac{0 - 1 + 0}{1 + 0} = -1$

7. (a) $\lim_{x\to\infty} \frac{2x+3}{5x+7} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2}{5}$ (b) $\frac{2}{5}$ (same process as part (a))

8. (a) $\lim_{x\to\infty} \frac{x+1}{x^2+3} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}} = 0$ (b) 0 (same process as part (a))

9. (a) $\lim_{x\to\infty} \frac{1-12x^3}{4x^2+12} = \lim_{x\to\infty} \frac{\frac{1}{x^2}-12x}{4+\frac{12}{x^2}} = -\infty$ (b) $\lim_{x\to-\infty} \frac{1-12x^3}{4x^2+12} = \lim_{x\to-\infty} \frac{\frac{1}{x^2}-12x}{4+\frac{12}{x^2}} = \infty$

10. (a) $\lim_{x\to\infty} \frac{7x^3}{x^3-3x^2+6x} = \lim_{x\to\infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}} = 7$ (b) 7 (same process as part (a))

11. (a) $\lim_{x\to\infty} \frac{3x^2-6x}{4x-8} = \lim_{x\to\infty} \frac{3x-6}{4-\frac{8}{x}} = \infty$ (b) $\lim_{x\to-\infty} \frac{3x^2-6x}{4x-8} = \lim_{x\to-\infty} \frac{3x-6}{4-\frac{8}{x}} = -\infty$

12. (a) $\lim_{x\to\infty} \frac{2x^5+3}{-x^2+x} = \lim_{x\to\infty} \frac{2x^3+\frac{3}{x^2}}{-1+\frac{1}{x}} = -\infty$ (b) $\lim_{x\to-\infty} \frac{2x^5+3}{-x^2+x} = \lim_{x\to-\infty} \frac{2x^3+\frac{3}{x^2}}{-1+\frac{1}{x}} = \infty$

13. (a) $\lim_{x\to\infty} \frac{-2x^3-2x+3}{3x^3+3x^2-5x} = \lim_{x\to\infty} \frac{-2-\frac{2}{x^2}+\frac{3}{x^3}}{3+\frac{3}{x}-\frac{5}{x^2}} = -\frac{2}{3}$

(b) $-\frac{2}{3}$ (same process as part (a))

14. (a) $\lim_{x\to\infty} \frac{-x^4}{x^4-7x^3+7x^2+9} = \lim_{x\to\infty} \frac{-1}{1-\frac{7}{x}+\frac{7}{x^2}+\frac{9}{x^4}} = -1$

(b) -1 (same process as part (a))

15. $\lim_{x\to\infty} \frac{2\sqrt{x}+x^{-1}}{3x-7} = \lim_{x\to\infty} \frac{\left(\frac{2}{x^{1/2}}\right)+\left(\frac{1}{x^2}\right)}{3-\frac{7}{x}} = 0$

16. $\lim_{x\to\infty} \frac{2+\sqrt{x}}{2-\sqrt{x}} = \lim_{x\to\infty} \frac{\left(\frac{2}{x^{1/2}}\right)+1}{\left(\frac{2}{x^{1/2}}\right)-1} = -1$

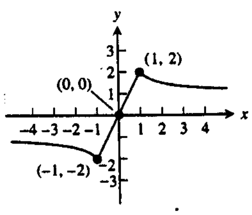
17. $\lim_{x\to-\infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}} = \lim_{x\to-\infty} \frac{1-x^{(1/5)-(1/3)}}{1+x^{(1/5)-(1/3)}} = \lim_{x\to-\infty} \frac{1-\left(\frac{1}{x^{2/15}}\right)}{1+\left(\frac{1}{x^{2/15}}\right)} = 1$

18. $\lim_{x\to\infty} \frac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}} = \lim_{x\to\infty} \frac{x+\frac{1}{x^2}}{1-\frac{1}{x}} = \infty$

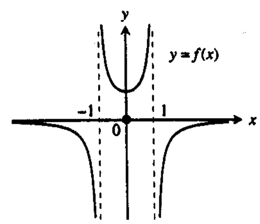
19. $\lim\limits_{x\to\infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}} = \lim\limits_{x\to\infty} \frac{2x^{1/15} - \frac{1}{x^{19/15}} + \frac{7}{x^{8/5}}}{1 + \frac{3}{x^{3/5}} + \frac{1}{x^{11/10}}} = \infty$

20. $\lim\limits_{x\to-\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4} = \lim\limits_{x\to-\infty} \frac{\frac{1}{x^{2/3}} - 5 + \frac{3}{x}}{2 + \frac{1}{x^{1/3}} - \frac{4}{x}} = -\frac{5}{2}$

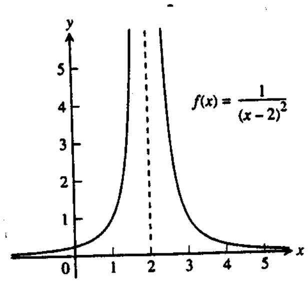
21. Here is one possibility.



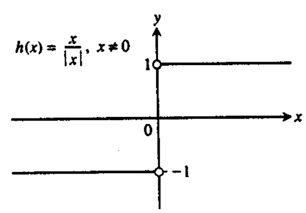
22. Here is one possibility.



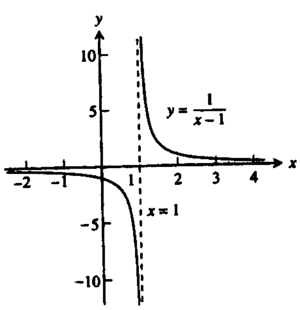
23. Here is one possibility.



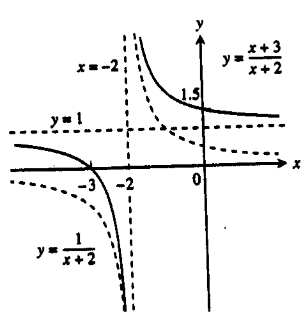
24. Here is one possibility.



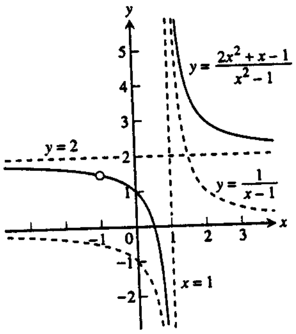
25. $y = \frac{1}{x-1}$



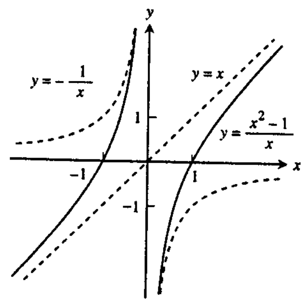
26. $y = \frac{x+3}{x+2} = 1 + \frac{1}{x+2}$



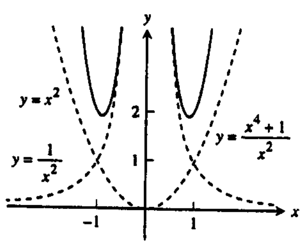
27. $y = \frac{2x^2 + x - 1}{x^2 - 1}$



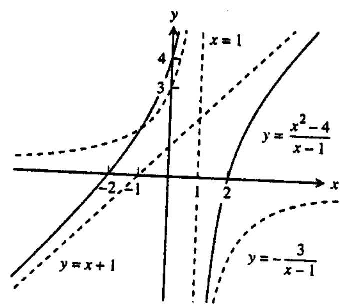
28. $y = \frac{x^2 - 1}{x} = x - \frac{1}{x}$



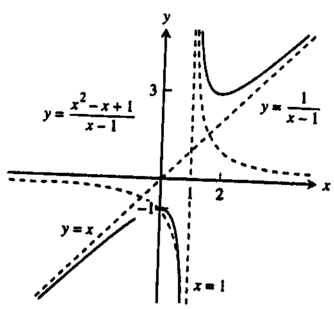
29. $y = \frac{x^4 + 1}{x^2} = x^2 + \frac{1}{x^2}$



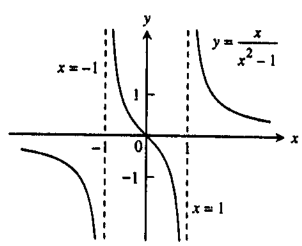
30. $y = \frac{x^2 - 4}{x - 1} = x + 1 - \frac{3}{x - 1}$



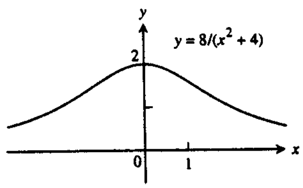
31. $y = \frac{x^2 - x + 1}{x - 1} = x + \frac{1}{x - 1}$



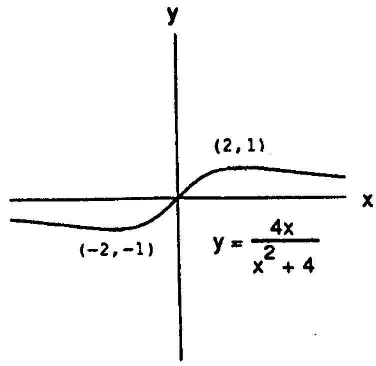
32. $y = \frac{x}{x^2 - 1}$



33. $y = \frac{8}{x^2+4}$



34. $y = \frac{4x}{x^2+4}$



35. An end behavior model is $\frac{2x^3}{x} = 2x^2$. (a)

36. An end behavior model is $\frac{x^5}{2x^2} = 0.5x^3$. (c)

37. An end behavior model is $\frac{2x^4}{-x} = -2x^3$. (d)

38. An end behavior model is $\frac{x^4}{-x^2} = -x^2$. (b)

39. (a) The function $y = e^x$ is a right end behavior model because $\lim_{x\to\infty} \frac{e^x - 2x}{e^x} = \lim_{x\to\infty} \left(1 - \frac{2x}{e^x}\right) = 1 - 0 = 1$.

(b) The function $y = -2x$ is a left end behavior model because $\lim_{x\to-\infty} \frac{e^x - 2x}{-2x} = \lim_{x\to-\infty} \left(-\frac{e^x}{2x} + 1\right) = 0 + 1 = 1$.

40. (a) The function $y = x^2$ is a right end behavior model because $\lim_{x\to\infty} \frac{x^2 + e^{-x}}{x^2} = \lim_{x\to\infty} \left(1 + \frac{e^{-x}}{x^2}\right) = 1 - 0 = 1$.

(b) The function $y = e^{-x}$ is a left end behavior model because $\lim_{x\to-\infty} \frac{x^2 + e^{-x}}{e^{-x}} = \lim_{x\to-\infty} \left(\frac{x^2}{e^{-x}} + 1\right)$

$= \lim_{x\to-\infty} (x^2e^x + 1) = 0 + 1 = 1$.

41. (a, b) The function $y = x$ is both a right end behavior model and a left end behavior model because

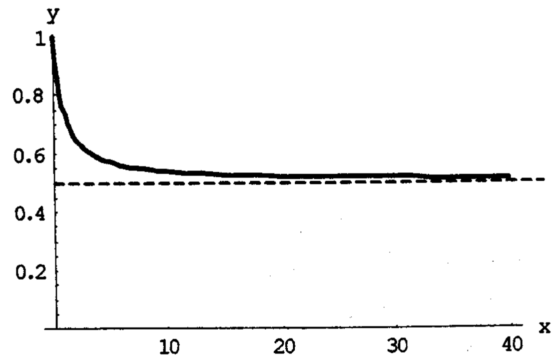
$\lim_{x\to\pm\infty} \left(\frac{x + \ln|x|}{x}\right) = \lim_{x\to\pm\infty} \left(1 + \frac{\ln|x|}{x}\right) = 1 - 0 = 1$.

42. (a, b) The function $y = x^2$ is both a right end behavior model and a left end behavior model because

$\lim_{x\to\pm\infty} \left(\frac{x^2 + \sin x}{x^2}\right) = \lim_{x\to\pm\infty} \left(1 + \frac{\sin x}{x^2}\right) = 1$.

43. $f(x) = \sqrt{x^2 + x + 1} - x$

(a)



The graph suggests that $\lim_{x \to \infty} f(x) = \frac{1}{2}$.

