RISK MEASURES IN FINITE STATE SPACES

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1. Incentives from practice

Consider a fund, a bank, an insurance company or an investor (in general a financial unit) whose profit-loss account is consisted by total profit P and total loss L. Then the deficit that this unit has to anticipate is equal to

$$X = P - L.$$

The set of monetary flows that can anticipate this deficit, is the following:

$$\{m \in \mathbb{R} | m + X \in \mathbb{R}_+\}.$$

The minimum monetary flow that can anticipate this deficit is equal to

$$\inf\{m \in \mathbb{R} | m + X \in \mathbb{R}_+\} = -X.$$

If there is a finite set of states of the world $\Omega = \{1, 2, ..., S\}$, for any state $s \in \Omega$ there is a deficit X_s. The minimum monetary flow that can anticipate all the possible deficits $X_s, s = 1, 2, ..., S$ is equal to

$$\inf\{m \in \mathbb{R} | m\mathbf{1} + X \in \mathbb{R}^S_+\} = -\inf\{X_s | s = 1, 2, ..., S\}.$$

The function $\rho : \mathbb{R}^S \to \mathbb{R}$ with $\rho(X) = -\inf\{X_s | s = 1, 2, ..., S\}$ is a first example of a **coherent risk measure** and $(\mathbb{R}^S_+, \mathbf{1})$ is a first example of an **insurance frame**, where $\mathbb{R}^S_+$ is an **acceptance set** and $\mathbf{1}$ is a **numeraire**, or else a **insurance instrument**. Note that $\rho(X) = \sup\{-X_s | s = 1, 2, ..., S\} = \sup\{e_s \cdot (-X) | s = 1, 2, ..., S\}$, where $e_s, s = 1, 2, ..., S$ are the vectors of the usual algebraic basis of $\mathbb{R}^S$. But these vectors are also the extreme points of the simplex Δ_{S-1}, which is the set of all the probability vectors over Ω. Hence, the expression

$$\rho(X) = \sup\{e_s \cdot (-X) | s = 1, 2, ..., S\},$$

is a first formulation of the famous so-called **duality representation theorem**.

Proposition 1.1. $\rho(X) = \sup\{e_s \cdot (-X) | s = 1, 2, ..., S\} = \sup\{\pi \cdot (-X) | \pi \in \Delta_{S-1}\}$.

Proof $\Delta_{S-1} = \overline{co}\{e_1, e_2, ..., e_S\}$. Hence $\rho(X) \leq \sup\{\pi \cdot (-X) | \pi \in \Delta_{S-1}\}$ by well-known properties of suprema. Conversely, for any $X \in \mathbb{R}^S$ and any $\pi \in \Delta_{S-1}$, $\pi \cdot (-X) \leq \sum_{s=1}^{S} \pi_s \sup\{e_s \cdot (-X) | s = 1, 2, ..., S\} = \rho(X)$. By taking supremum over π we get that $\sup\{\pi(-X) | \pi \in \Delta_{S-1}\} \leq \rho(X)$ and finally $\rho(X) = \sup\{\pi(-X) | \pi \in \Delta_{S-1}\}$. □

The capital $\rho(X)$ is also called **solvency capital**, because it comes from the simple solvency notion that we indicated above. Moreover, coherent risk measures and the Principle of the Minimum Capital is not irrelevant to Solvency II which is the continuation of Basel II in the actuarial frame.

Note: Great part of the following Sections relies on [14]

2. Financial positions, acceptance sets and coherent risk measures

Suppose that we live in an economic environment where there is a finite number of states $\Omega = \{1,\ 2,\ \ldots,\ S\}$ which may occur tomorrow (or after a certain period of time T). A *financial position* X is a random variable $X : \Omega \to \mathbb{R}$ which denotes either the terminal value of a portfolio or an investment at time-period T (if we put today to be the time-period 0) or the claim due to the occurence of one of the states in Ω. An *acceptance set* $\mathcal{A}$ is a set of financial positions, claimed by the regulator as the 'safe' ones. However, *financial risk* according to the frame posed in [5] is the variability of the terminal value of some portfolio due to the presence of uncertainty. The essential properties that an acceptance set must satisfy, posed as axioms in [5] are the following:

(*i*) $\mathcal{A} + \mathcal{A} \subseteq \mathcal{A}$
(*ii*) $\lambda\mathcal{A} \subseteq \mathcal{A}$ for any $\lambda \in \mathbb{R}_+$
(*iii*) $\mathbb{R}^S_+ \subseteq \mathcal{A}$

Note that by (i), (ii) $\mathcal{A}$ is a wedge. If $\mathcal{A}$ is a cone, another property which concludes coherence of $\mathcal{A}$ is

$$\mathcal{A} \cap (-\mathbb{R}^S_+) = \{0\}.$$

The three properties mentioned are directly related to the properties of Sub-additivity, Positive Homogeneity and Monotonicity of a coherent risk measure, as it is clear from the next definition of it. Also, a *numeraire asset* $e \in \mathbb{R}^S_+$ is used for *insuring* the financial positions with respect to some acceptance set $\mathcal{A}$. Insuring the financial position $x \in \mathbb{R}^S$ with respect to the pair $(\mathcal{A}, e)$ is the determination of $\lambda \in \mathbb{R}$ shares of e such that

$$x + \lambda e \in \mathcal{A}.$$

Here we suppose that the numeraire asset is the *riskless investment* $\mathbf{1} = (1, 1, ..., 1) \in \mathbb{R}^S$. A monetary $(\mathcal{A}, e)$-risk measure is a function $\rho : \mathbb{R}^S \to \mathbb{R}$ which indicates the minimum amount of shares $\rho(X)$ of e needed so that $X \in \mathbb{R}^S$ to become acceptable (insurable). The acceptance set of ρ is the set of positions

$$\mathcal{A}_\rho = \{X \in \mathbb{R}^S | \rho(X) \leq 0\},$$

being actually a *coherent acceptance set*, or else an acceptance set which satisfies the properties $(i) - (iii)$ of the primary axioms in [5] whenever ρ is a coherent risk measure, according to the definition mentioned below. The *risk measure measure associated with the pair* $(\mathcal{A}, e)$ is defined as follows:

$$\rho_{\mathcal{A},e}(x) = \inf\{m \in \mathbb{R} | x + m \cdot e \in \mathcal{A}\},$$

for any $x \in \mathbb{R}^S$. As it is well known (see [5]), a risk measure $\rho : \mathbb{R}^S \to \mathbb{R}$ which satisfies the properties $(X, Y \in \mathbb{R}^S)$

(*i*) $\rho(X + a\mathbf{1}) = \rho(X) - a$ (Translation Invariance)
(*ii*) $\rho(X + Y) \leq \rho(X) + \rho(Y)$ (Sub-additivity)
(*iii*) $\rho(\lambda X) = \lambda \rho(X), \lambda \in \mathbb{R}_+$ (Positive Homogeneity) and
(*iv*) $X \leq Y$ in terms of the usual partial ordering of $\mathbb{R}^S$, implies $\rho(Y) \leq \rho(X)$ (Monotonicity)

is called *coherent*.

In order to conclude on the relation between acceptance sets and risk measures we may prove a first proposition

Proposition 2.1. *If $\mathcal{A}$ is a closed cone of $\mathbb{R}^S$ containing $\mathbb{R}^S_+$, then the risk measure $\rho_{\mathcal{A},\mathbf{1}}$ is a coherent risk measure which doesn't take infinite values.*

Proof We have to verify that the properties of a coherent risk measure hold for $\rho_{\mathcal{A},\mathbf{1}}$. Note that according to the [11, Pr. 3.1.3], $\mathbf{1}$ is an order-unit of $\mathbb{R}^S$. Also, for any $X \in \mathbb{R}^S$ there is some $m \in \mathbb{R}$ such that $X + m\mathbf{1} \in \mathcal{A}$. That is because

$$\mathbb{R}^S = \cup_{n=1}^{\infty}[-n\mathbf{1}\,,\ n\mathbf{1}]_{\mathcal{A}}\,,$$

where $[\,.\,,.\,]_{\mathcal{A}}$ denotes the corresponding order-interval with respect to the partial ordering induced on $\mathbb{R}^S$ by $\mathcal{A}$. Hence for some $n(X) \in \mathbb{N}$,

$$X \in [-n(X)\mathbf{1}\,,\ n(X)\mathbf{1}]_{\mathcal{A}}\,,$$

which implies $X + n(X)\mathbf{1} \in \mathcal{A}$ and the set $\{m \in \mathbb{R}\,|\,X + m\mathbf{1} \in \mathcal{A}\}$ is non-empty for any $X \in \mathbb{R}^S$. This also indicates that $\rho_{\mathcal{A},\mathbf{1}}(X) \neq \infty$ for any $X \in \mathbb{R}^S$.

In order to show that $\rho_{\mathcal{A},\mathbf{1}}(X) \neq -\infty$ for every $X \in \mathbb{R}^S$, we have to show that for any $X \in \mathbb{R}^S$, the set of real numbers $\{m \in \mathbb{R}\,|\,m\mathbf{1} + X \in \mathcal{A}\}$ is lower bounded. Since $\mathbf{1} \in int\mathbb{R}^S_+$, $\mathbf{1} \in int\mathcal{A}$, which implies that $\mathbf{1}$ is an order-unit of $\mathbb{R}^S$ being partially ordered by the wedge $\mathcal{A}$. As we mentioned before, this implies that for any $X \in \mathbb{R}^S$ there is some $n(X) \in \mathbb{N}$ such that

$$X \in [-n(X)\mathbf{1}\,,\ n(X)\mathbf{1}]\,,$$

where $[\,.\,,.\,]$ denotes an order-interval with respect to the partial ordering induced by $\mathcal{A}$ on $\mathbb{R}^S$. But on the other hand for any $X \in \mathbb{R}^S$ there is some nonzero $m_0(X) \in \mathbb{R}_+$ such that

$$X - m_0(X)\mathbf{1} \notin \mathcal{A}\,.$$

If $X = 0$ this m_0 is any $\varepsilon > 0$. If $X \neq 0$ we suppose that such a positive real number does not exist, so we would have for any $\lambda > 0$, that

$$X - \lambda\mathbf{1} \in \mathcal{A}\,.$$

Since $\mathbf{1} \in int\mathcal{A}$ we get the equality $\mathbb{R}^S = \cup_{n=1}^{\infty}[-n\mathbf{1}\,,\ n\mathbf{1}]_{\mathcal{A}}$. Suppose that $n(X) \in \mathbb{N}$ is some natural number such that

$$X \in [-n(X)\mathbf{1}\,,\ n(X)\mathbf{1}]\,.$$

Since $X \geq n(X)e$ by assumption, we get that $n(X)\mathbf{1} \leq X \leq n(X)\mathbf{1}$ in terms of the partial ordering induced by $\mathcal{A}$ on $\mathbb{R}^S$ and because $\mathcal{A}$ is a cone, it implies

$$X = n(X)\mathbf{1}\,.$$

But $X \in [-(n(X)+1)\mathbf{1}\,,\ (n(X)+1)\mathbf{1}]$ and by the same assumption $X \geq (n(X)+1)\mathbf{1}$ which implies by the same way that

$$X = (n(X)+1)\mathbf{1}\,,$$

which is a contradiction because if this is the case, we find $X = 0$. Hence, for any $X \neq 0$ there exists some nonzero $m_0(X) \in \mathbb{R}_+$ such that

$$X - m_0(X)\mathbf{1} \notin \mathcal{A}\,.$$

After all, we remark that $-m_0(X)$ is a lower bound for the set $\{m \in \mathbb{R}\,|\,m\mathbf{1} + X \in \mathcal{A}\}$. If we suppose that there is some $k < -m_0(X)$ such that

$$k\mathbf{1} + X \in \mathcal{A}\,,$$

then we come to a contradiction since then we would have

$$X - m_0(X)\mathbf{1} = X + k\mathbf{1} + (-m_0(X) - k)\mathbf{1} \in \mathcal{A}$$

from the well-known properties of a wedge, since $X + k\mathbf{1} \in \mathcal{A}$ by assumption and $(-m_0(X) - k)\mathbf{1} \in \mathcal{A}$ since $\mathbf{1} \in \mathcal{A}$ and $(-m_0(X) - k) > 0$. Finally, we found that $\rho_{\mathcal{A},\mathbf{1}}$ cannot take the value $-\infty$ if $\mathcal{A}$ is a cone.