(b)

x	f(x) to 6 decimal places
0	1.000000
10	0.535654
100	0.503731
1000	0.500375
10000	0.500037
100000	0.500004
1000000	0.500000

The table of values also suggest that $\lim_{x \to \infty} f(x) = \frac{1}{2}$

Proof: $\lim_{x \to \infty} \left(\sqrt{x^2+x+1} - x\right) = \lim_{x \to \infty} \left[\left(\sqrt{x^2+x+1} - x\right)\left(\frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x}\right)\right] = \lim_{x \to \infty} \left(\frac{1+x}{\sqrt{x^2+x+1}+x}\right)$

$= \lim_{x \to \infty} \left(\frac{1+1/x}{\sqrt{1+1/x+1/x^2}+1}\right) = \frac{1}{2}$

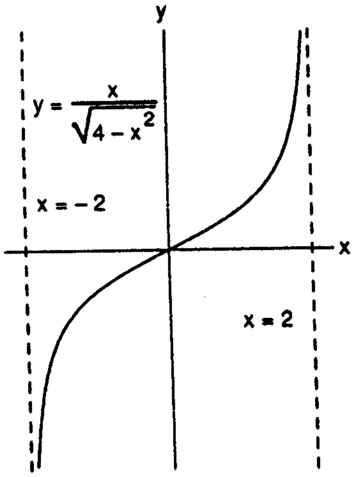
44. $\lim_{x \to \infty} \sqrt{x^2+x} - \sqrt{x^2-x} = \lim_{x \to \infty} \left[\sqrt{x^2+x} - \sqrt{x^2-x}\right] \cdot \left[\frac{\sqrt{x^2+x} + \sqrt{x^2-x}}{\sqrt{x^2+x} + \sqrt{x^2-x}}\right] = \lim_{x \to \infty} \frac{(x^2+x)-(x^2-x)}{\sqrt{x^2+x} + \sqrt{x^2-x}}$

$= \lim_{x \to \infty} \frac{2x}{\sqrt{x^2+x} + \sqrt{x^2-x}} = \lim_{x \to \infty} \frac{2}{\sqrt{1+\frac{1}{x}} + \sqrt{1-\frac{1}{x}}} = \frac{2}{1+1} = 1$

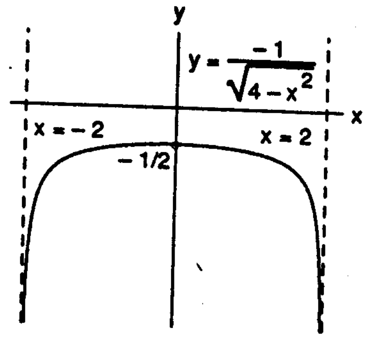
45. At most 2 horizontal asymptotes: one for $x \to \infty$ and possibly another for $x \to -\infty$.

46. At most the degree of the denominator, which is zero at a vertical asymptote. A polynomial of degree n has at most n real roots (or zeros).

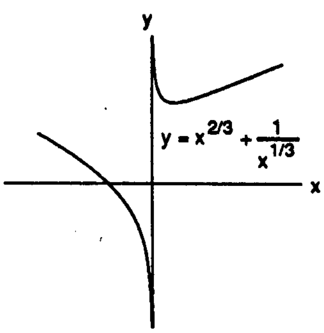
47. $y = \frac{x}{\sqrt{4 - x^2}}$



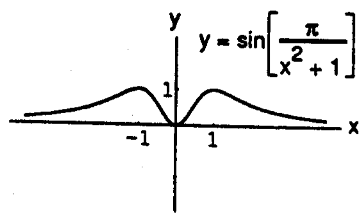
48. $y = \frac{-1}{\sqrt{4 - x^2}}$



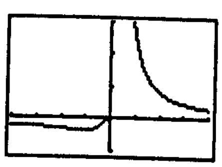
49. $y = x^{2/3} + \frac{1}{x^{1/3}}$



50. $y = \sin\left(\frac{\pi}{x^2 + 1}\right)$



51.



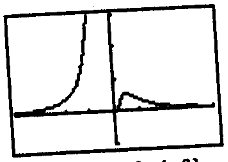
[−4, 4] by [−1, 3]

The graph of $y = f\left(\frac{1}{x}\right) = \frac{1}{x}e^{1/x}$ is shown.

$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = \infty$

$\lim_{x\to -\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = 0$

52.



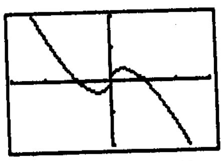
[−4, 4] by [−1, 3]

The graph of $y = f\left(\frac{1}{x}\right) = \frac{1}{x^2}e^{-1/x}$ is shown.

$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = 0$

$\lim_{x\to -\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = \infty$

53.



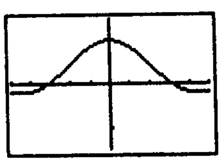
[−3, 3] by [−2, 2]

The graph of $y = f\left(\frac{1}{x}\right) = x \ln\left|\frac{1}{x}\right|$ is shown.

$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = 0$

$\lim_{x\to -\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = 0$

54.



[−5, 5] by [−1.5, 1.5]

The graph of $y = f\left(\frac{1}{x}\right) = \frac{\sin x}{x}$ is shown.

$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = 1$

$\lim_{x\to -\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = 1$

55. $\lim\limits_{x\to-\infty} \frac{\cos\frac{1}{x}}{1+\frac{1}{x}} = \lim\limits_{\theta\to 0^-} \frac{\cos\theta}{1+\theta} = \frac{1}{1} = 1, \quad \left(\theta = \frac{1}{x}\right)$

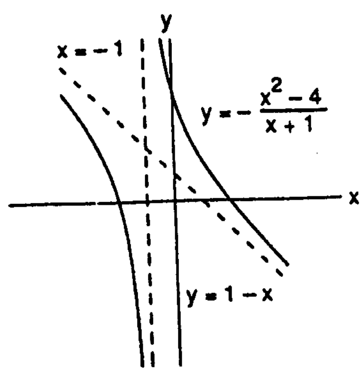
56. $\lim\limits_{x\to\infty} \left(\frac{1}{x}\right)^{1/x} = \lim\limits_{z\to 0^+} z^z = 1, \quad \left(z = \frac{1}{x}\right)$

57. $\lim\limits_{x\to\pm\infty} \left(3+\frac{2}{x}\right)\left(\cos\frac{1}{x}\right) = \lim\limits_{\theta\to 0} (3+2\theta)(\cos\theta) = (3)(1) = 3, \quad \left(\theta = \frac{1}{x}\right)$

58. $\lim\limits_{x\to\infty} \left(\frac{3}{x^2} - \cos\frac{1}{x}\right)\left(1+\sin\frac{1}{x}\right) = \lim\limits_{\theta\to 0^+} \left(3\theta^2 - \cos\theta\right)(1+\sin\theta) = (0-1)(1+0) = -1, \quad \left(\theta = \frac{1}{x}\right)$

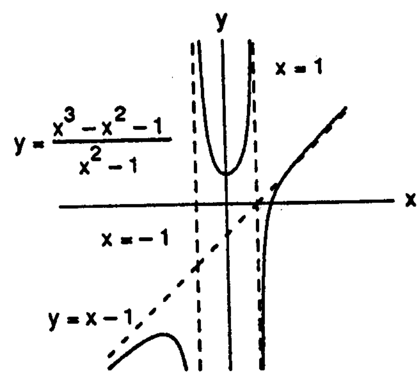
59. $y = -\frac{x^2-4}{x+1} = 1 - x + \frac{3}{x+1}$

The graph of the function mimics each term as it becomes dominant.

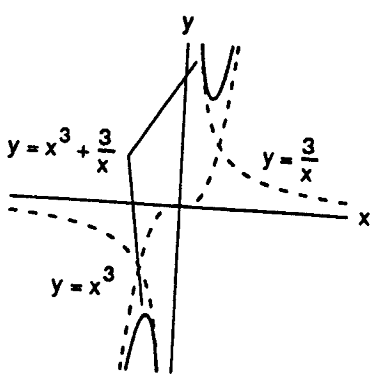


60. $y = \frac{x^3-x^2-1}{x^2-1} = x - 1 + \frac{x-2}{x^2-1}$

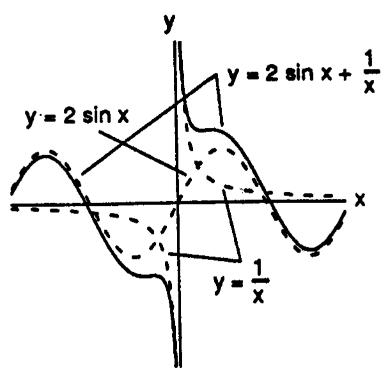
The graph of the function mimics each term as it becomes dominant.



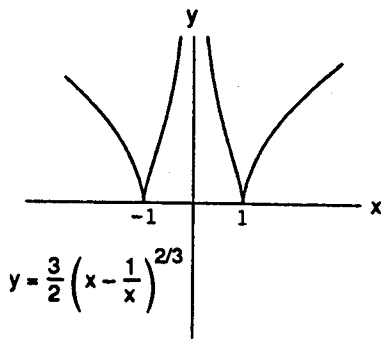
61. The graph of the function mimics each term as it becomes dominant.



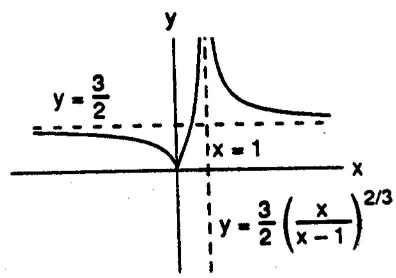
62. The graph of the function mimics each term as it becomes dominant.



63. (a) $y \to \infty$ (see the accompanying graph)
    (b) $y \to \infty$ (see the accompanying graph)
    (c) cusps at $x = \pm 1$ (see the accompanying graph)



64. (a) $y \to 0$ and a cusp at $x = 0$ (see the accompanying graph)
    (b) $y \to \frac{3}{2}$ (see the accompanying graph)
    (c) a vertical asymptote at $x = 1$ and contains the point $\left(-1, \frac{3}{2\sqrt[3]{4}}\right)$ (see the accompanying graph)



1.4 CONTINUITY

1. No, discontinuous at $x = 2$, not defined at $x = 2$

2. No, discontinuous at $x = 3$, $1 = \lim_{x \to 3^-} g(x) \neq g(3) = 1.5$

3. Continuous on $[-1, 3]$

4. No, discontinuous at $x = 1$, $1.5 = \lim_{x \to 1^-} k(x) \neq \lim_{x \to 1^+} k(x) = 0$

5. (a) Yes
   (b) Yes, $\lim_{x \to -1^+} f(x) = 0$
   (c) Yes
   (d) Yes

6. (a) Yes, $f(1) = 1$
   (b) Yes, $\lim_{x \to 1} f(x) = 2$
   (c) No
   (d) No

7. (a) No
   (b) No

8. $[-1,0)\cup(0,1)\cup(1,2)\cup(2,3)$

9. $f(2)=0$, since $\lim_{x\to 2^-} f(x) = -2(2)+4 = 0 = \lim_{x\to 2^+} f(x)$

10. $f(1)$ should be changed to $2 = \lim_{x\to 1} f(x)$

11. The function $f(x)$ is not continuous at $x=0$ because $\lim_{x\to 0} f(x) = 0$, $f(0)=1$ and, therefore, $\lim_{x\to 0} f(x) \neq f(0)$. The function $f(x)$ is not continuous at $x=1$ because $\lim_{x\to 1} f(x)$ does not exist since $\lim_{x\to 1^-} f(x) = -1$ and $\lim_{x\to 1^+} f(x) = 0$. The discontinuity at $x=0$ is removable because the function would be continuous there if the value of $f(0)$ were 0 instead of 1. The discontinuity at $x=1$ is not removable because $\lim_{x\to 1} f(x)$ does not exist and the discontinuity cannot be removed by defining or redefining $f(1)$.

12. The function $f(x)$ is not continuous at $x=1$ because $\lim_{x\to 1} f(x)$ does not exist since $\lim_{x\to 1^-} f(x) = -2$ and $\lim_{x\to 1+} f(x) = 0$. The function $f(x)$ is not continuous at $x=2$ because $\lim_{x\to 2} f(x) = 1$, $f(2)=0$ and, therefore, $\lim_{x\to 2} f(x) \neq f(2)$. The discontinuity at $x=1$ is not removable because $\lim_{x\to 1} f(x)$ does not exist and the discontinuity cannot be removed by defining or redefining $f(1)$. The discontinuity at $x=2$ is removable because the function would be continuous there if the value of $f(2)$ were 1 instead of 0.

13. Discontinuous only when $x-2=0 \Rightarrow x=2$ $\Rightarrow$ continuous on $(-\infty,2)\cup(2,\infty)$

14. Discontinuous only when $(x+2)^2=0 \Rightarrow x=-2$ $\Rightarrow$ continuous on $(-\infty,-2)\cup(-2,\infty)$

15. Discontinuous only when $t^2-4t+3=0 \Rightarrow (t-3)(t-1)=0 \Rightarrow t=3$ or $t=1 \Rightarrow$ continuous on $(-\infty,1)\cup(1,3)\cup(3,\infty)$

16. Continuous everywhere. ($|t|+1 \neq 0$ for all t; limits exist and are equal to function values.)

17. Discontinuous only at $\theta=0 \Rightarrow$ continuous on $(-\infty,0)\cup(0,\infty)$

18. Discontinuous when $\frac{\pi\theta}{2}$ is an odd integer multiple of $\frac{\pi}{2}$, i.e., $\frac{\pi\theta}{2} = (2n-1)\frac{\pi}{2}$, n an integer $\Rightarrow \theta = 2n-1$, n an integer (i.e., θ is an odd integer). Continuous everywhere else $\Rightarrow$ continuous on $((2n-1)\pi/2, (2n+1)\pi/2)$ for n an integer.

19. Discontinuous when $2v+3<0$ or $v<-\frac{3}{2} \Rightarrow$ continuous on the interval $\left[-\frac{3}{2},\infty\right)$.

20. Discontinuous when $3x-1<0$ or $x<\frac{1}{3} \Rightarrow$ continuous on the interval $\left[\frac{1}{3},\infty\right)$.

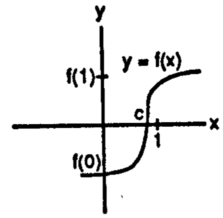
21. $\lim_{x\to\pi} \sin(x-\sin x) = \sin(\pi-\sin\pi) = \sin(\pi-0) = \sin\pi = 0$; continuous at $x=\pi$

22. $\lim_{t\to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right) = \sin\left(\frac{\pi}{2}\cos(\tan(0))\right) = \sin\left(\frac{\pi}{2}\cos(0)\right) = \sin\left(\frac{\pi}{2}\right) = 1$; continuous at $t=0$

23. $\lim_{y\to 1} \sec\left(y\sec^2 y - \tan^2 y - 1\right) = \lim_{y\to 1} \sec\left(y\sec^2 y - \sec^2 y\right) = \lim_{y\to 1} \sec\left((y-1)\sec^2 y\right) = \sec\left((1-1)\sec^2 1\right)$ $= \sec 0 = 1$; continuous at $y=1$

24. $\lim_{\theta\to 0} \tan\left[\frac{\pi}{4}\cos\left(\sin\theta^{1/3}\right)\right] = \tan\left[\frac{\pi}{4}\cos(\sin(0))\right] = \tan\left(\frac{\pi}{4}\cos(0)\right) = \tan\left(\frac{\pi}{4}\right) = 1$; continuous at $\theta=0$.

25. f(x) is continuous on $[0,1]$ and $f(0) < 0$, $f(1) > 0$ $\Rightarrow$ by the Intermediate Value Theorem f(x) takes on every value between f(0) and f(1) $\Rightarrow$ the equation $f(x) = 0$ has at least one solution between $x = 0$ and $x = 1$.



26. $\cos x = x \Rightarrow (\cos x) - x = 0$. If $x = -\frac{\pi}{2}$, $\cos\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) > 0$. If $x = \frac{\pi}{2}$, $\cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2} < 0$. Thus $\cos x - x = 0$ for some x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ according to the Intermediate Value Theorem.

27. All five statements ask for the same information because of the intermediate value property of continuous functions.

    (a) A root of $f(x) = x^3 - 3x - 1$ is a point c where $f(c) = 0$. The roots are approximately $x_1 = -1.53$, $x_2 = -0.347$, $x_3 = 1.88$, the points where f(x) changes sign.

    (b) The points where $y = x^3$ crosses $y = 3x + 1$ have the same y-coordinate, or $y = x^3 = 3x + 1 \Rightarrow y = f(x) = x^3 - 3x - 1 = 0$.

    (c) $x^3 - 3x = 1 \Rightarrow x^3 - 3x - 1 = 0$. The solutions to the equation are the roots of $f(x) = x^3 - 3x - 1$.

    (d) The points where $y = x^3 - 3x$ crosses $y = 1$ have common y-coordinates, or $y = x^3 - 3x = 1 \Rightarrow y = f(x) = x^3 - 3x - 1 = 0$.

    (e) The solutions of $x^3 - 3x - 1 = 0$ are those points where $f(x) = x^3 - 3x - 1$ has value 0.

28. Answers may vary. Note that f is continuous for every value of x.

    (a) $f(0) = 10$, $f(1) = 1^3 - 8(1) + 10 = 3$. Since $3 < \pi < 10$, by the Intermediate Value Theorem, there exists a c so that $0 < c < 1$ and $f(c) = \pi$.

    (b) $f(0) = 10$, $f(-4) = (-4)^3 - 8(-4) + 10 = -22$. Since $-22 < -\sqrt{3} < 10$, by the Intermediate Value Theorem, there exists a c so that $-4 < c < 0$ and $f(c) = -\sqrt{3}$.

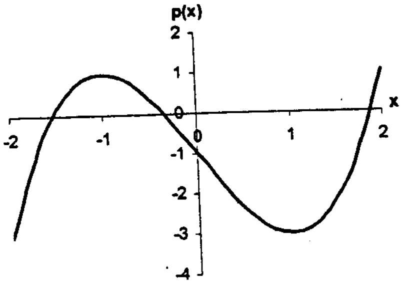
    (c) $f(0) = 10$, $f(1000) = (1000)^3 - 8(1000) + 10 = 999{,}992{,}010$. Since $10 < 5{,}000{,}000 < 999{,}992{,}010$, by the Intermediate Value Theorem, there exists a c so that $0 < c < 1000$ and $f(c) = 5{,}000{,}000$.

29. Answers may vary. For example, $f(x) = \frac{\sin(x-2)}{x-2}$ is discontinuous at $x = 2$ because it is not defined there. However, the discontinuity can be removed because f has a limit (namely 1) as $x \to 2$.

30. Answers may vary. For example, $g(x) = \frac{1}{x+1}$ has a discontinuity at $x = -1$ because $\lim_{x \to -1} g(x)$ does not exist. ($\lim_{x \uparrow -1} g(x) = -\infty$ and $\lim_{x \downarrow -1} g(x) = +\infty$.)

31. Noting that $r = 0$ is triple zero, the polynomial can be rewritten as $x^3(x^2 - x - 5)$. Therefore, the roots of the quintic polynomial are $r_1 = \frac{1 - \sqrt{21}}{2} \approx -1.791$, $r_2 = r_3 = r_4 = 0$, and $r_5 = \frac{1 + \sqrt{21}}{2} \approx 2.791$.

32. The graph shows that the polynomial has three zeros between -2 and 2, any one a candidate for r. By zooming in, the choices for r are estimated at -1.532, -0.347, or 1.879.



33. (a) Suppose x_0 is rational $\Rightarrow f(x_0) = 1$. Choose $\epsilon = \frac{1}{2}$. For any $\delta > 0$ there is an irrational number x (actually infinitely many) in the interval $(x_0 - \delta, x_0 + \delta) \Rightarrow f(x) = 0$. Then $0 < |x - x_0| < \delta$ but $|f(x) - f(x_0)| = 1 > \frac{1}{2} = \epsilon$, so $\lim_{x \to x_0} f(x)$ fails to exist $\Rightarrow$ f is discontinuous at x_0 rational. On the other hand, x_0 irrational $\Rightarrow f(x_0) = 0$ and there is a rational number x in $(x_0 - \delta, x_0 + \delta) \Rightarrow f(x) = 1$. Again $\lim_{x \to x_0} f(x)$ fails to exist $\Rightarrow$ f is discontinuous at x_0 irrational. That is, f is discontinuous at every point.

(b) f is neither right-continuous nor left-continuous at any point x_0 because in every interval $(x_0 - \delta, x_0)$ or $(x_0, x_0 + \delta)$ there exist both rational and irrational real numbers. Thus neither limits $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ exist by the same arguments used in part (a).

34. Yes. Both $f(x) = x$ and $g(x) = x - \frac{1}{2}$ are continuous on $[0,1]$. However $\frac{f(x)}{g(x)}$ is undefined at $x = \frac{1}{2}$ since $g\left(\frac{1}{2}\right) = 0 \Rightarrow \frac{f(x)}{g(x)}$ is discontinuous at $x = \frac{1}{2}$.

35. Yes, because of the Intermediate Value Theorem. If f(a) and f(b) did have different signs then f would have to equal zero at some point between a and b since f is continuous on $[a, b]$.

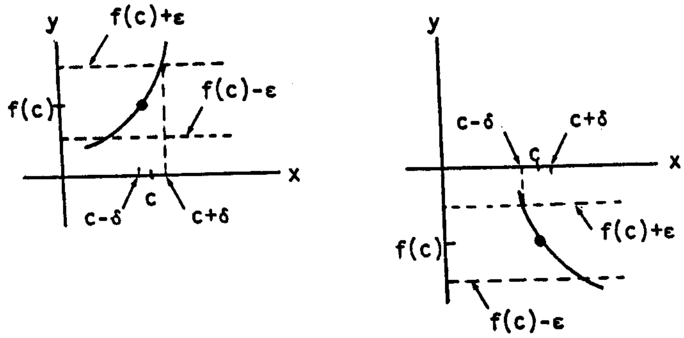
36. Let f(x) be the new position of point x and let $d(x) = f(x) - x$. The displacement function d is negative if x is the left-hand point of the rubber band and positive if x is the right-hand point of the rubber band. By the Intermediate Value Theorem, $d(x) = 0$ for some point in between. That is, $f(x) = x$ for some point x, which is then in its original position.

37. If $f(0) = 0$ or $f(1) = 1$, we are done (i.e., $c = 0$ or $c = 1$ in those cases). Then let $f(0) = a > 0$ and $f(1) = b < 1$ because $0 \le f(x) \le 1$. Define $g(x) = f(x) - x \Rightarrow g$ is continuous on $[0,1]$. Moreover, $g(0) = f(0) - 0 = a > 0$ and $g(1) = f(1) - 1 = b - 1 < 0 \Rightarrow$ by the Intermediate Value Theorem there is a number c in $(0,1)$ such that $g(c) = 0 \Rightarrow f(c) - c = 0$ or $f(c) = c$.

38. Let $\epsilon = \frac{|f(c)|}{2} > 0$. Since f is continuous at $x = c$ there is a $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$ $\Rightarrow f(c) - \epsilon < f(x) < f(c) + \epsilon$.

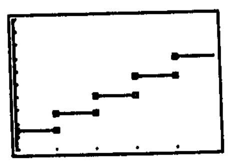
If $f(c) > 0$, then $\epsilon = \frac{1}{2}f(c) \Rightarrow \frac{1}{2}f(c) < f(x) < \frac{3}{2}f(c) \Rightarrow f(x) > 0$ on the interval $(c - \delta, c + \delta)$.

If $f(c) < 0$, then $\epsilon = -\frac{1}{2}f(c) \Rightarrow \frac{3}{2}f(c) < f(x) < \frac{1}{2}f(c) \Rightarrow f(x) < 0$ on the interval $(c - \delta, c + \delta)$.



39. (a) Luisa's salary is $\$36{,}500 = \$36{,}500(1.035)^0$ for the first year ($0 \le t < 1$), $\$36{,}500(1.035)$ for the second year ($1 \le t < 2$), $\$36{,}500(1.035)^2$ for the third year ($2 \le t < 3$), and so on. This corresponds to $y = 36{,}500(1.035)^{\text{int } t}$.

(b)



[0, 4.98] by [35,000, 45,000]

The function is continuous at all points in the domain $[0, 5)$ except at $t = 1, 2, 3, 4$.

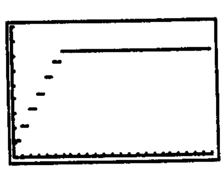
40. (a) We require:

$$f(x) = \begin{cases} 0, & x = 0 \\ 1.10, & 0 < x \le 1 \\ 2.20, & 1 < x \le 2 \\ 3.30, & 2 < x \le 3 \\ 4.40, & 3 < x \le 4 \\ 5.50, & 4 < x \le 5 \\ 6.60, & 5 < x \le 6 \\ 7.25, & 6 < x \le 24. \end{cases}$$

This may be written more compactly as

$$f(x) = \begin{cases} -1.10\text{int}(-x), & 0 \le x \le 6 \\ 7.25, & 6 < x \le 24 \end{cases}$$

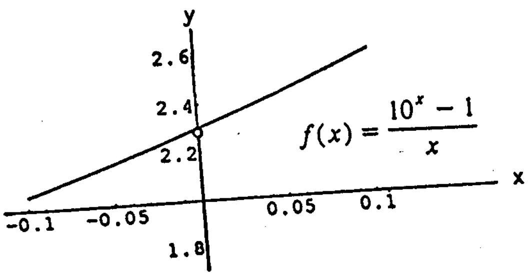
(b)



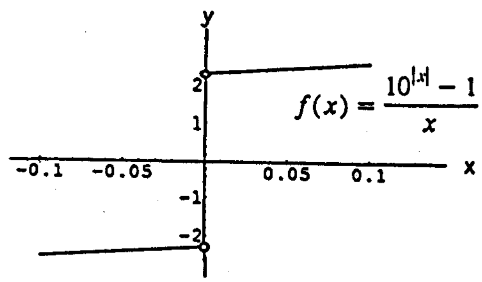
[0, 24] by [0, 9]

This is continuous for all values of x in the domain $[0, 24]$ except for $x = 0, 1, 2, 3, 4, 5, 6$.