About the properties of a coherent risk measure, we have the following:

(*i*) (Translation Invariance):

$$\begin{aligned}\rho_{\mathcal{A},\mathbf{1}}(X + ae) &= \inf\{m \in \mathbb{R} \,|\, (X + a\mathbf{1}) + m\mathbf{1} \in \mathcal{A}\} \\ &= \inf\{(m + a) - a \,|\, X + (m + a)\mathbf{1} \in \mathcal{A}\} \\ &= \inf\{k \in \mathbb{R} \,|\, X + k\mathbf{1} \in \mathcal{A}\} - a = \rho_{\mathcal{A},\mathbf{1}}(X) - a\,,\end{aligned}$$

for any $X \in \mathbb{R}^S$ and any $a \in \mathbb{R}$.

(*ii*) (Sub-additivity):

If $m_1 \in \{m \in \mathbb{R} \,|\, X + m\mathbf{1} \in \mathcal{A}\}$ and $m_2 \in \{m \in \mathbb{R} \,|\, Y + m\mathbf{1} \in \mathcal{A}\}$ then

$$m_1 + m_2 \in \{k \in \mathbb{R} \,|\, (X + Y) + k\mathbf{1} \in \mathcal{A}\}\,.$$

It means

$$\rho_{\mathcal{A},\mathbf{1}}(X + Y) \leq m_1 + m_2\,.$$

Hence $\rho_{\mathcal{A},\mathbf{1}}(X + Y) - m_1 \leq m_2$ which implies

$$\rho_{\mathcal{A},\mathbf{1}}(X + Y) - m_1 \leq \rho_{\mathcal{A},\mathbf{1}}(Y)$$

for any such m_1. In the same way, by $\rho_{\mathcal{A},\mathbf{1}}(X + Y) - \rho_{\mathcal{A},\mathbf{1}}(Y) \leq m_1$ we obtain

$$\rho_{\mathcal{A},\mathbf{1}}(X + Y) - \rho_{\mathcal{A},\mathbf{1}}(Y) \leq \rho_{\mathcal{A},\mathbf{1}}(X)$$

and the required property holds for any $X, Y \in \mathbb{R}^S$.

The above proof of sub-additivity holds in case where $X, Y \in \mathbb{R}^S$ are such that $\rho_{\mathcal{A},\mathbf{1}}(X + Y),\ \rho_{\mathcal{A},\mathbf{1}}(X),\ \rho_{\mathcal{A},\mathbf{1}}(Y) \in \mathbb{R}$. If $\rho_{\mathcal{A},\mathbf{1}}(X + Y) = -\infty$ and at least one of $\rho_{\mathcal{A},\mathbf{1}}(X), \rho_{\mathcal{A},\mathbf{1}}(Y)$ is equal to $-\infty$ the sub-additivity holds. Note that if $\rho_{\mathcal{A},\mathbf{1}}(X+Y) = -\infty$ and $\rho_{\mathcal{A},\mathbf{1}}(X), \rho_{\mathcal{A},\mathbf{1}}(Y) \in \mathbb{R}$ the sub-additivity property is also true. Finally we note that if for example $\rho_{\mathcal{A},\mathbf{1}}(X) = -\infty$ and $\rho_{\mathcal{A},\mathbf{1}}(Y) \in \mathbb{R}$, then $\rho_{\mathcal{A},\mathbf{1}}(X + Y) = -\infty$. This is true because since $\rho_{\mathcal{A},\mathbf{1}}(X) = -\infty$ there is a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers, such that $a_n \to -\infty$ and $X + a_n\mathbf{1} \in \mathcal{A}$. Also, since $\rho_{\mathcal{A},\mathbf{1}}(Y) \in \mathbb{R}$ consider some $a \in \mathbb{R}$ with $Y + a\mathbf{1} \in \mathcal{A}$. Then since for the sequence $(d_n)_{n\in\mathbb{N}}$ with $d_n = a_n + a$ for any $n \in \mathbb{N}$ $(X+Y)+(a_n+a)\mathbf{1} \in \mathcal{A}$ from the property $\mathcal{A}+\mathcal{A} \subseteq \mathcal{A}$ of $\mathcal{A}$, while $lim_n(a_n+a) = -\infty$, we get $\rho_{\mathcal{A},\mathbf{1}}(X + Y) = -\infty$. In case where $\mathcal{A}$ is a cone, this point may be omitted since $\rho_{\mathcal{A},\mathbf{1}}$ takes only finite values.

(*iii*) (Positive Homogeneity):

For $\lambda = 0$ we have that $\rho_{\mathcal{A},\mathbf{1}}(0) \leq 0$, since $0 \in \{m \in \mathbb{R} \,|\, 0 + m\mathbf{1} \in \mathcal{A}\}$. If we suppose that $\rho_{\mathcal{A},\mathbf{1}}(0) < 0$, then for $\delta = -\rho_{\mathcal{A},\mathbf{1}}(0)$ and by the definition of $\rho_{\mathcal{A},\mathbf{1}}(0)$ as the infimum of a subset of the real numbers, there is some $m_\delta \in \{m \in \mathbb{R} \,|\, 0+m\mathbf{1} \in \mathcal{A}\}$ such that

$$m_\delta < \rho_{\mathcal{A},\mathbf{1}}(0) + \delta = 0\,.$$

Hence $m_\delta \mathbf{1} \in \mathcal{A}$ with $m_\delta < 0$. But from the properties of a wedge, we also have that $m_\delta \mathbf{1} \in -\mathcal{A}$. Then,

$$\mathbf{1} \in \mathcal{A} \cap (-\mathcal{A})$$

because $\mathcal{A}$ is a wedge, a contradiction since $\mathbf{1}$ is an interior point of $\mathcal{A}$. The contradiction is implied by the assumption that $\rho_{\mathcal{A},\mathbf{1}}(0) < 0$. This implies

$$\rho_{\mathcal{A},\mathbf{1}}(0) = 0\,,$$

and the positive homogeneity if $\lambda = 0$ holds.

If $\lambda > 0$ we remark that

$$\{\lambda m \in \mathbb{R} \,|\, m \in \mathbb{R} \text{ is such that } X + m\mathbf{1} \in \mathcal{A}\} \subseteq \{k \in \mathbb{R} \,|\, \lambda X + k\mathbf{1} \in \mathcal{A}\}\,,$$

which implies the inequality

$$\rho_{\mathcal{A},\mathbf{1}}(\lambda X) \leq \lambda \rho_{\mathcal{A},\mathbf{1}}(X)\,. \tag{2.1}$$

We can also see that

$$\left\{\frac{k}{\lambda} \in \mathbb{R} \,|\, k \in \mathbb{R} \text{ is such that } \lambda X + k\mathbf{1} \in \mathcal{A}\right\} \subseteq \{m \in \mathbb{R} \,|\, X + m\mathbf{1} \in \mathcal{A}\}\,.$$

This last remark implies

$$\rho_{\mathcal{A},\mathbf{1}}(X) \leq \frac{\rho_{\mathcal{A},\mathbf{1}}(\lambda X)}{\lambda}\,,$$

and from the above inequality, together with the relation (2.1), the required property is established for any $X \in \mathbb{R}^S$ and any $\lambda \in \mathbb{R}_+$.

(iv) (Monotonicity):

If $Y \geq X$ with respect to the partial ordering induced on $\mathbb{R}^S$ by $\mathcal{A}$ we notice that

$$\{m \in \mathbb{R} \,|\, X + m\mathbf{1} \in \mathcal{A}\} \subseteq \{m \in \mathbb{R} \,|\, Y + m\mathbf{1} \in \mathcal{A}\}\,. \tag{2.2}$$

Indeed, if $m_1 \in \{m \in \mathbb{R} \,|\, X + m\mathbf{1} \in \mathcal{A}\}$, then

$$X + m_1\mathbf{1} \in \mathcal{A}\,, \tag{2.3}$$

and since $Y \geq X$,

$$Y - X \in \mathcal{A}\,. \tag{2.4}$$

By the relations (2.3) and (2.4) we find that

$$Y + m_1\mathbf{1} = (Y - X) + (X + m_1\mathbf{1}) \in \mathcal{A}$$

taking in account the properties of a cone. Hence $m_1 \in \{m \in \mathbb{R} \,|\, Y + m\mathbf{1} \in \mathcal{A}\}$ and the (2.2) is true. Therefore

$$\rho_{\mathcal{A},\mathbf{1}}(Y) \leq \rho_{\mathcal{A},\mathbf{1}}(X)\,,$$

and the required property holds for any $X, Y \in \mathbb{R}^S$.