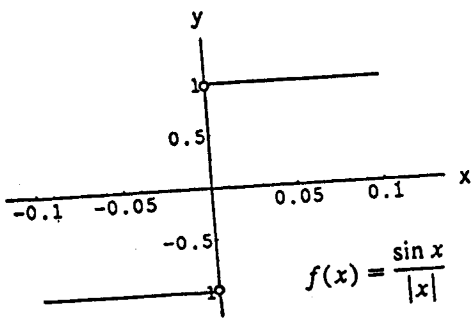
41. The function can be extended: $f(0) \approx 2.3$.



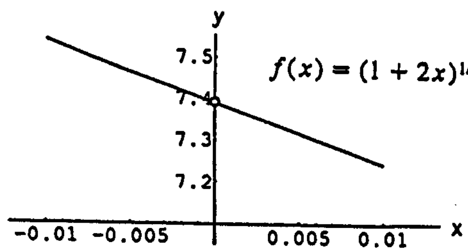
42. The function cannot be extended to be continuous at $x = 0$. If $f(0) \approx 2.3$, it will be continuous from the right. Or if $f(0) \approx -2.3$, it will be continuous from the left.



43. The function cannot be extended to be continuous at $x = 0$. If $f(0) = 1$, it will be continuous from the right. Or if $f(0) = -1$, it will be continuous from the left.



44. The function can be extended: $f(0) \approx 7.39$.



45. $x \approx 1.8794, -1.5321, -0.3473$

46. $x \approx 1.4516, -0.8546, 0.4030$

47. $x \approx 1.7549$

48. $x \approx 1.5596$

49. $x \approx 3.5156$

50. $x \approx -3.9059, 3.8392, 0.0667$

51. $x \approx 0.7391$

52. $x \approx -1.8955, 0, 1.8955$

1.5 TANGENT LINES

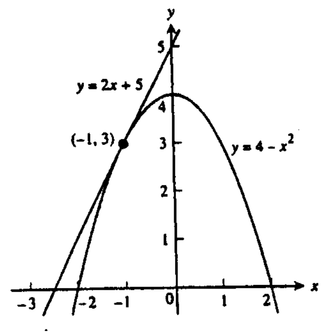
1. P_1: $m_1 = 1$, P_2: $m_2 = 5$

2. P_1: $m_1 = -2$, P_2: $m_2 = 0$

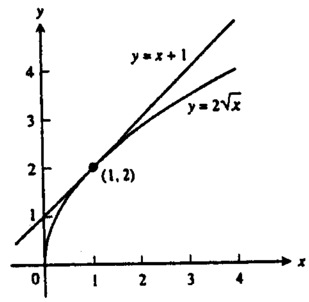
3. P_1: $m_1 = \frac{5}{2}$, P_2: $m_2 = -\frac{1}{2}$

4. P_1: $m_1 = 3$, P_2: $m_2 = -3$

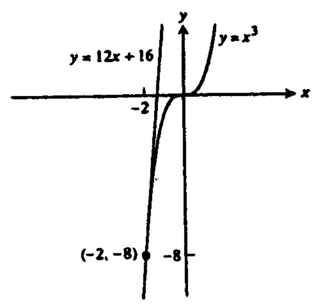
5. $m = \lim_{h\to 0} \frac{[4-(-1+h)^2]-(4-(-1)^2)}{h}$
$= \lim_{h\to 0} \frac{-(1-2h+h^2)+1}{h} = \lim_{h\to 0} \frac{h(2-h)}{h} = 2;$
at $(-1,3)$: $y = 3+2(x-(-1)) \Rightarrow y = 2x+5$, tangent line



6. $m = \lim_{h\to 0} \frac{2\sqrt{1+h}-2\sqrt{1}}{h} = \lim_{h\to 0} \frac{2\sqrt{1+h}-2}{h} \cdot \frac{2\sqrt{1+h}+2}{2\sqrt{1+h}+2}$
$= \lim_{h\to 0} \frac{4(1+h)-4}{2h\left(\sqrt{1+h}+1\right)} = \lim_{h\to 0} \frac{2}{\sqrt{1+h}+1} = 1;$
at $(1,2)$: $y = 2+1(x-1) \Rightarrow y = x+1$, tangent line



7. $m = \lim_{h\to 0} \frac{(-2+h)^3-(-2)^3}{h} = \lim_{h\to 0} \frac{-8+12h-6h^2+h^3+8}{h}$
$= \lim_{h\to 0} \left(12-6h+h^2\right) = 12;$
at $(-2,-8)$: $y = -8+12(x-(-2)) \Rightarrow y = 12x+16$, tangent line



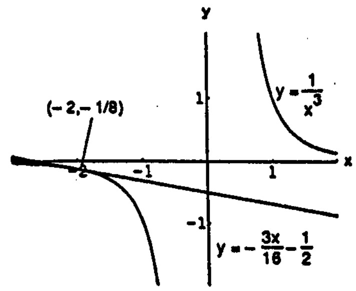
8. $m = \lim_{h\to 0} \frac{\frac{1}{(-2+h)^3} - \frac{1}{(-2)^3}}{h} = \lim_{h\to 0} \frac{-8-(-2+h)^3}{-8h(-2+h)^3}$

$= \lim_{h\to 0} \frac{-(12h - 6h^2 + h^3)}{-8h(-2+h)^3} = \lim_{h\to 0} \frac{12 - 6h + h^2}{8(-2+h)^3}$

$= \frac{12}{8(-8)} = -\frac{3}{16};$

at $\left(-2, -\frac{1}{8}\right)$: $y = -\frac{1}{8} - \frac{3}{16}(x - (-2))$

$\Rightarrow y = -\frac{3}{16}x - \frac{1}{2}$, tangent line



9. $m = \lim_{h\to 0} \frac{[(1+h) - 2(1+h)^2] - (-1)}{h} = \lim_{h\to 0} \frac{(1 + h - 2 - 4h - 2h^2) + 1}{h} = \lim_{h\to 0} \frac{h(-3-2h)}{h} = -3;$

at $(1,-1)$: $y + 1 = -3(x-1)$, tangent line

10. $m = \lim_{h\to 0} \frac{[(1+h)^3 + 3(1+h)] - 4}{h} = \lim_{h\to 0} \frac{(1 + 3h + 3h^2 + h^3 + 3 + 3h) - 4}{h} = \lim_{h\to 0} \frac{h(6 + 3h + h^2)}{h} = 6;$

at $(1,4)$: $y - 4 = 6(t-1)$, tangent line

11. $m = \lim_{h\to 0} \frac{\frac{3+h}{(3+h)-2} - 3}{h} = \lim_{h\to 0} \frac{(3+h) - 3(h+1)}{h(h+1)} = \lim_{h\to 0} \frac{-2h}{h(h+1)} = -2;$

at $(3,3)$: $y - 3 = -2(u-3)$, tangent line

12. $m = \lim_{h\to 0} \frac{\sqrt{(8+h)+1} - 3}{h} = \lim_{h\to 0} \frac{\sqrt{9+h} - 3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} = \lim_{h\to 0} \frac{(9+h) - 9}{h(\sqrt{9+h}+3)} = \lim_{h\to 0} \frac{h}{h(\sqrt{9+h}+3)}$

$= \frac{1}{\sqrt{9}+3} = \frac{1}{6}$; at $(8,3)$: $y - 3 = \frac{1}{6}(x-8)$, tangent line

13. At $x = 3$, $y = \frac{1}{2} \Rightarrow m = \lim_{h\to 0} \frac{\frac{1}{(3+h)-1} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{2-(2+h)}{2h(2+h)} = \lim_{h\to 0} \frac{-h}{2h(2+h)} = -\frac{1}{4}$, slope

14. At $x = 0$, $y = -1 \Rightarrow m = \lim_{h\to 0} \frac{\frac{h-1}{h+1} - (-1)}{h} = \lim_{h\to 0} \frac{(h-1)+(h+1)}{h(h+1)} = \lim_{h\to 0} \frac{2h}{h(h+1)} = 2$, slope

15. At a horizontal tangent the slope $m = 0 \Rightarrow 0 = m = \lim_{h\to 0} \frac{[(x+h)^2 + 4(x+h) - 1] - (x^2 + 4x - 1)}{h}$

$= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2 + 4x + 4h - 1) - (x^2 + 4x - 1)}{h} = \lim_{h\to 0} \frac{(2xh + h^2 + 4h)}{h} = \lim_{h\to 0} (2x + h + 4) = 2x + 4;$

$2x + 4 = 0 \Rightarrow x = -2$. Then $f(-2) = 4 - 8 - 1 = -5 \Rightarrow (-2,-5)$ is the point on the graph where there is a horizontal tangent.

16. $0 = m = \lim_{h\to 0} \frac{[(x+h)^3 - 3(x+h)] - (x^3 - 3x)}{h} = \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h) - (x^3 - 3x)}{h}$

$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} = \lim_{h\to 0} (3x^2 + 3xh + h^2 - 3) = 3x^2 - 3$; $3x^2 - 3 = 0 \Rightarrow x = -1$ or $x = 1$. Then $f(-1) = 2$ and $f(1) = -2 \Rightarrow (-1, 2)$ and $(1, -2)$ are the points on the graph where a horizontal tangent exists.

17. $-1 = m = \lim_{h\to 0} \frac{\frac{1}{(x+h)-1} - \frac{1}{x-1}}{h} = \lim_{h\to 0} \frac{(x-1)-(x+h-1)}{h(x-1)(x+h-1)} = \lim_{h\to 0} \frac{-h}{h(x-1)(x+h-1)} = -\frac{1}{(x-1)^2}$
$\Rightarrow (x-1)^2 = 1 \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$. If $x = 0$, then $y = -1$ and $m = -1$ $\Rightarrow y = -1 - (x - 0) = -(x+1)$. If $x = 2$, then $y = 1$ and $m = -1 \Rightarrow y = 1 - (x-2) = -(x-3)$.

18. $\frac{1}{4} = m = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h\to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$

$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$. Thus, $\frac{1}{4} = \frac{1}{2\sqrt{x}} \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4 \Rightarrow y = 2$. The tangent line is

$y = 2 + \frac{1}{4}(x-4) = \frac{x}{4} + 1$.

19. $\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(100 - 4.9(2+h)^2) - (100 - 4.9(2)^2)}{h} = \lim_{h\to 0} \frac{-4.9(4 + 4h + h^2) + 4.9(4)}{h}$
$= \lim_{h\to 0} (-19.6 - 4.9h) = -19.6$. The minus sign indicates the object is falling <u>downward</u> at a speed of 19.6 m/sec.

20. $\lim_{h\to 0} \frac{f(10+h) - f(10)}{h} = \lim_{h\to 0} \frac{3(10+h)^2 - 3(10)^2}{h} = \lim_{h\to 0} \frac{3(20h + h^2)}{h} = 60$ ft/sec.

21. $\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\pi(3+h)^2 - \pi(3)^2}{h} = \lim_{h\to 0} \frac{\pi[9 + 6h + h^2 - 9]}{h} = \lim_{h\to 0} \pi(6+h) = 6\pi$

22. $\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\frac{4\pi}{3}(2+h)^3 - \frac{4\pi}{3}(2)^3}{h} = \lim_{h\to 0} \frac{\frac{4\pi}{3}[12h + 6h^2 + h^3]}{h} = \lim_{h\to 0} \frac{4\pi}{3}[12 + 6h + h^2] = 16\pi$

23. $\lim_{h\to 0} \frac{s(1+h) - s(1)}{h} = \lim_{h\to 0} \frac{1.86(1+h)^2 - 1.86(1)^2}{h} = \lim_{h\to 0} \frac{1.86 + 3.72h + 1.86h^2 - 1.86}{h} = \lim_{h\to 0} (3.72 + 1.86h)$
$= 3.72$

24. $\lim_{h\to 0} \frac{s(2+h) - s(2)}{h} = \lim_{h\to 0} \frac{11.44(2+h)^2 - 11.44(2)^2}{h} = \lim_{h\to 0} \frac{45.76 + 45.76h + 11.44h^2 - 45.76}{h}$

$= \lim_{h\to 0} (45.76 + 11.44h) = 45.76$

25. Slope at origin $= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0 \Rightarrow$ yes, f(x) does have a tangent at the origin with slope 0.

26. $\lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0} \frac{h\sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} \sin\frac{1}{h}$. Since $\lim_{h\to 0} \sin\frac{1}{h}$ does not exist, f(x) has no tangent at the origin.

27. $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-1-0}{h} = \infty$, and $\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{1-0}{h} = \infty$. Therefore, $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \infty \Rightarrow$ yes, the graph of f has a vertical tangent at the origin.

28. $\lim_{h\to 0^-} \frac{U(0+h)-U(0)}{h} = \lim_{h\to 0^-} \frac{0-1}{h} = \infty$, and $\lim_{h\to 0^+} \frac{U(0+h)-U(0)}{h} = \lim_{h\to 0^+} \frac{1-1}{h} = 0 \Rightarrow$ no, the graph of f does not have a vertical tangent at (0, 1) because the limit does not exist.

29. (a) $\frac{\Delta f}{\Delta x} = \frac{f(0)-f(-2)}{0-(-2)} = \frac{1-e^{-2}}{2} \approx 0.432$ (b) $\frac{\Delta f}{\Delta x} = \frac{f(3)-f(1)}{3-1} = \frac{e^3-e}{2} \approx 8.684$

30. (a) $\frac{\Delta f}{\Delta x} = \frac{f(4)-f(1)}{4-1} = \frac{\ln 4 - 0}{3} = \frac{\ln 4}{3} \approx 0.462$

(b) $\frac{\Delta f}{\Delta x} = \frac{f(103)-f(100)}{103-100} = \frac{\ln 103 - \ln 100}{3} = \frac{1}{3}\ln\frac{103}{100} = \frac{1}{3}\ln 1.03 \approx 0.0099$

31. (a) $\frac{\Delta f}{\Delta t} = \frac{f(3\pi/4)-f(\pi/4)}{(3\pi/4)-(\pi/4)} = \frac{-1-1}{\pi/2} = -\frac{4}{\pi} \approx -1.273$

(b) $\frac{\Delta f}{\Delta t} = \frac{f(\pi/2)-f(\pi/6)}{(\pi/2)-(\pi/6)} = \frac{0-\sqrt{3}}{\pi/3} = -\frac{3\sqrt{3}}{\pi} \approx -1.654$

32. (a) $\frac{\Delta f}{\Delta t} = \frac{f(\pi)-f(0)}{\pi-0} = \frac{1-3}{\pi} = -\frac{2}{\pi} \approx -0.637$

(b) $\frac{\Delta f}{\Delta t} = \frac{f(\pi)-f(-\pi)}{\pi-(-\pi)} = \frac{1-1}{2\pi} = 0$

33. (a) $\frac{2.1-1.5}{1995-1993} = 0.3$

The rate of change was 0.3 billion dollars per year.

(b) $\frac{3.1-2.1}{1997-1995} = 0.5$

The rate of change was 0.5 billion dollars per year.

(c) $y = 0.0571x^2 - 0.1514x + 1.3943$

[0, 10] by [0, 4]

(d) $\frac{y(5)-y(3)}{5-3} \approx 0.31$

$\frac{y(7)-y(5)}{7-5} \approx 0.53$

According to the regression equation, the rates were 0.31 billion dollars per year and 0.53 billion dollars per year.

(e) $$\lim_{h\to 0} \frac{y(7+h)-y(7)}{h} = \lim_{h\to 0} \frac{[0.0571(7+h)^2 - 0.1514(7+h) + 1.3943] - [0.0571(7)^2 - 0.1514(7) + 1.3943]}{h}$$
$$= \lim_{h\to 0} \frac{0.0571(14h+h^2) - 0.1514h}{h}$$
$$= \lim_{h\to 0} [0.0571(14) - 0.1514 + 0.0571h]$$
$$\approx 0.65$$

The funding was growing at a rate of about 0.65 billion dollars per year.

34. (a)

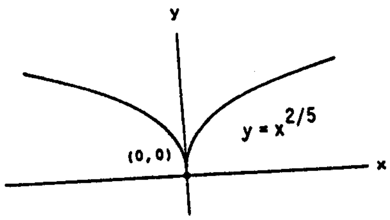
[7, 18] by [0, 900]

(b)

Q from year	Slope
1988	$\frac{440-225}{17-8} \approx 23.9$
1989	$\frac{440-289}{17-9} \approx 18.9$
1990	$\frac{440-270}{17-10} \approx 24.3$
1991	$\frac{440-493}{17-11} \approx -8.8$
1992	$\frac{440-684}{17-12} \approx -48.8$
1993	$\frac{440-763}{17-13} \approx -80.8$
1994	$\frac{440-651}{17-14} \approx -70.3$
1995	$\frac{440-600}{17-15} \approx -80.0$
1996	$\frac{440-296}{17-16} \approx 144.0$

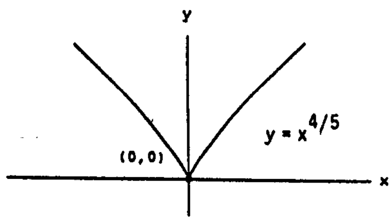
(c) As Q gets closer to 1997, the slopes do not seem to be approaching a limit value. The years 1995–97 seem to be very unusual and unpredictable.

35. (a) The graph appears to have a cusp at $x = 0$.



(b) $\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^{2/5} - 0}{h} = \lim_{h \to 0^-} \frac{1}{h^{3/5}} = -\infty$ and $\lim_{h \to 0^+} \frac{1}{h^{3/5}} = \infty \Rightarrow$ limit does not exist $\Rightarrow$ the graph of $y = x^{2/5}$ does not have a vertical tangent at $x = 0$.

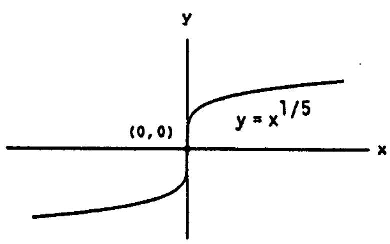
36. (a) The graph appears to have a cusp at $x = 0$.



(b) $\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^{4/5} - 0}{h} = \lim_{h \to 0^-} \frac{1}{h^{1/5}} = -\infty$ and $\lim_{h \to 0^+} \frac{1}{h^{1/5}} = \infty \Rightarrow$ limit does not exist

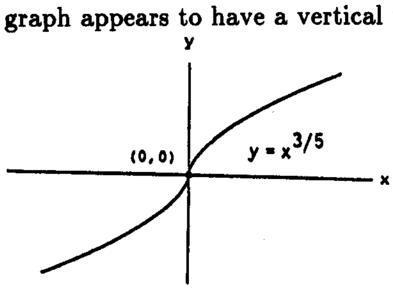
$\Rightarrow y = x^{4/5}$ does not have a vertical tangent at $x = 0$.

37. (a) The graph appears to have a vertical tangent at $x = 0$.



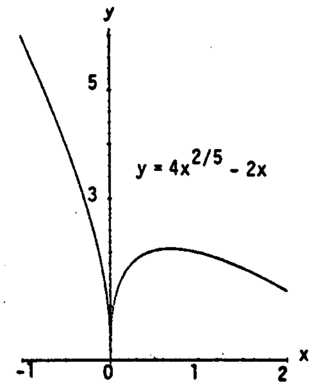
(b) $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{1/5} - 0}{h} = \lim_{h \to 0} \frac{1}{h^{4/5}} = \infty \Rightarrow y = x^{1/5}$ has a vertical tangent at $x = 0$.

38. (a) The graph appears to have a vertical tangent at $x = 0$.



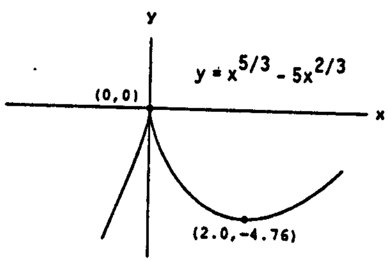
(b) $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{3/5} - 0}{h} = \lim_{h \to 0} \frac{1}{h^{2/5}} = \infty \Rightarrow$ the graph of $y = x^{3/5}$ has a vertical tangent at $x = 0$.

39. (a) The graph appears to have a cusp at $x = 0$.



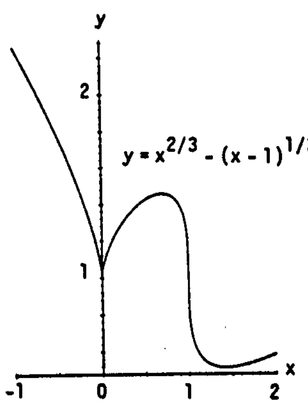
(b) $\lim\limits_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0^-} \frac{4h^{2/5}-2h}{h} = \lim\limits_{h\to 0^-} \frac{4h^{2/5}-2h}{h} = \lim\limits_{h\to 0^-} \frac{4}{h^{3/5}} - 2 = -\infty$ and $\lim\limits_{h\to 0^+} \frac{4}{h^{3/5}} - 2$ $= \infty \Rightarrow$ limit does not exist $\Rightarrow$ the graph of $y = 4x^{2/5} - 2x$ does not have a vertical tangent at $x = 0$.

40. (a) The graph appears to have a cusp at $x = 0$.



(b) $\lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{h^{5/3}-5h^{2/3}}{h} = \lim\limits_{h\to 0} h^{2/3} - \frac{5}{h^{1/3}} = 0 - \lim\limits_{h\to 0} \frac{5}{h^{1/3}}$ does not exist $\Rightarrow$ the graph of $y = x^{5/3} - 5x^{2/3}$ does not have a vertical tangent.

41. (a) The graph appears to have a vertical tangent at $x = 1$ and a cusp at $x = 0$.



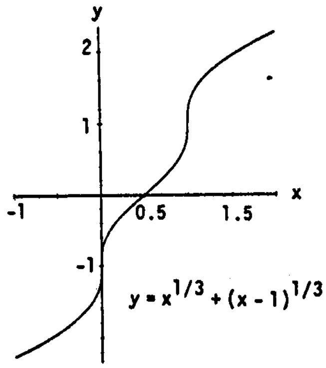
(b) $x = 1$: $\lim\limits_{h\to 0} \frac{(1+h)^{2/3} - (1+h-1)^{1/3} - 1}{h} = \lim\limits_{h\to 0} \frac{(1+h)^{2/3} - h^{1/3} - 1}{h} = -\infty$

$\Rightarrow y = x^{2/3} - (x-1)^{1/3}$ has a vertical tangent at $x = 1$;

$x = 0$: $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^{2/3}-(h-1)^{1/3}-(-1)^{1/3}}{h} = \lim_{h\to 0}\left[\frac{1}{h^{1/3}} - \frac{(h-1)^{1/3}}{h} + \frac{1}{h}\right]$

does not exist $\Rightarrow y = x^{2/3} - (x-1)^{1/3}$ does not have a vertical tangent at $x = 0$.

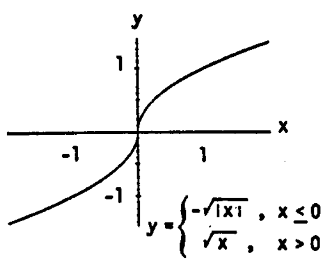
42. (a) The graph appears to have vertical tangents at $x = 0$ and $x = 1$.



(b) $x = 0$: $\lim_{h\to 0} \frac{(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^{1/3}+(h-1)^{1/3}-(-1)^{1/3}}{h} = \infty \Rightarrow y = x^{1/3} + (x-1)^{1/3}$ has a vertical tangent at $x = 0$;

$x = 1$: $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^{1/3}+(1+h-1)^{1/3}-1}{h} = \infty \Rightarrow y = x^{1/3} + (x-1)^{1/3}$ has a vertical tangent at $x = 1$.

43. (a) The graph appears to have a vertical tangent at $x = 0$.

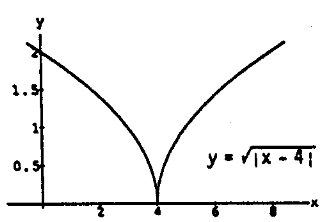


(b) $\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{x\to 0^+} \frac{\sqrt{h}-0}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty$;

$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-\sqrt{|h|}-0}{h} = \lim_{h\to 0^-} \frac{-\sqrt{|h|}}{-|h|} = \lim_{h\to 0^-} \frac{1}{\sqrt{|h|}} = \infty$

$\Rightarrow$ y has a vertical tangent at $x = 0$.