□

For the completeness of the above proof, the following lemma is needed

Lemma 2.2. *If A is a wedge of the normed linear space L which is not a subspace of it, then if $X \in intA$ (where $intA$ denotes the norm-interior of A), then $X \notin A \cap (-A)$.*

Proof of Lemma 2.2 Let us suppose that there is a $X \in intA$ with $X \in A \cap (-A)$. Then there exists some $\delta > 0$ such that $X + B(0,\delta) \subseteq A$, where $B(0,\delta)$ denotes the open ball centered at 0 with radius equal to δ. Since A is not a subspace, then there exists some $X_0 \in (-A) \setminus A = (-A) \cap A^c$ where A^c is the set-theoretic complement of A. (If $(-A) \cap A^c = \oslash$ then $A = (-A)$ which means that A is a subspace of L, a contradiction). Since $B(0,\delta)$ is an absorbing set, there is some $\lambda_\delta > 0$ such that $\lambda_\delta X_0 \in B(0,\delta)$. Then $X + \lambda_\delta X_0 \in (-A) \setminus A$. This is true, because $(-A)$ is a wedge of L and since $X_0 \in -A$ then $\lambda_\delta X_0 \in -A$, while $X \in -A$, too. This implies $X + \lambda_\delta X_0 \in -A$. On the other hand, since we supposed that $X \in A \cap (-A)$, we have that $-X \in A$. If $X + \lambda_\delta X_0 \in A$, then we would have that $\lambda_\delta X_0 \in A + (-X) \subseteq A + A \subseteq A$. By the properties of a wedge we would also have that $X_0 = \frac{1}{\lambda_\delta}\lambda_\delta X_0 \in \frac{1}{\lambda_\delta}A \subseteq A$. But $X + \lambda_\delta X_0 \in X + B(0,\delta) \subseteq A$ from the fact that $X \in intA$, which implies a contradiction. The contradiction came as a consequence of the assumption that such a point X exists. □

Hence $\rho_{\mathcal{A},\mathbf{1}}(X)$ for any $X \in \mathbb{R}^S$ is actually the minimum amount of money which makes X acceptable.

The second important proposition is the following.

Proposition 2.3. *If ρ is coherent, then $\rho = \rho_{\mathcal{A}_\rho}$.*

Proof. For any $X \in \mathbb{R}^S$,

$$\begin{aligned} \rho_{\mathcal{A}_\rho}(X) &= \inf\{\alpha \in \mathbb{R} | X + \alpha\mathbf{1} \in \mathcal{A}_\rho\} \\ &= \inf\{\alpha \in \mathbb{R} | \rho(X + \alpha\mathbf{1}) \leq 0\} = \\ &= \inf\{\alpha \in \mathbb{R} | \rho(X) - \alpha \leq 0\} = \rho(X). \end{aligned}$$

□

Hence the interpretation of $\rho(X)$ as the minimum amount of money needed so that X is acceptable, holds for every coherent risk measure and for its own acceptance set $\mathcal{A}_\rho$. Namely, the meaning of the properties of a coherent risk measure from the aspect of insurance is the following:

- (i) Translation Invariance: If we incorporate an amount of money α in a financial position X, then the reduction to the amount of money needed so that the investment consisted by the position X and the amount α to be acceptable with respect to ρ is equal to α.
- (ii) Sub-additivity: Diversification of the investments reduces the risk (the total premium).
- (iii) Positive Homogeneity : The risk premium with respect to ρ for a position consisted by a long position on some shares $\lambda \geq 0$ on this position X is the premium of X with respect to ρ multiplied by λ.
- (iv) Monotonicity: Greater payoff means less need for insurance premium.

Proposition 2.4. *If $\mathcal{A}$ is a closed wedge of $\mathbb{R}^S$, then $\mathcal{A}_{\rho_{(\mathcal{A},1)}} = \mathcal{A}$.*

Proof. Obviously $\mathcal{A} \subseteq \mathcal{A}_{\rho_{(\mathcal{A},\mathbf{1})}}$ because $0 \in \{m \in \mathbb{R} | m\mathbf{1} + U \in \mathcal{A}\}$ if $U \in \mathcal{A}$, hence

$$\rho_{(\mathcal{A},E)}(U) \leq 0\,,$$

and $U \in \mathcal{A}_{\rho_{(\mathcal{A},\mathbf{1})}}$. Suppose there is some $Z_0 \in \mathcal{A}_{\rho_{(\mathcal{A},\mathbf{1})}} \setminus \mathcal{A}$. Then $\rho_{(\mathcal{A},\mathbf{1})}(Z_0) \leq 0$. If $\rho_{(\mathcal{A},\mathbf{1})}(Z_0) = 0$, then for any $\epsilon > 0$, there is some $m_\epsilon > 0$ with $m_\epsilon < 0 + \epsilon = \epsilon$ such that $m_\epsilon\mathbf{1} + Z_0 \in \mathcal{A}$. If we put $\epsilon_n = 1/n$ for any $n \in \mathbb{N}$ we take a sequence $(m_n)_{n\in\mathbb{N}}$ of real numbers, such that

$m_n E + Z_0 \in \mathcal{A}$ and $0 < m_n < 1/n$. The limit of the last sequence of elements of $\mathcal{A}$ is Z_0 and since $\mathcal{A}$ is closed, $Z_0 \in \mathcal{A}$. But this is a contradiction, since we supposed that Z_0 is not an element of $\mathcal{A}$. Then we conclude that such a Z_0 does not exist in case where $\rho_{(\mathcal{A},\mathbf{1})}(Z_0) = 0$. If $\rho_{(\mathcal{A},\mathbf{1})}(Z_0) < 0$, then for $\epsilon_0 = -\rho_{(\mathcal{A},\mathbf{1})}(Z_0)$ we have that there is some

$$m_{\epsilon_0} < \rho_{(\mathcal{A},\mathbf{1})}(Z_0) + \epsilon_0 = 0\,,$$

such that $m_{\epsilon_0}\mathbf{1} + Z_0 \in \mathcal{A}$. Hence $Z_0 \in \mathcal{A} - m_{\epsilon_0}\mathbf{1} \subseteq \mathcal{A} + \mathcal{A} \subseteq \mathcal{A}$. But this is a contradiction, since we supposed that Z_0 is not an element of $\mathcal{A}$. Then we conclude that such a Z_0 also does not exist in case where $\rho_{(\mathcal{A},\mathbf{1})}(Z_0) < 0$. Hence $\mathcal{A}_{\rho_{(\mathcal{A},\mathbf{1})}} = \mathcal{A}$ is true. □

3. Properties of the coherent risk measures

For a coherent risk measure ρ, we name E^*_ρ the functional defined on $\mathbb{R}^S$ such that

$$E^*_\rho(X) = \rho(-X) \tag{3.1}$$

for every $X \in \mathbb{R}^S$ (we use the notation met in the proof of [5, Th. 4.1]). This functional enjoys the following properties ($X, Y \in \mathbb{R}^S$)

- (*i*) $E^*_\rho(aX + b\mathbf{1}) = aE^*_\rho(X) + b$, $a \in \mathbb{R}_+, b \in \mathbb{R}$ (Positive Affine Homogeneity)
- (*ii*) $E^*_\rho(X + Y) \leq E^*_\rho(X) + E^*_\rho(Y)$ (Sub-additivity)
- (*iii*) $X \leq Y$ in terms of the usual partial ordering of $\mathbb{R}^S$ implies $E^*_\rho(X) \leq E^*_\rho(Y)$ (Monotonicity)

The list of these properties may be found in [9], where such functionals are characterized as *upper expectations* (see [9, Pr. 2.1]), which are also used in the proof of [5, Pr. 4.1] for the characterization of a coherent risk measure as the 'worst expectation' on some non-empty set of probability vectors on the state space $\{1,\ 2,\ \ldots,\ S\}$. In [5, Pr. 2.2] is indicated that every coherent risk measure ρ is a continuous function, hence the acceptance set

$$\mathcal{A}_\rho = \{X \in \mathbb{R}^S \,|\, \rho(X) \leq 0\}\,, \tag{3.2}$$

associated with the risk measure ρ, is closed. In [5, Pr. 2.2] there is an assertion that this is a consequence of the convexity of ρ.

3.1. **The continuity of coherent risk measures in Euclidean spaces.**

In the following Proposition we present a proof for the continuity of ρ deduced from its properties as coherent measure. For this purpose we are going to use the properties of the associated functional E^*_ρ.

Proposition 3.1. *If ρ is a coherent risk measure, then the function $\rho : \mathbb{R}^S \to \mathbb{R}$ is Lipschitz.*

Proof. We are going to prove that E^*_ρ, defined by (3.1), is a Lipschitz function. First, note that

$$E^*_\rho(X) \leq E^*_\rho(Y) + E^*_\rho(X - Y)$$

for any $X,\ Y \in \mathbb{R}^S$, from the sub-additivity of E^*_ρ. We can write

$$E^*_\rho(Y) \leq E^*_\rho(X) + E^*_\rho(Y - X)$$

for the same reason. We are going to show that

$$E^*_\rho(X - Y) \leq A\|X - Y\|_1$$

and

$$E_\rho^*(Y - X) \le A\|Y - X\|_1$$

for some real number $A > 0$, where $\|.\|_1$ denotes the usual ℓ_1 norm for the vectors of $\mathbb{R}^S$. It suffices to show that for any $X \in \mathbb{R}^S$,

$$E_\rho^*(X) \le A\|X\|_1$$

holds for some $A > 0$. Consider the usual basis $\{e_1, e_2, \ldots, e_S\}$ of $\mathbb{R}^S$ which corresponds to the so-called set of *Arrow securities*.

Let us suppose that $E_\rho^*(e_s) = 0$ for any state $s \in \{1, 2, \ldots, S\}$, then

$$E_\rho^*(\mathbf{1}) = E_\rho^*(\sum_{s=1}^{S} e_s) \le \sum_{s=1}^{S} E_\rho^*(e_s) = 0 \tag{3.3}$$

by sub-additivity of E_ρ^*. But $\mathbf{1} \ge 0$ and from the monotonicity of E_ρ^* we get

$$E_\rho^*(\mathbf{1}) \ge 0\,. \tag{3.4}$$

Hence by (3.3) and (3.4) we obtain $E_\rho^*(\mathbf{1}) = 0$. But from positive affine homogeneity of E_ρ^* we get

$$E_\rho^*(\mathbf{1}) = E_\rho^*(0 + 1 \cdot \mathbf{1}) = 0E_\rho^*(0) + 1 = 1$$

thus we arrive at a contradiction. Therefore, a state s_0 such that $E_\rho^*(e_{s_0}) > 0$ exists. Note that from monotonicity we get $E_\rho^*(e_s) \ge 0$ for every state $s = 1, 2, \ldots, S$. We put

$$A := \max\{E_\rho^*(e_s) \mid s = 1, 2, \ldots, S\} > 0\,. \tag{3.5}$$

Then

$$E_\rho^*(X) = E_\rho^*\left(\sum_{s=1}^{S} X_s e_s\right) \le E_\rho^*(|X|) = E_\rho^*\left(\sum_{s=1}^{S} |X_s| e_s\right)$$

from monotonicity and

$$E_\rho^*(\sum_{s=1}^{S} |X_s| e_s) \le \sum_{s=1}^{S} |X_s| E_\rho^*(e_s)$$

from the sub-additivity of E_ρ^*. But then we get

$$E_\rho^*(X) \le A\|X\|_1$$

with A is the constant defined in (3.5), hence

$$\begin{aligned} E_\rho^*(X) &\le E_\rho^*(Y) + A\|X - Y\|_1 \\ E_\rho^*(Y) &\le E_\rho^*(X) + A\|X - Y\|_1 \end{aligned}$$

so

$$|E_\rho^*(X) - E_\rho^*(Y)| \le A\|X - Y\|_1 \tag{3.6}$$

for any $X, Y \in \mathbb{R}^S$ therefore E_ρ^* is Lipschitz. Now put $-X$ instead of X and $-Y$ instead of Y in the relation (3.6) and by (3.1) the conclusion holds. □

As a consequence we get the following corollary.