44. (a) The graph appears to have a cusp at $x = 4$.



(b) $\lim\limits_{h\to 0^+} \frac{f(4+h)-f(4)}{h} = \lim\limits_{h\to 0^+} \frac{\sqrt{|4-(4+h)|}-0}{h} = \lim\limits_{h\to 0^+} \frac{\sqrt{|h|}}{h} = \lim\limits_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty;$

$\lim\limits_{h\to 0^-} \frac{f(4+h)-f(4)}{h} = \lim\limits_{h\to 0^-} \frac{\sqrt{|4-(4+h)|}}{h} = \lim\limits_{h\to 0^-} \frac{\sqrt{|h|}}{-\sqrt{h^2}} = -\infty$

$\Rightarrow y = \sqrt{4-x}$ does not have a vertical tangent at $x = 4$.

45-48. Example CAS commands:

Maple:

```
f:=x -> cos(x) + 4*sin(2*x);
x0:=Pi:
dq:=h -> (f(x0 + h) - f(x0))/h;
slope:=limit(dq(h),h=0);
L:=x -> f(x0) + slope*(x - x0);
y1:=f(x0) + dq(3)*(x - x0):
y2:=f(x0) + dq(2)*(x - x0):
y3:=f(x0) + dq(1)*(x - x0):
plot ({f(x),y1,y2,y3,L(x)},x = x0 - 1..x0 + 3);
```

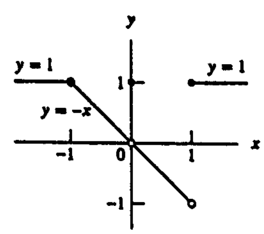
Mathematica:

```
Clear [f,m,x,y]
x0 = Pi; f[x_] := Cos[x] + 4 Sin[2x]
Plot[ f[x], {x,x0 - 1,x0 + 3) ]
dq[h_] := (f[x0 + h] - f[x0])/h
m = Limit[ dq[h], h -> 0 ]
y := f[x0] + m (x - x0)
y1 := f[x0] + dq[1] (x - x0)
y2 := f[x0] + dq[2] (x - x0)
y3 := f[x0] + dq[3] (x - x0)
Plot[ {f[x],y,y1,y2,y3}, {x,x0 - 1,x0 + 3} ]
```

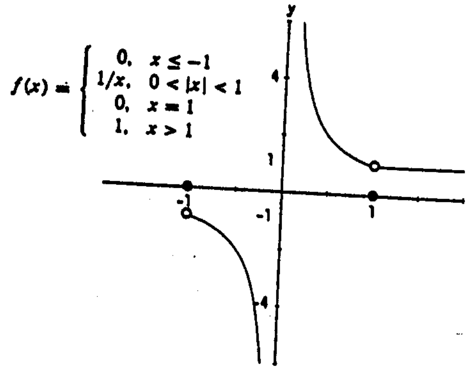
CHAPTER 1 PRACTICE EXERCISES

1. At $x = -1$: $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^+} f(x) = 1 \Rightarrow \lim\limits_{x\to -1} f(x)$
$= 1 = f(-1) \Rightarrow$ f is continuous at $x = -1$.
At $x = 0$: $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = 0 \Rightarrow \lim\limits_{x\to 0} f(x) = 0$.
But $f(0) = 1 \neq \lim\limits_{x\to 0} f(x) \Rightarrow$ f is discontinuous at $x = 0$.
At $x = 1$: $\lim\limits_{x\to 1^-} f(x) = -1$ and $\lim\limits_{x\to 1^+} f(x) = 1 \Rightarrow \lim\limits_{x\to 1} f(x)$
does not exist $\Rightarrow$ f is discontinuous at $x = 1$.

$$f(x) = \begin{cases} 1, & x \le -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \ge 1. \end{cases}$$



2. At $x = -1$: $\lim\limits_{x\to -1^-} f(x) = 0$ and $\lim\limits_{x\to -1^+} f(x) = -1 \Rightarrow \lim\limits_{x\to -1} f(x)$
does not exist $\Rightarrow$ f is discontinuous at $x = -1$.



At $x = 0$: $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = \infty \Rightarrow \lim_{x\to 0} f(x)$ does not exist $\Rightarrow$ f is discontinuous at $x = 0$.

At $x = 1$: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 1 \Rightarrow \lim_{x\to 1} f(x) = 1$. But $f(1) = 0 \neq \lim_{x\to 1} f(x) \Rightarrow$ f is discontinuous at $x = 1$.

3. (a) $\lim_{t\to t_0} (3f(t)) = 3 \lim_{t\to t_0} f(t) = 3(-7) = -21$

(b) $\lim_{t\to t_0} (f(t))^2 = \left(\lim_{t\to t_0} f(t)\right)^2 = (-7)^2 = 49$

(c) $\lim_{t\to t_0} (f(t)\cdot g(t)) = \lim_{t\to t_0} f(t)\cdot \lim_{t\to t_0} g(t) = (-7)(0) = 0$

(d) $\lim_{t\to t_0} \frac{f(t)}{g(t)-7} = \frac{\lim_{t\to t_0} f(t)}{\lim_{t\to t_0} (g(t)-7)} = \frac{-7}{\lim_{t\to t_0} g(t) - \lim_{t\to t_0} 7} = \frac{-7}{0-7} = 1$

(e) $\lim_{t\to t_0} \cos(g(t)) = \cos\left(\lim_{t\to t_0} g(t)\right) = \cos 0 = 1$

(f) $\lim_{t\to t_0} |f(t)| = \left|\lim_{t\to t_0} f(t)\right| = |-7| = 7$

(g) $\lim_{t\to t_0} (f(t)+g(t)) = \lim_{t\to t_0} f(t) + \lim_{t\to t_0} g(t) = -7 + 0 = -7$

(h) $\lim_{t\to t_0} (1/f(t)) = \frac{1}{\lim_{t\to t_0} f(t)} = \frac{1}{-7} = -\frac{1}{7}$

4. (a) $\lim_{x\to 0} -g(x) = -\lim_{x\to 0} g(x) = -\sqrt{2}$

(b) $\lim_{x\to 0} (g(x)\cdot f(x)) = \lim_{x\to 0} g(x)\cdot \lim_{x\to 0} f(x) = (\sqrt{2})\left(\frac{1}{2}\right) = \frac{\sqrt{2}}{2}$

(c) $\lim_{x\to 0} (f(x)+g(x)) = \lim_{x\to 0} f(x) + \lim_{x\to 0} g(x) = \frac{1}{2} + \sqrt{2}$

(d) $\lim_{x\to 0} \frac{1}{f(x)} = \frac{1}{\lim_{x\to 0} f(x)} = \frac{1}{\frac{1}{2}} = 2$

(e) $\lim_{x\to 0} (x + f(x)) = \lim_{x\to 0} x + \lim_{x\to 0} f(x) = 0 + \frac{1}{2} = \frac{1}{2}$

(f) $\lim_{x\to 0} \frac{f(x)\cdot \cos x}{x-1} = \frac{\lim_{x\to 0} f(x)\cdot \lim_{x\to 0} \cos x}{\lim_{x\to 0} x - \lim_{x\to 0} 1} = \frac{\left(\frac{1}{2}\right)(1)}{0-1} = -\frac{1}{2}$

5. Since $\lim_{x\to 0} x = 0$ we must have that $\lim_{x\to 0} (4-g(x)) = 0$. Otherwise, if $\lim_{x\to 0} (4-g(x))$ is a finite positive number, we would have $\lim_{x\to 0^-} \left[\frac{4-g(x)}{x}\right] = -\infty$ and $\lim_{x\to 0^+} \left[\frac{4-g(x)}{x}\right] = \infty$ so the limit could not equal 1 as

$x \to 0$. Similar reasoning holds if $\lim_{x\to 0}(4-g(x))$ is a finite negative number. We conclude that $\lim_{x\to 0} g(x) = 4$.

6. $2 = \lim_{x\to -4}\left[x \lim_{x\to 0} g(x)\right] = \lim_{x\to -4} x \cdot \lim_{x\to -4}\left[\lim_{x\to 0} g(x)\right] = -4 \lim_{x\to -4}\left[\lim_{x\to 0} g(x)\right] = -4 \lim_{x\to 0} g(x)$ (since $\lim_{x\to 0} g(x)$ is a constant) $\Rightarrow \lim_{x\to 0} g(x) = \frac{2}{-4} = -\frac{1}{2}$.

7. (a) $\lim_{x\to c} f(x) = \lim_{x\to c} x^{1/3} = c^{1/3} = f(c)$ for every real number $c \Rightarrow f$ is continuous on $(-\infty,\infty)$

(b) $\lim_{x\to c} g(x) = \lim_{x\to c} x^{3/4} = c^{3/4} = g(c)$ for every nonnegative real number $c \Rightarrow g$ is continuous on $[0,\infty)$

(c) $\lim_{x\to c} h(x) = \lim_{x\to c} x^{-2/3} = \frac{1}{c^{2/3}} = h(c)$ for every nonzero real number $c \Rightarrow h$ is continuous on $(-\infty,0)$ and $(0,\infty)$

(d) $\lim_{x\to c} k(x) = \lim_{x\to c} x^{-1/6} = \frac{1}{c^{1/6}} = k(c)$ for every positive real number $c \Rightarrow h$ is continuous on $(0,\infty)$

8. (a) $\bigcup_{n\in I}\left(\left(n-\frac{1}{2}\right)\pi, \left(n+\frac{1}{2}\right)\pi\right)$, where I = the set of all integers.

(b) $\bigcup_{n\in I}(n\pi, (n+1)\pi)$, where I = the set of all integers.

(c) $(-\infty,\infty)$

(d) $(-\infty,0)\cup(0,\infty)$

9. (a) $\lim_{x\to 0}\frac{x^2-4x+4}{x^3+5x^2-14x} = \lim_{x\to 0}\frac{(x-2)(x-2)}{x(x+7)(x-2)} = \lim_{x\to 0}\frac{x-2}{x(x+7)}$, $x \neq 2$; the limit does not exist because $\lim_{x\to 0^-}\frac{x-2}{x(x+7)} = \infty$ and $\lim_{x\to 0^+}\frac{x-2}{x(x+7)} = -\infty$

(b) $\lim_{x\to 2}\frac{x^2-4x+4}{x^3+5x^2-14x} = \lim_{x\to 2}\frac{(x-2)(x-2)}{x(x+7)(x-2)} = \lim_{x\to 2}\frac{x-2}{x(x+7)}$, $x \neq 2 = \frac{0}{2(9)} = 0$

10. (a) $\lim_{x\to 0}\frac{x^2+x}{x^5+2x^4+x^3} = \lim_{x\to 0}\frac{x(x+1)}{x^3(x^2+2x+1)} = \lim_{x\to 0}\frac{x+1}{x^2(x+1)(x+1)} = \lim_{x\to 0}\frac{1}{x^2(x+1)}$, $x \neq 0$ and $x \neq -1$.

Now $\lim_{x\to 0^-}\frac{1}{x^2(x+1)} = \infty$ and $\lim_{x\to 0^+}\frac{1}{x^2(x+1)} = \infty \Rightarrow \lim_{x\to 0}\frac{x^2+x}{x^5+2x^4+x^3} = \infty$.

(b) $\lim_{x\to -1}\frac{x^2+x}{x^5+2x^4+x^3} = \lim_{x\to -1}\frac{x(x+1)}{x^3(x^2+2x+1)} = \lim_{x\to -1}\frac{1}{x^2(x+1)}$, $x \neq 0$ and $x \neq -1$. The limit does not exist because $\lim_{x\to -1^-}\frac{1}{x^2(x+1)} = -\infty$ and $\lim_{x\to -1^+}\frac{1}{x^2(x+1)} = \infty$.

11. $\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1}\frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \lim_{x\to 1}\frac{1}{1+\sqrt{x}} = \frac{1}{2}$

12. $\lim_{x\to a}\frac{x^2-a^2}{x^4-a^4} = \lim_{x\to a}\frac{(x^2-a^2)}{(x^2+a^2)(x^2-a^2)} = \lim_{x\to a}\frac{1}{x^2+a^2} = \frac{1}{2a^2}$

13. $\lim\limits_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim\limits_{h\to 0} \frac{(x^2+2hx+h^2)-x^2}{h} = \lim\limits_{h\to 0} (2x+h) = 2x$

14. $\lim\limits_{x\to 0} \frac{(x+h)^2 - x^2}{h} = \lim\limits_{x\to 0} \frac{(x^2+2hx+h^2)-x^2}{h} = \lim\limits_{x\to 0} (2x+h) = h$

15. $\lim\limits_{x\to 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \lim\limits_{x\to 0} \frac{2-(2+x)}{2x(2+x)} = \lim\limits_{x\to 0} \frac{-1}{4+2x} = -\frac{1}{4}$

16. $\lim\limits_{x\to 0} \frac{(2+x)^3 - 8}{x} = \lim\limits_{x\to 0} \frac{(x^3+6x^2+12x+8)-8}{x} = \lim\limits_{x\to 0} (x^2+6x+12) = 12$

17. $\lim\limits_{x\to\infty} \frac{2x+3}{5x+7} = \lim\limits_{x\to\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2+0}{5+0} = \frac{2}{5}$

18. $\lim\limits_{x\to-\infty} \frac{2x^2+3}{5x^2+7} = \lim\limits_{x\to-\infty} \frac{2+\frac{3}{x^2}}{5+\frac{7}{x^2}} = \frac{2+0}{5+0} = \frac{2}{5}$

19. $\lim\limits_{x\to-\infty} \frac{x^2-4x+8}{3x^3} = \lim\limits_{x\to-\infty} \left(\frac{1}{3x} - \frac{4}{3x^2} + \frac{8}{3x^3}\right) = 0 - 0 + 0 = 0$

20. $\lim\limits_{x\to\infty} \frac{1}{x^2-7x+1} = \lim\limits_{x\to\infty} \left(\frac{\frac{1}{x^2}}{1-\frac{7}{x}+\frac{1}{x^2}}\right) = \frac{0}{1-0+0} = 0$

21. $\lim\limits_{x\to-\infty} \frac{x^2-7x}{x+1} = \lim\limits_{x\to-\infty} \left(\frac{x-7}{1+\frac{1}{x}}\right) = -\infty$

22. $\lim\limits_{x\to\infty} \frac{x^4+x^3}{12x^3+128} = \lim\limits_{x\to\infty} \left(\frac{x+1}{12+\frac{128}{x^3}}\right) = \infty$

23. $\lim\limits_{x\to\infty} \frac{|\sin x|}{\text{int } x} \le \lim\limits_{x\to\infty} \frac{1}{\text{int } x} = 0$ since int $x \to \infty$ as $x \to \infty$

24. $\lim\limits_{\theta\to\infty} \frac{|\cos\theta - 1|}{\theta} \le \lim\limits_{\theta\to\infty} \frac{|-2|}{\theta} = 0$

25. $\lim\limits_{x\to\infty} \frac{x+\sin x+2\sqrt{x}}{x+\sin x} = \lim\limits_{x\to\infty} \left(\frac{1+\frac{\sin x}{x}+\frac{2}{\sqrt{x}}}{1+\frac{\sin x}{x}}\right) = \frac{1+0+0}{1+0} = 1$

26. $\lim\limits_{x\to\infty} \frac{x^{2/3}+x^{-1}}{x^{2/3}+\cos^2 x} = \lim\limits_{x\to\infty} \left(\frac{1+x^{-5/3}}{1+\frac{\cos^2 x}{x^{2/3}}}\right) = \frac{1+0}{1+0} = 1$

27. $\lim\limits_{x\to\infty} e^{-x^2} = \lim\limits_{x\to\infty} \frac{1}{e^{x^2}} = 0$

28. Letting $u = \frac{1}{x}$ gives $\lim\limits_{x\to-\infty} e^{1/x} = \lim\limits_{u\to 0^-} e^u = 1$.

29. (a) $f(-1) = -1$ and $f(2) = 5 \Rightarrow f$ has a root between -1 and 2 by the Intermediate Value Theorem.
(b), (c) root is 1.32471795724

30. (a) $f(-2) = -2$ and $f(0) = 2 \Rightarrow$ f has a root between -2 and 0 by the Intermediate Value Theorem.
(b), (c) root is -1.76929235424

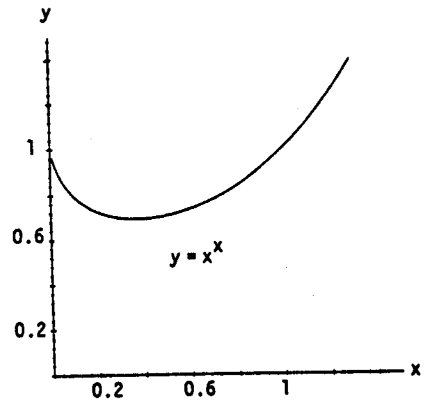
CHAPTER 1 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. (a)

x	0.1	0.01	0.001	0.0001	0.00001
x^x	0.7943	0.9550	0.9931	0.9991	0.9999

Apparently, $\lim_{x\to 0^+} x^x = 1$

(b)

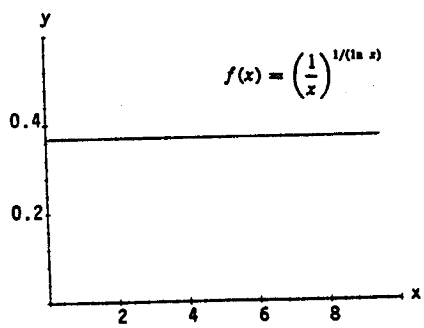


2. (a)

x	10	100	1000
$\left(\frac{1}{x}\right)^{1/(\ln x)}$	0.3678	0.3678	0.3678

Apparently, $\lim_{x\to\infty} \left(\frac{1}{x}\right)^{1/(\ln x)} = 0.3678 = \frac{1}{e}$

(b)



3. $\lim_{v\to c^-} L = \lim_{v\to c^-} L_0\sqrt{1-\frac{v^2}{c^2}} = L_0\sqrt{1-\frac{\lim_{v\to c^-} v^2}{c^2}} = L_0\sqrt{1-\frac{c^2}{c^2}} = 0$

The left-hand limit was needed because the function L is undefined if $v > c$ (the rocket cannot move faster than the speed of light).

4. $\left|\frac{\sqrt{x}}{2}-1\right| < 0.2 \Rightarrow -0.2 < \frac{\sqrt{x}}{2} - 1 < 0.2 \Rightarrow 0.8 < \frac{\sqrt{x}}{2} < 1.2 \Rightarrow 1.6 < \sqrt{x} < 2.4 \Rightarrow 2.56 < x < 5.76.$

$\left|\frac{\sqrt{x}}{2}-1\right| < 0.1 \Rightarrow -0.1 < \frac{\sqrt{x}}{2} - 1 < 0.1 \Rightarrow 0.9 < \frac{\sqrt{x}}{2} < 1.1 \Rightarrow 1.8 < \sqrt{x} < 2.2 \Rightarrow 3.24 < x < 4.84.$

5. $\left|10 + (t-70)\times 10^{-4} - 10\right| < 0.0005 \Rightarrow \left|(t-70)\times 10^{-4}\right| < 0.0005 \Rightarrow -0.0005 < (t-70)\times 10^{-4} < 0.0005$
$\Rightarrow -5 < t - 70 < 5 \Rightarrow 65^\circ < t < 75^\circ \Rightarrow$ Within 5° F.

6. Yes. Let R be the radius of the equator (earth) and suppose at a fixed instant of time we label noon as the zero point, 0, on the equator $\Rightarrow 0 + \pi R$ represents the midnight point (at the same exact time). Suppose x_1 is a point on the equator "just after" noon $\Rightarrow x_1 + \pi R$ is simultaneously "just after" midnight. It seems reasonable that the temperature T at a point just after noon is hotter than it would be at the diametrically opposite point just after midnight: That is, $T(x_1) - T(x_1 + \pi R) > 0$. At exactly the same moment in time pick x_2 to be a point just before midnight $\Rightarrow x_2 + \pi R$ is just before noon. Then $T(x_2) - T(x_2 + \pi R) < 0$. Assuming the temperature function T is continuous along the equator (which is reasonable), the Intermediate Value Theorem says there is a point c between 0 (noon) and πR (simultaneously midnight) such that $T(c) - T(c + \pi R) = 0$; i.e., there is always a pair of antipodal points on the earth's equator where the temperatures are the same.

7. (a) At x = 0: $\lim\limits_{a\to 0} r_+(a) = \lim\limits_{a\to 0} \frac{-1+\sqrt{1+a}}{a} = \lim\limits_{a\to 0} \left(\frac{-1+\sqrt{1+a}}{a}\right)\left(\frac{-1-\sqrt{1+a}}{-1-\sqrt{1+a}}\right)$

$= \lim\limits_{a\to 0} \frac{1-(1+a)}{a\left(-1-\sqrt{1+a}\right)} = \frac{-1}{-1-\sqrt{1+0}} = \frac{1}{2}$

At x = −1: $\lim\limits_{a\to -1^+} r_+(a) = \lim\limits_{a\to -1^+} \frac{1-(1+a)}{a\left(-1-\sqrt{1+a}\right)} = \lim\limits_{a\to -1} \frac{-a}{a\left(-1-\sqrt{1+a}\right)} = \frac{-1}{-1-\sqrt{0}} = 1$

(b) At x = 0: $\lim\limits_{a\to 0^-} r_-(a) = \lim\limits_{a\to 0^-} \frac{-1-\sqrt{1+a}}{a} = \lim\limits_{a\to 0^-} \left(\frac{-1-\sqrt{1+a}}{a}\right)\left(\frac{-1+\sqrt{1+a}}{-1+\sqrt{1+a}}\right)$

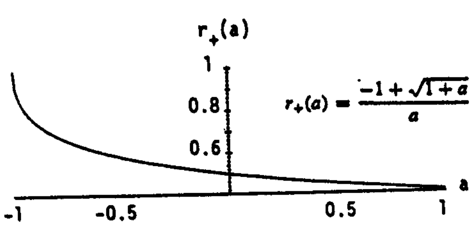
$= \lim\limits_{a\to 0^-} \frac{1-(1+a)}{a\left(-1+\sqrt{1+a}\right)} = \lim\limits_{a\to 0^-} \frac{a}{a\left(-1+\sqrt{1+a}\right)} = \lim\limits_{a\to 0^-} \frac{-1}{-1+\sqrt{1+a}} = \infty$ (because the

denominator is always negative); $\lim\limits_{a\to 0^+} r_-(a) = \lim\limits_{a\to 0^+} \frac{-1}{-1+\sqrt{1+a}} = -\infty$ (because the denominator

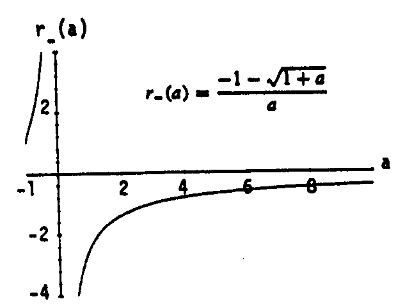
is always positive). Therefore, $\lim\limits_{a\to 0} r_-(a)$ does not exist.

At x = −1: $\lim\limits_{a\to -1^+} r_-(a) = \lim\limits_{a\to -1^+} \frac{-1-\sqrt{1+a}}{a} = \lim\limits_{a\to -1^+} \frac{-1}{-1+\sqrt{1+a}} = 1$

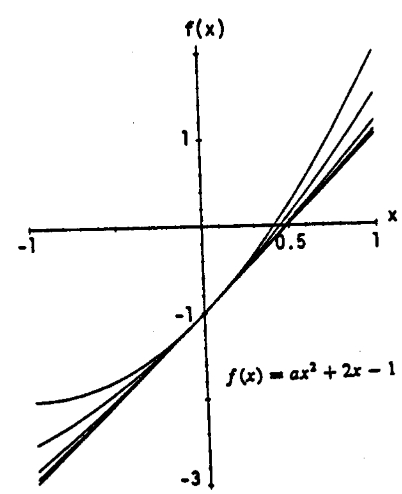
(c)

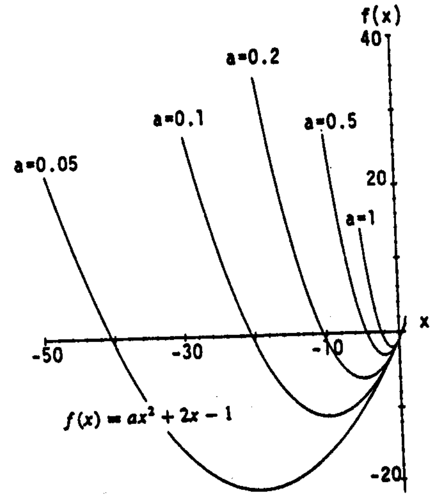


Graph not to scale



(d)





8. (a) Since $x \to 0^+$, $0 < x^3 < x < 1 \Rightarrow (x^3 - x) \to 0^- \Rightarrow \lim_{x\to 0^+} f(x^3 - x) = \lim_{y\to 0^-} f(y) = B$ where $y = x^3 - x$.