Corollary 3.2. *The acceptance set of any coherent risk measure $\rho : \mathbb{R}^S \to \mathbb{R}$ is closed.*

Proof. From (3.6) we see that the acceptance set is closed as the inverse map of a closed set of the real numbers through the continuous function ρ.

□

The closedness of the acceptance set of any coherent risk measure in a Euclidean space of risks is directly related to the fact that the space is finite-dimensional, as it is indicated in the proof of Proposition 3.1.

3.2. **The representability of coherent risk measures in Euclidean spaces.** The results of Proposition 3.1 and Corollary 3.2 are mentioned in [7, p. 5, (6)] as essential properties of a coherent risk measure when $L = L^\infty$.

We are going to give a complete proof of the [5, Th. 4.1] on the characterization of a coherent risk measure through a set of probability measures on the (finite) state space. Actually, we shall follow the lines of the relevant proof about L^∞-spaces in [7, Th. 2.3]. This Theorem, which is shown for *sub(super)modular functions* defined on L^∞, enlights several things about the set of probability vectors $\mathcal{P}_\rho$, which represents every ρ in the way that [5, Th. 4.1] indicates. For this reason we suggest its proof in a detailed form.

As $\mathbb{R}^S_+ \subseteq \mathcal{A}_\rho$, the acceptance set of a coherent risk measure is a generating wedge since

$$\mathbb{R}^S = \mathbb{R}^S_+ - \mathbb{R}^S_+ \subseteq \mathcal{A}_\rho - \mathcal{A}_\rho \subseteq \mathbb{R}^S \,.$$

Hence $\mathcal{A}^0_\rho$ is a cone. We also remark that if K is a wedge of $\mathbb{R}^S$, then K^0 is a closed wedge. From what we have mentioned above,

$$\mathcal{A}^0_\rho \subseteq \mathbb{R}^S_+ = (\mathbb{R}^S_+)^0 \,. \tag{3.7}$$

Hence the set

$$B = \{y \in \mathcal{A}^0_\rho \,|\, \sum_{s=1}^{S} y_s = 1\} = \Delta_{S-1} \cap \mathcal{A}^0_\rho \tag{3.8}$$

where

$$\Delta_{S-1} = \{y \in \mathbb{R}^S_+ \,|\, \sum_{s=1}^{S} y_s = 1\} \,.$$

Hence B is a bounded, closed and convex set, since the simplex Δ_{S-1} is bounded, closed and convex and $\mathcal{A}^0_\rho$ is a closed and convex subset of $\mathbb{R}^S$.

Proposition 3.3. *If $\rho : \mathbb{R}^S \to \mathbb{R}$ is a coherent risk measure, then it is representable as follows*

$$\rho(X) = \sup\{\pi(-X) \,|\, \pi \in B\} \,, \tag{3.9}$$

for any $X \in \mathbb{R}^S$, where B is given by (3.8).

Proof. We remark that $\rho[X + \rho(X)\mathbf{1}] = 0$ for any $X \in \mathbb{R}^S$. Hence follows

$$X + \rho(X)\mathbf{1} \in \mathcal{A}_\rho \,.$$

Therefore by (3.7) and (3.8) we see that $\pi\,(X + \rho(X)\mathbf{1}) \geq 0$ for any $\pi \in B$. Hence for any $X \in \mathbb{R}^S$,

$$\pi(X) + \rho(X) \geq 0$$

for any $\pi \in B$. It follows $\rho(X) \geq \pi(-X)$ for any $\pi \in B$ which implies

$$\rho(X) \geq \sup\{\pi(-X) \,|\, \pi \in B\}\,. \tag{3.10}$$

We have to show that the inverse inequality is also true. Let us consider some $\varepsilon > 0$. Then we obtain

$$\rho[X + (\rho(X) - \varepsilon)\mathbf{1}] = \varepsilon > 0$$

by the translation invariance property of the coherent risk measures. Hence $X + (\rho(X) - \varepsilon)\mathbf{1}$ does not belong to the acceptance set $\mathcal{A}_\rho$ for any X. Now we apply the Finite Dimensional Separation Theorem on the disjoint convex sets $\{X + (\rho(X) - \varepsilon)\mathbf{1}\}$ and $\mathcal{A}_\rho$. So from the fact that $0 \in \mathcal{A}_\rho$, for any X there exists some $\pi_0 \in B$ (which depends on X and $\varepsilon > 0$) such that

$$\pi_0\,(X + (\rho(X) - \varepsilon)\mathbf{1}) = \pi_0(X) + \rho(X) - \varepsilon < 0\,.$$

The sets separated are the following: the first one is the singleton $\{X + (\rho(X) - \varepsilon)\mathbf{1}\}$ being a compact set and the second one is acceptance set $\mathcal{A}_\rho$ which is closed in $\mathbb{R}^S$ from the continuity of ρ (Corollary 3.2). Hence, there is some $\delta > 0$ and some $\alpha \in \mathbb{R}$ such that for this $\pi_0 \neq 0$ we obtain

$$\pi_0(X + (\rho(X) - \varepsilon)\mathbf{1}) \leq \alpha < \alpha + \delta \leq \pi_0(Y),$$

for any $Y \in \mathcal{A}_\rho$. Then π_0 takes positive values on $\mathcal{A}_\rho$. This holds because if we suppose that there is some $Y \in \mathcal{A}_\rho$ such that $\pi_0(Y) < 0$ then we come to a contradiction. Indeed, since $\lambda Y \in \mathcal{A}_\rho$, then for $\lambda \to +\infty$,

$$\pi_0(\lambda Y) \to -\infty$$

which contradicts the above separation. Since $\pi_0 \in \mathcal{A}_\rho^0 \subseteq \mathbb{R}_+^S$ and $\pi_0 \neq 0$,

$$\frac{1}{\sum_{s=1}^{S} \pi_0(s)} \pi_0 \in B\,,$$

and we denote the last vector also by π_0. We remark that the supremum in (3.10) is finite for any X since B is a compact set and $-X$ may be viewed as a continuous linear functional of $\mathbb{R}^S$ which actually takes a maximum value on the compact set B.

Hence for this X and this $\varepsilon > 0$, we obtain

$$\rho(X) - \varepsilon < \pi_0(-X) \leq \sup\{\pi(-X) \,|\, \pi \in B\}\,. \tag{3.11}$$

From the inequality (3.11) and for this X if we put $\varepsilon_n = 1/n$ for any $n \in \mathbb{N}$, we get

$$\rho(X) - \frac{1}{n} \leq \sup\{\pi(-X) \,|\, \pi \in B\}$$

and by taking the limit of the sequence $\rho(X) - 1/n\,,\ n \in \mathbb{N}$ we get

$$\rho(X) \leq \sup\{\pi(-X) \,|\, \pi \in B\}$$

as desired.

□

The set B is actually the desired set of probabilities indicated in [5, Th. 4.1]. The representability of a coherent risk measure ρ when the set of states is the finite set $\{1,\ 2,\ \ldots,\ S\}$, indicates that a risk measure is coherent if and only if there is a set of probability vectors $\mathcal{P}$ for the set of states $\{1,\ 2,\ \dots,\ S\}$ such that

$$\rho(X) = \sup_{P\in\mathcal{P}} E_P(-X)\,,$$

where $E_P(-X)$ denotes the expectation of $-X$ under the probability vector P. The one direction of representability which is not so obvious is indicated by the Proposition 3.3. The other, much easier, is presented for completeness in the following.

Proposition 3.4. *The function $\rho_{\mathcal{P}} : \mathbb{R}^S \to \mathbb{R}$, where*

$$\rho_{\mathcal{P}}(X) = \sup_{P\in\mathcal{P}} E_P(-X) \tag{3.12}$$

and $\mathcal{P}$ is a set of probability vectors for the set of states, is a coherent risk measure.

Proof. Let us check the four properties of the coherent measures.

(i) (Translation Invariance):

$$\begin{aligned}\rho_{\mathcal{P}}(X + a\mathbf{1}) &= \sup\{E_P(-X - a\mathbf{1}) \,|\, P \in \mathcal{P}\} = \sup\{E_P(-X) - aE_P(\mathbf{1}) \,|\, P \in \mathcal{P}\}\\ &= \sup\{E_P(-X) - a \,|\, P \in \mathcal{P}\} = \sup\{E_P(-X) \,|\, P \in \mathcal{P}\} - a = \rho_{\mathcal{P}}(X) - a\,,\end{aligned}$$

for any $X \in \mathbb{R}^S$ and any $a \in \mathbb{R}$.

(ii) (Sub-additivity):

$$\begin{aligned}\rho_{\mathcal{P}}(X + Y) &= \sup\{E_P(-X - Y) \,|\, P \in \mathcal{P}\} = \sup\{E_P(-X) + E_P(-Y) \,|\, P \in \mathcal{P}\}\\ &\leq \sup\{E_P(-X) \,|\, P \in \mathcal{P}\} + \sup\{E_P(-Y) \,|\, P \in \mathcal{P}\} = \rho_{\mathcal{P}}(X) + \rho_{\mathcal{P}}(Y)\,,\end{aligned}$$

for any $X, Y \in \mathbb{R}^S$.

(iii) (Positive Homogeneity):

$$\begin{aligned}\rho_{\mathcal{P}}(\lambda X) &= \sup\{E_P(-\lambda X) \,|\, P \in \mathcal{P}\} = \sup\{\lambda E_P(-X) \,|\, P \in \mathcal{P}\}\\ &= \lambda \sup\{E_P(-X) \,|\, P \in \mathcal{P}\} = \lambda \rho_{\mathcal{P}}(X)\end{aligned}$$

for any $X \in \mathbb{R}^S$ and any $\lambda \in \mathbb{R}_+$.

(iv) (Monotonicity): If $Y \geq X$ in the usual partial ordering of $\mathbb{R}^S$, then $-X \geq -Y$, hence

$$E_P(-X) \geq E_P(-Y)$$

for any $P \in \mathcal{P}$. Taking suprema over $\mathcal{P}$ we get that

$$\rho_{\mathcal{P}}(X) = \sup_{P\in\mathcal{P}} E_P(-X) \geq \sup_{P\in\mathcal{P}} E_P(-Y) = \rho_{\mathcal{P}}(Y)\,.$$

□

3.3. **The Relevance property.** Finally, another question about coherent risk measures when the set of states of the world is finite is the so-called Relevance property, mentioned in [5], p. 210. The Relevance property for a risk measure $\rho : \mathbb{R}^S \to \mathbb{R}$ says that if $X \in -\mathbb{R}^S_+$ and $X \neq 0$, then $\rho(X) > 0$. If the risk measure is coherent, then a weaker property can be proved due to the properties of it. The fact that the set of states of the world is finite plays an important role in the deduction of this property. The property and its proof are given in the following proposition

Proposition 3.5. *If $\rho : \mathbb{R}^S \to \mathbb{R}$ is a coherent risk measure, then for any $X \in -\mathbb{R}^S_{++}$, $\rho(X) > 0$ holds.*

Proof. By subadditivity of ρ we take $\rho(0) = 0 \leq \rho(X) + \rho(-X)$. This implies $-\rho(-X) \leq \rho(X)$ and since $-X \in \mathbb{R}^S_{++} \subseteq \mathcal{A}_\rho$, we take $\rho(-X) \leq 0$. Hence $\rho(X) \geq 0$. But since $X \in -\mathbb{R}^S_{++}$, we pose $2k = \max\{X_s | s = 1, 2, ..., S\} < 0$ and then $\rho(X - k\mathbf{1}) = \rho(X) + k$ by the Translation Invariance property. But we notice that $X + k\mathbf{1} \in -\mathbb{R}^S_{++}$ since for any state s we have $2X_s < X_s \leq \max\{X_s | s = 1, 2, ..., S\}$. By the first remark we made, we take $\rho(X - k\mathbf{1}) = \rho(X) + k \geq 0$. Hence $\rho(X) \geq -k > 0$ and the conclusion is ready. □

Of course, the above version of the Relevance property can be deduced by the representation of a coherent risk measure. If $X \in -\mathbb{R}^S_{++}$, then $-X \in \mathbb{R}^S_{++}$ which means that there is some $\delta > 0$ such that $X_s \geq \delta > 0$ for any $s = 1, 2, ..., S$. From the representation theorem 3.3, we get that for any $\pi \in \mathcal{A}^0_\rho \cap \Delta_{S-1}$ $\pi(-X) = \pi \cdot (-X) \geq \delta > 0$. Hence $\rho(X) = \sup\{\pi(-X) | \pi \in \mathcal{A}^0_\rho \cap \Delta_{S-1}\} \geq \delta > 0$.

4. Convex risk measures and their dual representation

The following proofs rely on the equivalent ones of [8], [14]. We remind that by Δ_{S-1} we denote the simplex of the positive cone of $\mathbb{R}^S$.

Definition 4.1. *A risk measure $\rho : \mathbb{R}^S \to \mathbb{R}$ satisfies the Convexity property if it is a convex function*

$$\rho(\lambda U_1 + (1-\lambda)U_2) \leq \lambda\rho(U_1) + (1-\lambda)\rho(U_2),$$

for any $\lambda \in [0, 1]$ and $U_1, U_2 \in \mathbb{R}^S$.

Definition 4.2. *If $\rho : \mathbb{R}^S \to \mathbb{R}$ satisfies the properties of Translation Invariance, Convexity and Monotonicity, then it is called* ***convex risk measure****.*

Theorem 4.3. *If $\rho : \mathbb{R}^S \to \mathbb{R}$ is a convex risk measure, then*

$$\rho(U) = \sup\{\pi(-U) - a(\pi) \,|\, \pi \in \Delta_{S-1}\}\,, \tag{4.13}$$

for any $U \in \mathbb{R}^S$ and $a : \Delta_{S-1} \to \overline{\mathbb{R}}$ is a 'penalty function' associated with ρ, with $a(\pi) \in (-\infty, \infty]$ for any $\pi \in \Delta_{S-1}$. On the other hand, every ρ defined through (4.13), is a convex risk measure.

Proof. If we consider a convex risk measure ρ, there exists a penalty function a such that ρ has a representation like the one indicated in (4.13). To see this, we remark that for any $\pi \in B$ we define

$$a(\pi) = \sup\{\pi(-X) - \rho(X) \,|\, X \in \mathbb{R}^S\}\,. \tag{4.14}$$

Then as in the proof of [8, Th. 5], we denote

$$\hat{a}(\pi) = \sup\{\pi(-X) \,|\, X \in \mathcal{A}_\rho\}\,. \tag{4.15}$$

We will prove that $a(\pi) = \hat{a}(\pi)$ for any $\pi \in \Delta_{S-1}$. We remark that $a(\pi) \geq \hat{a}(\pi)$. This holds because for any $X \in \mathcal{A}_\rho$, $\pi(-X) - \rho(X) \geq \pi(-X)$. Hence

$$\sup\{\pi(-X) - \rho(X) | X \in \mathbb{R}^S\} \geq \sup\{\pi(-X) - \rho(X) | X \in \mathcal{A}_\rho\} \geq \sup\{\pi(-X) | X \in \mathcal{A}_\rho\}.$$

To prove the inverse, we take $X \in \mathbb{R}^S$ and we consider $X' = X + \rho(X)\mathbf{1} \in \mathcal{A}_\rho$. Hence

$$\hat{a}(\pi) \geq \pi(-X') = \pi(-X) - \rho(X)\,,$$

so $a(\pi) = \hat{a}(\pi)$ by taking suprema over all $X \in \mathbb{R}^S$. We remark that $a(\pi) \in (-\infty\,,\, +\infty]$ for any $\pi \in \Delta_{S-1}$. Next, we remark that for any $Y \in \mathbb{R}^S$ and by the expression (4.14) of a, we have

$$\rho(Y) \geq \sup\{\pi(-Y) - a(\pi) \,|\, \pi \in \Delta_{S-1}\}$$

for any $Y \in \mathbb{R}^S$. In order to prove the desired equality, we have the following: Suppose that there is some $Y_0 \in \mathbb{R}^S$, such that

$$\rho(Y_0) > \sup\{\pi(-Y_0) - a(\pi) \,|\, \pi \in \Delta_{S-1}\}\,.$$

Hence there exists some $m \in \mathbb{R}$ such that

$$\rho(Y_0) > m > \sup_{\pi \in B}\{\pi(-Y_0) - a(\pi)\}\,.$$

From the last remark we take that

$$\rho(Y_0 + m\mathbf{1}) = \rho(Y_0) - m > 0$$

and that $Y_0 + m\mathbf{1} \notin \mathcal{A}_\rho$. The singleton $\{Y_0 + m\mathbf{1}\}$ is a convex, compact set and $\mathcal{A}_\rho$ is by assumption a closed set of $\mathbb{R}^S$ which is also convex, since ρ is a convex risk measure. Since these two sets are disjoint, from the Strong Separation Theorem for convex sets in locally convex spaces, there is some $\ell \in \mathbb{R}^S, \ell \neq 0$, an $\alpha \in \mathbb{R}$ and a $\delta > 0$ such that

$$\ell(Y_0 + m\mathbf{1}) \geq \alpha + \delta > \alpha \geq \ell(X)$$

for any $X \in \mathcal{A}_\rho$. Hence we take that

$$\ell(Y_0 + m\mathbf{1}) > \sup\{\ell(X) \,|\, X \in \mathcal{A}_\rho\}\,.$$

The functional ℓ takes negative values on $\mathbb{R}^S_+$ since if there is some $X_0 \in \mathbb{R}^S_+$ such that $\ell(X_0) > 0$, then for any $\lambda \in \mathbb{R}_+$ we take $\lambda X_0 \in \mathbb{R}^S_+ \subseteq \mathcal{A}_\rho$. Then if $\lambda \to +\infty$ then

$$\ell(\lambda X_0) > \ell(Y_0 + me)$$

for λ big enough, being a contradiction according to the previous separation argument. Then since we have that $-\ell \in \mathbb{R}^S_+\,,\ \ell \neq 0$, we may suppose that

$$-\ell(\mathbf{1}) = 1$$

holds, or else $-\ell \in \Delta_{S-1}$. Hence the separation of the sets $\{Y_0 + m\mathbf{1}\}$ and $\mathcal{A}_\rho$ implies that

$$(-\ell)(-Y_0) - m > \sup_{X \in \mathcal{A}_\rho} (-\ell)(-X) = a(-\ell)\,.$$

Denote $-\ell$ by π_0 and we get

$$\pi_0(-Y_0) - a(\pi_0) > m\,,$$

which is a contradiction, since in this case

$$m > \sup\{\pi(-Y_0) - a(\pi) \,|\, \pi \in \Delta_{S-1}\} \geq \pi_0(-Y_0) - a(\pi_0) > m\,.$$

The contradiction was due to the assumption that some $Y_0 \in \mathbb{R}^S$ exists, such that

$$\rho(Y_0) > \sup\{\pi(-Y_0) - a(\pi) \,|\, \pi \in \Delta_{S-1}\}\,.$$

Hence for any $Y \in \mathbb{R}^S$ we get (4.13).

For the opposite direction, it suffices to show that any $\rho : \mathbb{R}^S \to \mathbb{R}$, defined through (4.13), is a convex risk measure. For this, we have to verify that ρ satisfies the properties of a convex risk measure:

(*i*) (Translation Invariance):

$$\begin{aligned}\rho(X+k\mathbf{1}) &= \sup\{\pi(-X-k\mathbf{1})-a(\pi)\,|\,\pi\in\Delta_{S-1}\}\\ &= \sup\{\pi(-X)-a(\pi)-k\pi(\mathbf{1})\,|\,\pi\in\Delta_{S-1}\}\\ &= \sup\{\pi(-X)-a(\pi)-k\,|\,\pi\in\Delta_{S-1}\}\\ &= \sup\{\pi(-X)-a(\pi)\,|\,\pi\in\Delta_{S-1}\}-k=\rho(X)-k\end{aligned}$$

for any $X\in\mathbb{R}^S$ and any $k\in\mathbb{R}$.

(*ii*) (Convexity): The function which maps every X to $\pi(-X)-a(\pi)$ for some $\pi\in\Delta_{S-1}$ is a convex real-valued function on $\mathbb{R}^S$, hence ρ is a convex function on $\mathbb{R}^S$ as the supremum of convex functions defined on $\mathbb{R}^S$.

(*iii*) (Monotonicity): If $Y\geq X$ in terms of the partial ordering of $\mathbb{R}^S$, then $-X\geq -Y$, hence

$$\pi(-X)\geq\pi(-Y)$$

for any $\pi\in\Delta_{S-1}$ since $\Delta_{S-1}\subseteq\mathbb{R}^S_+$. Hence

$$\pi(-X)-a(\pi)\geq\pi(-Y)-a(\pi)$$

for any $\pi\in\Delta_{S-1}$ and by taking suprema over the elements of Δ_{S-1} we get that

$$\rho(X)=\sup_{\pi\in\Delta_{S-1}}\{\pi(-X)-a(\pi)\}\geq\rho(Y)=\sup_{\pi\in\Delta_{S-1}}\{\pi(-Y)-a(\pi)\}\,.$$

□

Proposition 4.4. *Every $(\mathcal{A},\mathbf{1})$ -convex risk measure $\rho:\mathbb{R}^S\to\mathbb{R}$ is Lipschitz-continuous.*

If $\{f_i:\mathbb{R}^S\to\mathbb{R}\,,\ i\in I\}$ is a family of real-valued functions defined on $\mathbb{R}^S$, then we observe that

$$\sup_{i\in I}f_i(X)-\sup_{i\in I}f_i(Y)\leq\sup_{i\in I}\{f_i(X)-f_i(Y)\}\ ,$$

where $X,Y\in\mathbb{R}^S$ (for better interpretation, we may suppose that the family of functions $\{f_i:\mathbb{R}^S\to\overline{\mathbb{R}}\,,\ i\in I\}$ is such that $\sup_{i\in I}f_i(X)\neq\infty$ for any X). Indeed, this holds because if we denote by A the set $\{f_i(X)-f_i(Y)\,|\,i\in I\}$ and by D the set $\{f_i(Y)\,|\,i\in I\}$, then

$$\{f_i(X)\,|\,i\in I\}\subseteq A+D\,.$$

Hence

$$\sup_{i\in I}f_i(X)\leq\sup_{i\in I}\{f_i(X)-f_i(Y)\}+\sup_{i\in I}f_i(Y)\,.$$

We have seen that ρ has the representation (4.13). Then

$$\rho(X)-\rho(Y)=\sup\{\pi(-X)-a(\pi)\,|\,\pi\in\Delta_{S-1}\}-\sup\{\pi(-Y)-a(\pi)\,|\,\pi\in\Delta_{S-1}\}\,.$$

By the above remark we take that

$$\rho(X)-\rho(Y)\leq\sup\{\pi(-X)-\pi(-Y)\,|\,\pi\in\Delta_{S-1}\}=\sup\{\pi(Y-X)\,|\,\pi\in\Delta_{S-1}\}\,.$$

We actually have that $I=\Delta_{S-1}$ and $f_i=f_\pi$ for any $\pi\in\Delta_{S-1}$ where $f_\pi:\mathbb{R}^S\to\overline{\mathbb{R}}$ is such that $f_\pi(X)=\pi(-X)-a(\pi)$ for any $X\in\mathbb{R}^S$. Also, we suppose that $\pi(-X)-a(\pi)-(\pi(-Y)-a(\pi))=\pi(-X)-\pi(-Y)$ for any $X,Y\in L$ and any $\pi\in\Delta_{S-1}$, namely that $a(\pi)-a(\pi)=0$. This is a simple subtraction in the case where $a(\pi)\in\mathbb{R}$, but in the case where $a(\pi)=\infty$ we have the subtraction of two infinity values. But we may suppose that their difference is equal to zero, since we subtract infinities 'of the same form'. On the other hand we may say that if $a(\pi) = \infty$ then $-a(\pi) = -\infty$, hence $\pi(-X) - a(\pi) = \pi(-X)$ because if we add a real number to $-\infty$ we take the real number itself.

In order to complete the proof, we note that

$$\pi(Y - X) \leq |\pi(Y - X)| \leq \|\pi\| \cdot \|X - Y\| \leq \frac{1}{b}\|X - Y\|$$

for any $\pi \in B$, from the definition of B and the fact that $\mathbf{1}$ is a uniformly monotonic linear functional of $(\mathbb{R}^S_+)^0 = \mathbb{R}^S_+$ and $\pi(\mathbf{1}) \geq b\|\pi\|$. Hence

$$\rho(X) - \rho(Y) \leq \frac{1}{b}\|X - Y\|$$

and in the same way we may show

$$\rho(Y) - \rho(X) \leq \frac{1}{b}\|X - Y\|\,.$$

The last two inequalities imply that

$$|\rho(X) - \rho(Y)| \leq \frac{1}{b}\|X - Y\|$$

and the conclusion is ready. □

4.1. **Conjugate convex functions and convex risk measures.** The penalty function $a : \Delta_{S-1} \to (-\infty, \infty]$ associated with a risk measure ρ by the dual representation

$$\rho(X) = \sup\{\pi(-X) - a(\pi) | \pi \in \Delta_{S-1}\}.$$

Since

$$a(\pi) = \sup\{\pi(-X) - \rho(X) \,|\, X \in \mathbb{R}^S\},$$

then $a(\pi) = \rho^*(-\pi)$, where ρ^* is the convex conjugate of ρ, defined as follows:

$$\rho^*(f) = \sup\{f(x) - \rho(x) | x \in \mathbb{R}^S\},$$

for any $f \in \mathbb{R}^S$.

Note: The next two Sections are formulated by [22]

5. Portfolio theory and convex risk measures-Markowitz type problems

We name *'Markowitz type problems'* those portfolio selection problems in which the convex risk measures play a role equivalent to the one of variance in Markowitz model. Under the Markowitz framework, the portfolio selection is generally determined by the optimization principle 'maximize the expected payoff (return), while you should minimize the risk'. If the measurement of risk is determined by some convex risk measure ρ, then the 'expected return, risk pair' for any financial position $X \in \mathbb{R}^S$ is the pair

$$(\mathbb{E}(X), \rho(X))$$

of real numbers, where $\mathbb{E}(X)$ denotes the mean value of X under the statistical measure of the market μ and $\rho(X)$ is the risk of X with respect to ρ, or else the amount of money needed so that X is acceptable under ρ. If we consider a set of positions $\mathcal{X}$ among which an investor chooses her portfolio, then the **primarily admissible set of positions** with respect to ρ is

$$K_1(\mathcal{X}) = \{X \in \mathcal{X} | \mathbb{E}(X) \geq 0, \rho(X) \leq 0\} \tag{5.16}$$

and the **admissible set of positions at the risk level** c with respect to ρ is

$$K_2(\mathcal{X}) = \{X \in \mathcal{X} | \mathbb{E}(X) \geq 0, \rho(X) \leq c\}. \tag{5.17}$$

The **expected payoff-risk pairs of** $K_i(\mathcal{X}), i = 1, 2$ **with respect to** ρ or the **mean-risk efficiency set of** $\mathcal{X}$ **at the risk level** c with respect to ρ, is the following set of $\mathbb{R}^2$:

$$K_{(\mathbb{E},\rho),i}(\mathcal{X}) = \{(\mathbb{E}(X), \rho(X)) | X \in K_i(\mathcal{X})\},\ i = 1, 2 \tag{5.18}$$

The *efficiency frontier* of $K_i(\mathcal{X}), i = 1, 2$ is the set of Pareto efficient points $E(K_{(\mathbb{E},\rho),i}(\mathcal{X}), D)$ of $K_{(\mathbb{E},\rho),i}(\mathcal{X})$ with respect to the cone $D = \{x = (x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 0, x_2 \leq 0\}$ of $\mathbb{R}^2$. We have to mention that a desired level of mean payoff $\mathbb{E}(X)$ can be also considered in both of the cases $i = 1, 2$. This set is consisted by the pairs $(\mathbb{E}(X), \rho(X)), X \in K_i(\mathcal{X}), i = 1, 2$, such that there is not any position X_0 in $K_i(\mathcal{X})$ such that either $\mathbb{E}(X_0) > \mathbb{E}(X)$, while $\rho(X_0) \leq \rho(X)$, or $\mathbb{E}(X_0) \geq \mathbb{E}(X)$, while $\rho(X_0) < \rho(X)$, for any $X \in K_i(\mathcal{X})$. The solutions to the maximization problem

$$Maximize\ \mathbb{E}(X) - k\rho(X)\ subject\,to\ X \in K_i(\mathcal{X}),\ i = 1, 2, \tag{5.19}$$

are also elements of the efficiency frontier if $k \in \mathbb{R}_+, k \neq 0$, because the functional $(1, -k)$ is a strictly positive functional of D and the set of *positive proper efficient points* is a subset of the set of Pareto efficient points. The determination of the efficiency frontier $E(K_{(\mathbb{E},\rho),i}(\mathcal{X}), D)$, $i = 1, 2$, gives rise to the solution of the scalar optimization problem 5.19. For the solution set $S_k(K_{(\mathbb{E},\rho),i}(\mathcal{X}))$ of 5.19, we have that

$$S_k(K_{(\mathbb{E},\rho),i}(\mathcal{X})) \subseteq PosE(K_{(\mathbb{E},\rho),i}(\mathcal{X}), D) \subseteq E(K_{(\mathbb{E},\rho),i}(\mathcal{X}), D)),\ k > 0, i = 1, 2,$$

where $PosE(K_{(\mathbb{E},\rho),i}(\mathcal{X}), D)$ denotes the set of *positive proper efficient points* of $K_{(\mathbb{E},\rho),i}(\mathcal{X})$ with respect to the cone D. For the definition of this notion, see in the Appendix. The existence of Pareto efficient points is assured mainly by the theorem of J. Borwein, proved in [6] and is also stated in the book [10]. We may remark that due to the continuity of the maps $\mathbb{E}(.), \rho(.)$ defined on E, the properties of the boundedness and the closedness of $\mathcal{X}$ are transferred into the set $K_{(\mathbb{E},\rho),i}(\mathcal{X})$ which is examined for the existence of Pareto efficient points or positive proper efficient points. We may see that the following proposition holds.

The most essential assumption so that the efficiency frontier is well -defined is that the set of the admissible positions must be non-empty. A simple condition which implies the non-emptiness of this set is the assumption of the following proposition.

Proposition 5.1. *If $\mathcal{X} \cap \mathbb{R}^S_+ \neq \emptyset$, then $K_i(\mathcal{X})$ is non-empty for $i = 1, 2$.*

Proposition 5.2. *The set $K_{(\mathbb{E},\rho),i}(\mathcal{X}) = \{(\mathbb{E}(X), \rho(X)) | X \in K_i(\mathcal{X})\}$ is a convex subset of $\mathbb{R}^2$ if $\mathcal{X}$ is a convex set of $\mathbb{R}^2$ for both the cases $i = 1, 2$.*

Proposition 5.3. *The vector functional $\mathcal{T} = (\mathbb{E}(.), \rho(.)) : \mathbb{R}^S \to \mathbb{R}^2$ is continuous. Hence if $\mathcal{X}$ is closed in $\mathbb{R}^S$ and $\rho(0) = 0$, then $\mathcal{T}(\mathcal{X})$ is closed in $\mathbb{R}^2$ and if $\mathcal{X}$ is bounded in $\mathbb{R}^S$, then $\mathcal{T}(\mathcal{X})$ is bounded in $\mathbb{R}^2$.*

The set $\mathcal{T}^{-1}(E(K_{(\mathbb{E},\rho),i}(\mathcal{X}), D)), i = 1, 2$ is the **set of efficient portfolios of** $\mathcal{X}$ **under the expected payoff-risk conditions** $i = 1, 2$ **with respect to** ρ.

If we would like to answer the question about when the efficiency frontier of $K_i(\mathcal{X}), i = 1, 2$ is well -defined, then we have to determine cases where $E(K_{(\mathbb{E},\rho),i}(\mathcal{X}), D) \neq \emptyset$ as we mentioned before. The properties of $\mathcal{T}(\mathcal{X})$ are important for this purpose.

Let us see some propositions indicating when the efficiency frontier is non-empty.

Proposition 5.4. *If $\rho(0) = 0$, while $\mathcal{X}$ is closed and a D -upper bounded section of $K_{(\mathbb{E},\rho),i}(\mathcal{X})$ exists, then the efficiency frontier $E(K_{(\mathbb{E},\rho),i}(\mathcal{X}), D)$ is non-empty for both the cases $i = 1, 2$.*

Proposition 5.5. *If $\rho(0) = 0$, while $\mathcal{X}$ is closed and bounded, then the efficiency frontier $E(K_{(\mathbb{E},\rho),i}(\mathcal{X}), D)$ is non-empty for both the cases $i = 1, 2$.*

Theorem 5.6. *If $\rho(0) = 0$, while $\mathcal{X}$ is closed and a D -upper bounded section of $K_{(\mathbb{E},\rho),i}(\mathcal{X})$ exists, then efficient positions exist in $\mathcal{X}$ with respect to ρ in both of the cases $i = 1, 2$.*

Theorem 5.7. *If $\rho(0) = 0$, while $\mathcal{X}$ is closed and bounded, then efficient positions exist in $\mathcal{X}$ with respect to ρ in both of the cases $i = 1, 2$.*

Proposition 5.8. *If $\rho : \mathbb{R}^S \to \mathbb{R}$ is a coherent risk measure and $\mathcal{X} \cap \mathbb{R}^S_+ \neq \emptyset$, while $\mathcal{X}$ is $\mathbb{R}^S_+$-upper bounded by a positive multiple of an element in $\mathcal{X} \cap \mathbb{R}^S_+$, then $K_{(\mathbb{E},\rho),i}(\mathcal{X})$ is a D-upper bounded set.*

Proposition 5.9. *Suppose that $\rho : \mathbb{R}^S \to \mathbb{R}$ is convex with $\rho(0) = 0$. If the efficiency frontier $E(K_{(\mathbb{E},\rho),i}(\mathcal{X}), D), i = 1, 2$ is non -empty for a convex and closed financial position choice set $\mathcal{X}$, then for any norm-convergent sequence $X_n \to X_0$ in $K_i(\mathcal{X}), i = 1, 2$, where $(\mathbb{E}(X_n), \rho(X_n)), n \in \mathbb{N}$ are points of $E(K_{(\mathbb{E},\rho),i}(\mathcal{X}), D), i = 1, 2$, the point $(\mathbb{E}(X_0), \rho(X_0))$ also belongs to it.*

6. Portfolio theory and coherent risk measures-Utility type problems

We name 'utility type problems' those portfolio selection problems in which the coherent risk measures play a role equivalent to the one of a utility function in classical microeconomic theory. Hence a common class of portfolio selection problems is the *risk minimization problems*. The general form of these problems is the following:

$$Minimize\ \rho(X)\ subject\, to\ X \in \mathcal{X}, \tag{6.20}$$

where ρ is a coherent risk measure and $\mathcal{X} \subseteq \mathbb{R}^S$ is a non-empty set of financial positions. The problem 6.20 corresponds to the determination of the set of financial positions which minimize the risk with respect to ρ among the positions in $\mathcal{X}$. Among the conditions of normality for the constrained optimization problem 6.20, the more important are *convexity*, *closedness* and *boundedness* of $\mathcal{X}$. Various combinations of these properties are related to the existence and the uniqueness of solution to 6.20, combinations which are related to the form of the set $\mathcal{X}$ of constraints. For the determination of the solution to these problems standard optimization techniques are used, like the Kuhn-Tucker one. For the utility based problems of portfolio selection, see [23].

Let us remind some essential propositions on existence and uniqueness.

Proposition 6.1. *6.20 has a solution if $\mathcal{X}$ is closed and bounded.*

Proof. ρ as a coherent risk measure is continuous on $\mathbb{R}^S$. $\mathcal{X}$ is compact, hence by the Weierstrass theorem the conclusion is ready. □

Proposition 6.2. *If 6.20 has a solution and $\mathcal{X}$ is convex, then the solution set is an extreme set of it.*

Proof. The solution set is the following: $M = \{X \in C | \rho(X) = sup\{\rho(Y) | Y \in C\}$. Suppose that M is not an extreme set. Then, there are some $y_0, z_0 \in M$ and some $a_0 \in (0, 1)$ such that

$a_0y_0 + (1-a_0)z_0 \in M$, while $y_0, z_0 \notin M$. Since ρ is a convex function, $\rho(a_0y_0 + (1-a_0)z_0) \leq a_0)\rho(y_0) + (1-a_0)\rho(z_0)$. But since $z_0, y_0 \notin M$, $\rho(a_0y_0 + (1-a_0)z_0) > \rho(z_0) > (1-a_0)\rho(z_0)$, $\rho(a_0y_0 + (1-a_0)\rho(z_0) > \rho(y_0) > a_0\rho(y_0)$ which implies $\rho(a_0y_0 + (1-a_0)z_0) > a_0\rho(y_0) + (1-a_0)\rho(z_0)$, a contradiction.

□

Proposition 6.3. *If $\mathcal{X}$ is convex, closed and bounded, then 6.20 has a solution which is an extreme point of $\mathcal{X}$.*

Proof. ρ as a coherent risk measure is continuous on $\mathbb{R}^S$. $\mathcal{X}$ is compact and oonvex. The set $M = \{X \in C | \rho(X) = \sup\{\rho(Y) | Y \in C\}$ is an extreme subset of C and by previous proposition is non-empty. By Zorn's Lemma, there is an extreme point of C in M (see also Lemma 5.114 in [2]) □

7. Expected Shortfall

As an alternative against VaR, Expected Shortfall is a coherent risk measure ([1]) which becomes more common in financial practice. Here, we suppose that Ω is any set, not necessarily finite in order to introduce a certain example of coherent risk measure.

Definition 7.1. *The a-quantile of a $\mathcal{F}$-measurable random variable $X : \Omega \to \mathbb{R}$ is defined as follows:*

$$q_a(X) = \inf\{x \in \mathbb{R} | P(X \leq x) \geq a\}.$$

We remind that $VaR_a(X) = q_a(-X)$. The relation between VaR and solvency capital is that VaR is the minimum m such that $P(X + m \geq 0) \geq a$. The coherent risk measures are independent from the relation between solvency capital and both singificance level a and 'objective' probability measure P. The definition of Expected Shortfall is the following:

Definition 7.2. *If we consider a significance level $a \in (0,1)$ and a $\mathcal{F}$-measurable random variable $X : \Omega \to \mathbb{R}$. Suppose that $\mathbb{E}(X^-) < \infty$. Then,*

$$ES_a(X) = -\frac{1}{1-a}(\mathbb{E}(X \mathbf{1}_{\{-X \geq q_a(-X)\}}) + q_a(-X)(a - P(-X < q_a(-X))).$$

For these definitions, see ([24]) A famous expression of the ES_a is the following: $ES_a(X) = \frac{1}{1-a}\int_a^1 VaR_u(X)du$, see [1, Pr.3.2].

As it is also indicated by [24, Ex.2.4], VaR is not a coherent risk measure, because it does not satisfies the axiom of subadditivity. Another point of critiscism against VaR was that light and heavy tail distribution variables may have the same VaR, or else that VaR is *law invariant*: the distributions of X, Y need not be identical in order to be $VaR_a(X) = VaR_a(Y)$.

The previous Example considers two Pareto distributed, independent r.v. X, Y in $(-\infty, 1)$ while the joint distribution of (X, Y) is

$$P(X \leq x_1, Y \leq x_2) = (2-x_1)^{-1}(2-x_2)^{-1}, x_1, x_2 < 1.$$

This implies that

$$VaR_a(X_i) = \frac{1}{1-a} - 2, i = 1, 2,$$

where $X_1 = X, X_2 = Y$ and

$$P(X + Y \leq x) = \frac{2}{4-x} + \frac{2log(3-x)}{(4-x)^2}, x < 2.$$

For $a = 0.99$ we get $VaR_a(X) = VaR_a(Y) = 98$ and $VaR_a(X+Y) \cong 203.2$.

For the coherence of ES_a, see [1, Pr.3.1]. Also, note that for continuous random variables, $ES_a(X) = \mathbb{E}(-X|-X \leq VaR_a(X))$.

8. Appendix: notions from the theory of partially ordered linear spaces

In this paragraph, we give some essential notions and results from the theory of partially ordered linear spaces which are used in this paper. Let L be a (normed) linear space. A set $C \subseteq L$ satisfying $C + C \subseteq C$ and $\lambda C \subseteq C$ for any $\lambda \in \mathbb{R}_+$ is called *wedge*. A wedge for which $C \cap (-C) = \{0\}$ is called *cone*. A pair $(E, \geq)$ where E is a linear space and $\geq$ is a binary relation on E satisfying the following properties:

(i) $x \geq x$ for any $x \in L$ (reflexive)
(ii) If $x \geq y$ and $y \geq z$ then $x \geq z$, where $x, y, z \in E$ (transitive)
(iii) If $x \geq y$ then $\lambda x \geq \lambda y$ for any $\lambda \in \mathbb{R}_+$ and $x + z \geq y + z$ for any $z \in L$, where $x, y \in L$ (compatible with the linear structure of L),

is called *partially ordered linear space*. The binary relation $\geq$ in this case is a *partial ordering* on E. The set $P = \{x \in E | x \geq 0\}$ is called *(positive) wedge* of the partial ordering $\geq$ of E. Given a wedge C in E, the binary relation $\geq_C$ defined as follows:

$$x \geq_C y \Longleftrightarrow x - y \in C,$$

is a partial ordering on L, called *partial ordering induced by C on L*. If the partial ordering $\geq$ of the space E is *antisymmetric*, namely if $x \geq y$ and $y \geq x$ implies $x = y$, where $x, y \in E$, then C is a cone.

Then $(L, \geq)$ is called a *partially ordered linear space.*

A non-empty subset K of a vector space L is called *wedge* if it satisfies

(1) $K + K \subseteq K$
(2) $\lambda K \subseteq K$ for any real number $\lambda \geq 0$.

A non-empty subset C of X is called *cone* if it is a wedge which satisfies

$$C \cap (-C) = \{0\}. \tag{8.1}$$

Any cone C in a vector space L implies a *partial ordering relation* $\geq_C$ on it defined as follows:

$$x \geq_C y \Leftrightarrow x - y \in C. \tag{8.2}$$

When $x \geq_C y$ and $x - y \neq 0$ then we write $x > y$. When we write $y \leq x$ with respect to the same partial ordering relation implied by C on L, we mean that $x \geq y$. In this case the partially ordered linear space $(L, \geq_C)$ may be denoted by (L, C). On the other hand, if $(L, \geq)$ is a partially ordered linear space, then the set

$$P = \{x \in L \,|\, x \geq 0\}$$

is a cone and $\geq$, $\geq_P$ coincide.

A linear functional f of a vector space L is called *positive functional* of the partially ordered linear space (L, K) (or a *positive functional of K*) if

$$f(x) \geq 0\,,\ \forall x \in K\,,$$

and *strictly positive functional* of the ordered linear space (L, K) (or a *strictly positive functional of K*) if it is positive and

$$f(x) > 0\,,\ \forall x \in K \setminus (K \cap (-K))$$

if K is a wedge, while

$$f(x) > 0\,,\ \forall x \in K \setminus \{0\}$$

if K is a cone. If L is a normed linear space and C is a cone, a linear functional of L is *uniformly monotonic* if $f(x) \geq a\|x\|$ for any $x \in C \setminus \{0\}$ for some $a \in \mathbb{R}_+ \setminus \{0\}$.

In this article, since the spaces we use are also normed linear spaces, when we refer to positive or strictly positive functionals we mean continuous linear functionals (elements of the norm dual space L^*).

If K is a wedge of the normed linear space L, the *dual wedge* of K, denoted by K^0, represents the following subset of L^*:

$$K^0 = \{f \in L^* \mid f(x) \geq 0\,,\ \forall x \in K\}\,.$$

If K is a wedge with $K \subseteq C$ then $C^0 \subseteq K^0$. In the rest of this article we will denote by L_+^0 the dual wedge of L_+.

If (L, C) is a partially ordered linear space and C is a cone, then a *base* of C is any convex subset B of C such that for any $x \in C \setminus \{0\}$ there exists a unique $\lambda_x \in \mathbb{R}_+ \setminus \{0\}$ such that $\lambda_x x \in B$. If f is a strictly positive functional of L_+, the set

$$B_f = \{x \in C \mid f(x) = 1\}\,, \tag{8.3}$$

is the *base of* C *defined by* f. The base B_f is bounded if and only if f is uniformly monotonic. If B is a bounded base of C such that $0 \notin \overline{B}$ then C is called *well-based*. If C is well-based, then a bounded base of C defined by a $g \in L^*$ exists.

Let us denote by $[x, y]$ the order interval of L defined by $x\,,\ y$, i.e.

$$[x, y] = (x + K) \cap (y - K)\,.$$

An element $e \in L$ is called *order-unit* of (L, K) if

$$L = \cup_{n=1}^{\infty} [-ne, ne]\,.$$

If L is also a normed linear space, then according to [11, Pr. 3.1.3], e is an interior point of K in terms of the norm topology if and only if $[-e, e]$ is a neighborhood of zero. Hence an interior point of K is an order-unit of (L, K).

(L, K) is a *vector lattice* if for any $x, y \in L$, the supremum and the infimum of $\{x, y\}$ with respect to the partial ordering defined by K exist in L. In this case $\sup\{x, y\}$ and $\inf\{x, y\}$ are denoted by $x \vee y$, $x \wedge y$ respectively. If so,

$$|x| = \sup\{x, -x\}$$

is the *absolute value* of x and if L is also a normed space such that

$$\|\,|x|\,\| = \|x\|$$

for any $x \in L$, then (L, K) is called *normed lattice*.

A wedge K of L is called *generating* if the subspace $K - K$ it generates in L is equal to L itself. If K is a generating wedge in L, then K^0 is a cone of L^*.

Let us note that the usual partial ordering relation on $\mathbb{R}^S$ is defined as follows: $x \geq y$ if and only if $x_s \geq y_s$ for any $s = 1,\ 2,\ \ldots,\ S$, where $x, y \in \mathbb{R}^S$. The positive cone of this ordering is denoted by $\mathbb{R}_+^S$. Also, the usual partial ordering of an $L^p(\Omega, \mathcal{F}, \mu)$ space, where $(\Omega, \mathcal{F}, \mu)$ is a probability space is the following: $X \geq Y$ if and only if the set

$$\{\omega \in \Omega\ :\ X(\omega) \geq Y(\omega)\}$$

is a set of probability 1.

We remind the following essential result about wedges in Euclidean spaces: Consider some wedge K of the space $\mathbb{R}^S$. Then $K^{00} = (K^0)^0 = \overline{K}$. Throughout the article we suppose that the cone L_+ of L is $\sigma(L, L^*)$-closed and we need the following result.

Proposition 8.1. *L^0_+ is a $\sigma(L^*, L)$- closed subset of L^*.*

Proof. Consider a net $(p_\lambda)_{\lambda\in\Lambda} \subseteq L^0_+$ such that

$$p_\lambda \overset{\sigma(L^*,L)}{\to} p \,,$$

. Then $p_\lambda(X) \geq 0$ for any $X \in L_+$. From the weak-star convergence of $(p_\lambda)_{\lambda\in\Lambda}$ we get $p_\lambda(Y) \to p(Y)$ for any $Y \in L$, hence the same holds for any $Y \in L_+$. Hence for any $Y \in L_+$, $p(Y)$ is the limit of a convergent net consisted by positive real numbers. Finally, we get that $p(Y) \geq 0$ for any $Y \in L_+$ which implies $p \in L^0_+$. ☐

Proposition 8.2. *If the cone L_+ of L is $\sigma(L, L^*)$-closed and L is reflexive, then*

$$(L^0_+)^0 := L^{00}_+ = L_+ \,. \tag{8.4}$$

Proof. It suffices to show that $L_+ \subseteq L^{00}_+$ and $L^{00}_+ \subseteq L_+$.

Let $X \in L_+$. Then $X \in L^{00}_+$ since $L^{00}_+ = \{Y \in L^{**} \,|\, Y(\pi) \geq 0, \text{ for any } \pi \in L^0_+\}$. Since L is reflexive, $L^{00}_+ = \{Y \in L \,|\, \hat{Y}(\pi) \geq 0, \text{ for any } \pi \in L^0_+\}$. As far as $X \in L_+$, then $\hat{X}(\pi) = \pi(X) \geq 0$ for any $\pi \in L^0_+$, and the inclusion $L_+ \subseteq L^{00}_+$ is deduced.

For the converse inclusion let us suppose that there were some $Y_0 \in L^{00}_+$ for which $Y_0 \notin L_+$. Since L_+ is a weakly closed and convex set, while the singleton $\{Y_0\}$ is a weakly compact set, which is also convex. Then, from the Separation Theorem for disjoint convex sets in locally convex spaces (see for example [3, Th. 5.58]), there is some $\pi_0 \in L^*, \pi \neq 0$ which strongly separates them. In other words, there is some $\alpha \in \mathbb{R}$ and some $\delta > 0$ such that

$$\pi_0(Y_0) \leq \alpha < \alpha + \delta \leq \pi_0(Y) \tag{8.5}$$

for any $Y \in L_+$. Since L_+ is a wedge, π_0 takes positive values on the elements of it, since if we suppose that there is some $Y_1 \in L_+$ such that $\pi_0(Y_1) < 0$, then if $\lambda \to +\infty$, $\lambda Y_1 \in L_+$ and

$$\pi_0(\lambda Y_1) = \lambda \pi_0(Y_1) \to -\infty \,,$$

a contradiction from the separation argument. Hence, π_0 takes positive values on L_+, thus $\pi_0 \in L^0_+$. For $Y = 0$ we obtain from (8.5), $\pi_0(Y_0) < 0$, while we supposed that $Y_0 \in L^{00}_+$, which implies $\hat{Y_0}(\pi_0) = \pi_0(Y_0) \geq 0$. This is a contradiction deduced from the assumption that such a Y_0 exists. Namely, the inverse inclusion $L^{00}_+ \subseteq L$ is also true. ☐

Lemma 8.3. *If C is a closed wedge of some normed linear space L then the condition $C^0 = \{0\}$ is equivalent to $C = L$.*

Proof. If $C = L$ then if $y_0 \in C^0$ $y_0(x) \geq 0$ for any $x \in L$, which implies $y_0(-x) = -y_0(x) \geq 0$. Then for any $x \in L$, we get $y_0(x) = 0$ and by the properties of $\langle L, L^*\rangle$ as a dual pair (see for example Definition 5.79 in [2]) we get $y_0 = 0$. On the other hand, if C is such that $C^0 = \{0\}$, then if we suppose that there is some $x_0 \in L \setminus C$, then by applying the Strong Separation Theorem for disjoint convex sets in locally convex spaces, we have that $\{x_0\}$ is a compact convex set and C is a closed convex set in L which are disjoint. This implies the existence of some $f \in L^*, f \neq 0$ and of some $a \in \mathbb{R}, \delta > 0$ such that

$$f(x_0) \leq a < a + \delta \leq f(c),$$

for any $c \in C$. But f takes positive values on C since if there is some c_0 with $f(c_0) < 0$ then for $\lambda \to \infty$ we would take that $lim_{\lambda \to \infty} f(\lambda c_0) = -\infty$, while $\lambda c_0 \in C$ since C is a wedge. But in this case the separation condition would be violated and this is a contradiction. Hence $f \in C^0 = \{0\}$, a contradiction since $f \neq 0$ by the separation argument. We were led to a contradiction by supposing that a $x_0 \in L \setminus C$ exists. Hence $C = L$ and the conclusion is ready. □

9. Appendix: Vector Optimization, Convexity

Let L be a normed linear space partially ordered by the cone P and let C be a non-empty set of L. In an equivalent way, *the set of maximal points* of C with respect to P is the set of $y \in C$ such that $C \cap (y + P) = \{y\}$. The set of the maximal points of C with respect to P is denoted by $E_M(C, P)$ or $E(C, P)$, called *set of Pareto efficient points of C (with respect to P)*. The set which consists of the solution sets of any maximization problem of the form

$$Maximize\ f(x)\ subject\ to\ x \in C\,, \tag{9.6}$$

where $f \in E^*$ is a strictly positive functional of P is denoted by $PosE(C, P)$ and it is a subset of $E(C, P)$ (see [21]) called *set of positive proper Pareto efficient points of C (with respect to P)*.

Theorem 9.1. *(Essential theorem about existence of efficient points- [6]) Let C be a non-empty subset of a partially ordered topological linear space L partially ordered by a closed cone P. Then :*

- (*i*) *If the set C has a closed section which has an upper bound and the ordering cone P is Daniell, then there is at least one maximal element of C.*
- (*ii*) *If the set C has a closed and bounded section, the ordering cone P is Daniell and X is boundedly order complete, then there is at least one maximal element of C.*
- (*iii*) *If the set C has a compact section, then there is at least one maximal element of the set C.*

The subset $C \cap (y + P) = C_y$ of C is called *y-section of C* or else *section of C defined by y*.

A topological linear space L is L is *boundedly order complete* if for every bounded increasing net in the space X, the supremum of the elements of it exists. A cone P of a linear topological space L is called *Daniell cone* if every increasing net of L which is upper bounded converges to its supremum.

Note that every closed cone in a Euclidean space is Daniell and every Euclidean space partially ordered by a closed cone is boundedly order complete (nets in these case are replaced by sequences).

A subset F of a convex set C in L is called *extreme set* or else *face* of C, if whenever $x = az + (1 - a)y \in F$, where $0 < a < 1$ and $y, z \in C$ implies $y, z \in F$. If F is a singleton, F is called *extreme point* of C.

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