   (b) Since $x \to 0^-$, $-1 < x < x^3 < 0 \Rightarrow (x^3 - x) \to 0^+ \Rightarrow \lim_{x\to 0^-} f(x^3 - x) = \lim_{y\to 0^+} f(y) = A$ where $y = x^3 - x$.

   (c) Since $x \to 0^+$, $0 < x^4 < x^2 < 1 \Rightarrow (x^2 - x^4) \to 0^+ \Rightarrow \lim_{x\to 0^+} f(x^2 - x^4) = \lim_{y\to 0^+} f(y) = A$ where $y = x^2 - x^4$.

   (d) Since $x \to 0^-$, $-1 < x < 0 \Rightarrow 0 < x^4 < x^2 < 1 \Rightarrow (x^2 - x^4) \to 0^+ \Rightarrow \lim_{x\to 0^+} f(x^2 - x^4) = A$ as in part (c).

9. (a) True, because if $\lim_{x\to a} (f(x) + g(x))$ exists then $\lim_{x\to a} (f(x) + g(x)) - \lim_{x\to a} f(x) = \lim_{x\to a} [(f(x) + g(x)) - f(x)]$

   $= \lim_{x\to a} g(x)$ exists, contrary to assumption.

   (b) False; for example take $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$. Then neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists, but

   $\lim_{x\to 0} (f(x) + g(x)) = \lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x}\right) = \lim_{x\to 0} 0 = 0$ exists.

   (c) True, because $g(x) = |x|$ is continuous $\Rightarrow g(f(x)) = |f(x)|$ is continuous (it is the composite of continuous functions).

   (d) False; for example let $f(x) = \begin{cases} -1, & x \le 0 \\ 1, & x > 0 \end{cases} \Rightarrow f(x)$ is discontinuous at $x = 0$. However $|f(x)| = 1$ is continuous at $x = 0$.

10. $f(x) = x + 2\cos x \Rightarrow f(0) = 0 + 2\cos 0 = 2 > 0$ and $f(-\pi) = -\pi + 2\cos(-\pi) = -\pi - 2 < 0$. Since $f(x)$ is continuous on $[-\pi, 0]$, by the Intermediate Value Theorem, $f(x)$ must take on every value between $[-\pi - 2, 2]$. Thus there is some number c in $[-\pi, 0]$ such that $f(c) = 0$; i.e., c is a solution to $x + 2\cos x = 0$.

11. Show $\lim_{x\to 1} f(x) = \lim_{x\to 1} (x^2 - 7) = -6 = f(1)$.

    Step 1: $|(x^2 - 7) + 6| < \epsilon \Rightarrow -\epsilon < x^2 - 1 < \epsilon \Rightarrow 1 - \epsilon < x^2 < 1 + \epsilon \Rightarrow \sqrt{1-\epsilon} < x < \sqrt{1+\epsilon}$.

    Step 2: $|x - 1| < \delta \Rightarrow -\delta < x - 1 < \delta \Rightarrow -\delta + 1 < x < \delta + 1$.

    Then $-\delta + 1 = \sqrt{1-\epsilon}$ or $\delta + 1 = \sqrt{1+\epsilon}$. Choose $\delta = \min\left\{1 - \sqrt{1-\epsilon}, \sqrt{1+\epsilon} - 1\right\}$, then

$0 < |x-1| < \delta \Rightarrow |(x^2-7)-6| < \epsilon$ and $\lim_{x\to 1} f(x) = -6$. By the continuity test, f(x) is continuous at x = 1.

12. Show $\lim_{x\to \frac{1}{4}} g(x) = \lim_{x\to \frac{1}{4}} \frac{1}{2x} = 2 = g\left(\frac{1}{4}\right)$.

Step 1: $\left|\frac{1}{2x} - 2\right| < \epsilon \Rightarrow -\epsilon < \frac{1}{2x} - 2 < \epsilon \Rightarrow 2-\epsilon < \frac{1}{2x} < 2+\epsilon \Rightarrow \frac{1}{4-2\epsilon} > x > \frac{1}{4+2\epsilon}$.

Step 2: $\left|x - \frac{1}{4}\right| < \delta \Rightarrow -\delta < x - \frac{1}{4} < \delta \Rightarrow -\delta + \frac{1}{4} < x < \delta + \frac{1}{4}$.

Then $-\delta + \frac{1}{4} = \frac{1}{4+2\epsilon} \Rightarrow \delta = \frac{1}{4} - \frac{1}{4+2\epsilon} = \frac{\epsilon}{4(2+\epsilon)}$, or $\delta + \frac{1}{4} = \frac{1}{4-2\epsilon} \Rightarrow \delta = \frac{1}{4-2\epsilon} - \frac{1}{4} = \frac{\epsilon}{4(2-\epsilon)}$.

Choose $\delta = \frac{\epsilon}{4(2+\epsilon)}$, the smaller of the two values. Then $0 < \left|x - \frac{1}{4}\right| < \delta \Rightarrow \left|\frac{1}{2x} - 2\right| < \epsilon$ and $\lim_{x\to \frac{1}{4}} \frac{1}{2x} = 2$. By the continuity test, g(x) is continuous at $x = \frac{1}{4}$.

13. Show $\lim_{x\to 2} h(x) = \lim_{x\to 2} \sqrt{2x-3} = 1 = h(2)$.

Step 1: $\left|\sqrt{2x-3} - 1\right| < \epsilon \Rightarrow -\epsilon < \sqrt{2x-3} - 1 < \epsilon \Rightarrow 1-\epsilon < \sqrt{2x-3} < 1+\epsilon \Rightarrow \frac{(1-\epsilon)^2+3}{2} < x < \frac{(1+\epsilon)^2+3}{2}$.

Step 2: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta$ or $-\delta + 2 < x < \delta + 2$.

Then $-\delta + 2 = \frac{(1-\epsilon)^2+3}{2} \Rightarrow \delta = 2 - \frac{(1-\epsilon)^2+3}{2} = \frac{1-(1-\epsilon)^2}{2} = \epsilon - \frac{\epsilon^2}{2}$, or $\delta + 2 = \frac{(1+\epsilon)^2+3}{2}$ $\Rightarrow \delta = \frac{(1+\epsilon)^2+3}{2} - 2 = \frac{(1+\epsilon)^2-1}{2} = \epsilon + \frac{\epsilon^2}{2}$. Choose $\delta = \epsilon - \frac{\epsilon^2}{2}$, the smaller of the two values. Then, $0 < |x-2| < \delta \Rightarrow \left|\sqrt{2x-3} - 1\right| < \epsilon$, so $\lim_{x\to 2} \sqrt{2x-3} = 1$. By the continuity test, h(x) is continuous at x = 2.

14. Show $\lim_{x\to 5} F(x) = \lim_{x\to 5} \sqrt{9-x} = 2 = F(5)$.

Step 1: $\left|\sqrt{9-x} - 2\right| < \epsilon \Rightarrow -\epsilon < \sqrt{9-x} - 2 < \epsilon \Rightarrow 9-(2-\epsilon)^2 > x > 9-(2+\epsilon)^2$.

Step 2: $0 < |x-5| < \delta \Rightarrow -\delta < x-5 < \delta \Rightarrow -\delta + 5 < x < \delta + 5$.

Then $-\delta + 5 = 9-(2+\epsilon)^2 \Rightarrow \delta = (2+\epsilon)^2 - 4 = \epsilon^2 + 2\epsilon$, or $\delta + 5 = 9-(2-\epsilon)^2 \Rightarrow \delta = 4-(2-\epsilon)^2 = \epsilon^2 - 2\epsilon$. Choose $\delta = \epsilon^2 - 2\epsilon$, the smaller of the two values. Then, $0 < |x-5| < \delta \Rightarrow \left|\sqrt{9-x} - 2\right| < \epsilon$, so $\lim_{x\to 5} \sqrt{9-x} = 2$. By the continuity test, F(x) is continuous at x = 5.

15. (a) Let $\epsilon > 0$ be given. If x is rational, then $f(x) = x \Rightarrow |f(x) - 0| = |x-0| < \epsilon \Leftrightarrow |x-0| < \epsilon$; i.e., choose $\delta = \epsilon$. Then $|x-0| < \delta \Rightarrow |f(x) - 0| < \epsilon$ for x rational. If x is irrational, then $f(x) = 0 \Rightarrow |f(x) - 0| < \epsilon$ $\Leftrightarrow 0 < \epsilon$ which is true no matter how close irrational x is to 0, so again we can choose $\delta = \epsilon$. In either case, given $\epsilon > 0$ there is a $\delta = \epsilon > 0$ such that $0 < |x-0| < \delta \Rightarrow |f(x) - 0| < \epsilon$. Therefore, f is continuous at x = 0.

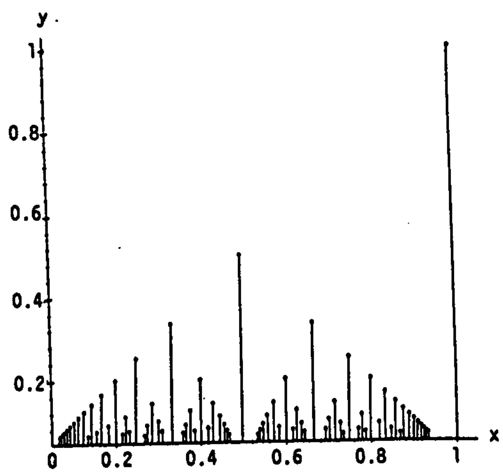
(b) Choose $x = c > 0$. Then within any interval $(c-\delta, c+\delta)$ there are both rational and irrational numbers. If c is rational, pick $\epsilon = \frac{c}{2}$. No matter how small we choose $\delta > 0$ there is an irrational number x in $(c-\delta, c+\delta) \Rightarrow |f(x) - f(c)| = |0-c| = c > \frac{c}{2} = \epsilon$. That is, f is not continuous at any rational $c > 0$. On the other hand, suppose c is irrational $\Rightarrow f(c) = 0$. Again pick $\epsilon = \frac{c}{2}$. No matter how small we choose $\delta > 0$ there is a rational number x in $(c - \delta, c + \delta)$ with $|x - c| < \frac{c}{2} = \epsilon \Leftrightarrow \frac{c}{2} < x < \frac{3c}{2}$. Then $|f(x) - f(c)| = |x - 0|$ $= |x| > \frac{c}{2} = \epsilon \Rightarrow$ f is not continuous at any irrational $c > 0$.

If $x = c < 0$, repeat the argument picking $\epsilon = \frac{|c|}{2} = \frac{-c}{2}$. Therefore f fails to be continuous at any nonzero value $x = c$.

16. (a) Let $c = \frac{m}{n}$ be a rational number in $[0,1]$ reduced to lowest terms $\Rightarrow f(c) = \frac{1}{n}$. Pick $\epsilon = \frac{1}{2n}$. No matter how small $\delta > 0$ is taken, there is an irrational number x in the interval $(c - \delta, c + \delta) \Rightarrow |f(x) - f(c)| = \left|0 - \frac{1}{n}\right|$ $= \frac{1}{n} > \frac{1}{2n} = \epsilon$. Therefore f is discontinuous at $x = c$, a rational number.

(b) Now suppose c is an irrational number $\Rightarrow f(c) = 0$. Let $\epsilon > 0$ be given. Notice that $\frac{1}{2}$ is the only rational number reduced to lowest terms with denominator 2 and belonging to $[0,1]$; $\frac{1}{3}$ and $\frac{2}{3}$ the only rational with denominator 3 belonging to $[0,1]$; $\frac{1}{4}$ and $\frac{3}{4}$ with denominator 4 in $[0,1]$; $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$ and $\frac{4}{5}$ with denominator 5 in $[0,1]$; etc. In general, choose N so that $\frac{1}{N} < \epsilon \Rightarrow$ there exist only finitely many rationals in $[0,1]$ having denominator $\le N$, say $r_1, r_2, \ldots, r_p$. Let $\delta = \min\{|c - r_i|: i = 1, \ldots, p\}$. Then the interval $(c - \delta, c + \delta)$ contains no rational numbers with denominator $\le N$. Thus, $0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| = |f(x) - 0|$ $= |f(x)| \le \frac{1}{N} < \epsilon \Rightarrow$ f is continuous at $x = c$ irrational.

(c) The graph looks like the markings on a typical ruler when the points $(x, f(x))$ on the graph of $f(x)$ are connected to the x-axis with vertical lines.



$f(x) = \begin{cases} 1/n & \text{if } x = m/n \text{ is a rational number in lowest terms} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

NOTES: