Max Planck and the birth of the quantum hypothesis

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Based on the functional dependence of entropy on energy, and on Wien's distribution for black-body radiation, Max Planck obtained a formula for this radiation by an interpolation relation that fitted the experimental measurements of thermal radiation at the Physikalisch Technishe Reichanstalt (PTR) in Berlin in the late 19th century. Surprisingly, his purely phenomenological result turned out to be not just an approximation, as would have been expected, but an *exact* relation. To obtain a physical interpretation for his formula, Planck then turned to Boltzmann's 1877 paper on the statistical interpretation of entropy, which led him to introduce the fundamental concept of energy *discreteness* into physics. A novel aspect of our account that has been missed in previous historical studies of Planck's discovery is to show that Planck could have found his phenomenological formula partially derived in Boltzmann's paper in terms of a variational parameter. But the dependence of this parameter on temperature is not contained in this paper, and it was first derived by Planck. © *2016 American Association of Physics Teachers.*
[http://dx.doi.org/10.1119/1.4955146]

I. INTRODUCTION

One of the most interesting episodes in the history of science was Max Planck's introduction of the quantum hypothesis, at the beginning of the 20th century. The emergence of this revolutionary concept in physics is a fascinating story that has been described previously,[1–12] but important aspects of this discovery are generally not found in the description of Planck's ideas in physics textbooks that discuss quantum mechanics. In particular, most physics textbooks do not mention how the concept of *discreteness* in energy, the revolutionary concept introduced by Planck to describe the spectrum of black-body radiation, originated in the first place.[8] From Planck's articles and correspondence on his theory of the spectrum of black-body radiation, it is clear that he took this concept directly from Boltzmann, who in his seminal 1877 paper on statistical mechanics discretized energy as a purely *mathematical device* in order to be able to *count* the possible configurations of a molecular gas in thermal equilibrium.[13] But this important connection between Planck's and Boltzmann's work has not been mentioned even in physics textbooks that emphasize a historical approach.[14] For example, in the description of Planck's discovery in his biography of Einstein, Abraham Pais concludes that:[15]

> "His [Planck's] reasoning was mad, but his madness has that divine quality that only the greatest transitional figures can bring to science."

This comment does not provide any more enlightenment on the origin of the idea of quantization in physics than Richard Feynman's succinct statement in his well-known Lectures on Physics, that:[16]

> "…by fiddling around [Planck] found a simple derivation [for his formula]."

Most accounts of Planck's discovery in physics textbooks are historically inaccurate, and Martin Klein's early analysis of Planck's work[3] debunked some myths contained in these books. For example, one of the most common myths is that Planck was responding to the problem in the classical theory of black-body radiation known as the *ultraviolet catastrophe*; this occurs when the equipartition theorem for a system in thermal equilibrium is applied to the spectral distribution of thermal radiation. But at the time, Planck appears to have been unaware of this problem, which was named by Ehrenfest several years after Planck's discovery. Indeed, the application of the equipartition theorem to black-body radiation was made by Lord Rayleigh[17] at about the same time that Planck obtained his famous formula for the black-body spectrum. There isn't any evidence that Planck was aware of Rayleigh's result, which agreed with new experiments for the long wavelength end of the spectrum observed at that time. Klein concluded that:[3]

> "it was probably a very good thing that Planck was not constrained in his thinking by the tight classical web which Rayleigh had woven."

In Boltzmann's 1877 paper, the mean energy of his fictitious molecular ensemble with discrete energies in multiples of a unit ϵ is obtained in terms of an undetermined variational parameter, but he calculated the temperature dependence of this parameter only in the limit relevant to classical mechanics. It has remained unnoticed that his result corresponds to Planck's formula for the black-body radiation spectrum (see Ref. 13, p. 181, and Appendix A).

In essence, Planck's approach to the theory of black-body radiation was based on the following steps. Taking advantage of Kirchhoff's theorem that the black-body distribution is a universal function independent of the nature of the source of radiation, Planck's first step was to obtain a relation for the energy distribution of this radiation in thermal equilibrium with an ensemble of microscopic Hertzian oscillators with variable frequency ν. By applying Wien's distribution that fitted the high frequency end of this radiation, and Maxwell's equations for the electromagnetic field, Planck obtained an expression for the mean energy of these oscillators. The measurements were made by careful experiments at the Physikalisch Technische Reichanstalt (PTR), which was the center for infrared radiation studies in Berlin at the end of the 19th century (see Fig. 1). After it was discovered that Wien's distribution did not fit new data at lower
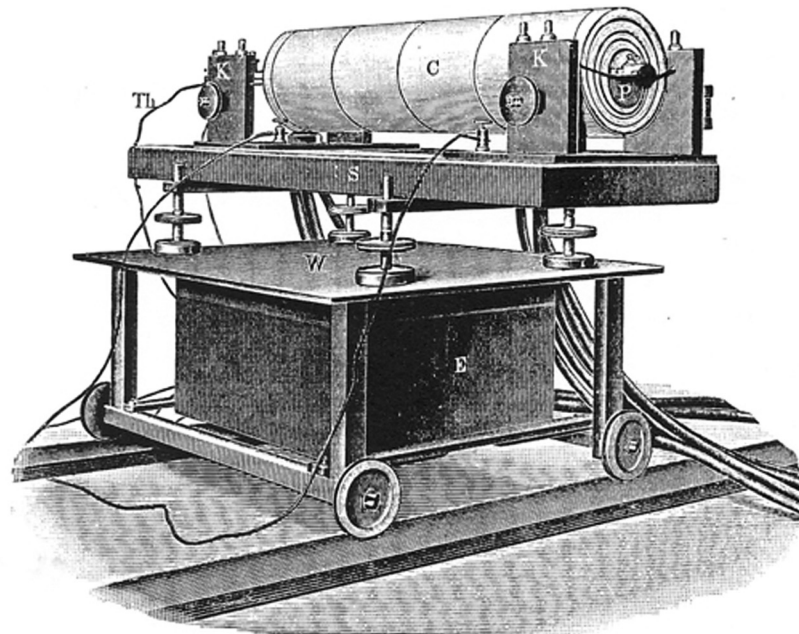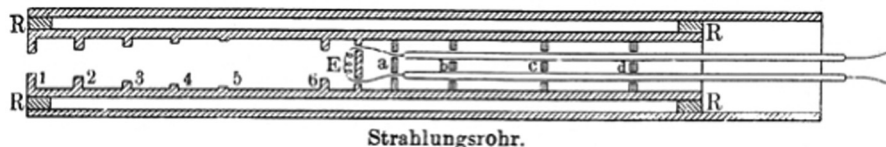
Schwarzer Körper nach Lummer-Kurlbaum.

Strahlungsrohr.

Fig. 1. Apparatus of Lummer and Kurlbaum to measure the spectrum of black-body radiation. An electrical current heats the filament E located in a tube inside the cylinder C to a fixed temperature T, giving rise to black-body radiation inside this cylinder. The spectrum of this radiation is observed by some radiation exiting through the hole at one end along the axis of the cylinder.

frequencies, Planck obtained a new distribution formula by an *interpolation* based on his application of the relationship between entropy and energy for a system in thermal equilibrium. Finally, to obtain a theoretical interpretation for his new formula, Planck turned to the seminal 1877 paper of Boltzmann, which formulates the relation between entropy and statistics.[13]

In his paper, Boltzmann introduced a relation between the entropy of a molecular gas and the *number* of microscopic configurations, or *complexions* (as he called them), of the molecules. He defined the state of thermal equilibrium to be the maximum number of these configurations subject to the constraint of a fixed number of molecules and total energy. At first sight, it is surprising that Boltzmann's ideas, based on purely classical concepts applying to systems having continuous energy, could have served as the springboard for Planck's quantum hypothesis of discrete energy levels. But to implement his statistical ideas, Boltzmann took for his initial example a fictitious model of a gas, whose molecules had *discrete* energies in integer multiples of an energy element of magnitude ϵ. For Boltzmann, this discretization of energy was purely a mathematical artifact that he introduced for the purpose of *counting* the number of configurations of the molecules. Subsequently, as would be expected, he took the limit of continuous molecular energy for which ϵ vanished. But when Planck applied Boltzmann's discrete model to his ensemble of Hertzian oscillators in thermal equilibrium with radiation, he did not take this continuum limit. Instead, he set Boltzmann's energy elements to a fixed value

$\epsilon = h\nu$, where ν is the frequency of his oscillators and h is a new universal constant, now known as Planck's constant, that relates frequency and energy. It was very fortunate for Planck that Boltzmann initially considered energy as the only degree of freedom of the molecules in his ensemble because that made possible Planck's direct extension to an ensemble of linear harmonic oscillators.[18]

Planck was aware that with his procedure he was violating the tenets of continuum physics. In his December 19, 1900 paper, presented at a meeting of the German Physical Society, he wrote that:[19]

> "If E [the total energy] is considered to be a continuous divisible quantity this distribution is possible in infinitely many ways. We consider, however—this is the most essential point of the whole calculation—E to be composed of a well-defined number of equal parts [of magnitude ϵ] and use thereto the constant of nature $h = 6.55 \times 10^{-27}$ erg $\times$ sec [setting $\epsilon = h\nu$]."

There were, however, inconsistencies in Planck's introduction of discrete energy for his Hertzian oscillators because in his derivation of the relationship between the black-body energy spectrum and the mean energy of these oscillators, Planck applied continuum mechanics and Maxwell's equations for electromagnetism. For example, an obvious question would have been to explain how Planck's oscillators could be restricted to discrete energies while changing energy by emitting and absorbing electromagnetic waves in a continuous

manner. This problem did not affect Boltzmann because he could assume that initially his molecules transfer energy in discrete units, but in the end, he took a continuum limit.

Many years later, Einstein commented that:[20]

> "…all my attempts…to adapt the theoretical foundations of physics to the edge failed completely. It was as if the ground had been pulled from under one, with no firm foundation to be seen anywhere."

In 1905, however, he resolved the conundrum by assuming that electromagnetic radiation also consisted of discrete energy quanta.[21] But Planck did not show such concern, and instead, for several years he attempted to incorporate his new results within the realm of continuum classical physics. Taking again another idea from Boltzmann's 1877 paper, Planck later considered the energy of the oscillator to be continuous, and $\epsilon = h\nu$ to be the magnitude of cells of equal probability in the phase space of the oscillators. Otherwise, the derivation of his formula proceeds in precisely the same form as before. In 1906, and again as late as 1909, he presented his derivation in lectures that he gave during his visit at Columbia University. But had Planck closely followed Boltzmann's statistical method, he could have realized earlier that a continuum energy interpretation of his formulae was not feasible. In a card to Ehrenfest in the spring of 1915, Planck wrote, "I hate discontinuity of energy even more than discontinuity of emission."

The main purpose of this paper is to clarify the relationship between Boltzmann's and Planck's work by providing a thorough mathematical discussion that is often absent in the literature on this subject. In the following Secs. II–IV, Planck's work is discussed as described in some of his publications, his autobiographical recollections,[22] his Nobel speech,[23] and in some of his correspondence. Section II reviews Planck's original serendipitous derivation of his well-known formula for black-body radiation, which he referred to as his "lucky intuition." Section III describes his application of Boltzmann's principles of statistical mechanics, and Sec. IV describes some of Planck's recollections on how he discovered his fundamental radiation formula. The relationship between Boltzmann's work and Planck's application of it is given in Appendix A, which also contains some new mathematical insights concerning this relation. Finally, Appendix B discusses some of the controversies among historians of science about Planck's role in the introduction of the quantum.

II. PLANCK'S PHENOMENOLOGICAL DERIVATION OF HIS BLACK-BODY FORMULA

An insightful description of how Planck obtained his famous formula for the spectrum of black-body radiation can be found in his scientific autobiography.[22] This account was written many years after the occurrence of this event, and may suffer from the usual lapses of memory and the absence of original documents and correspondence. Planck's own papers and correspondence were destroyed when Berlin was bombed in WWII. It appears, however, to be consistent with Planck's original publications. Therefore, here Planck will speak for himself, while, for clarification, some of the mathematical details will be filled in (keeping his original notation) in a form close to his original articles.

Planck wrote:[22]

> "While a host of outstanding physicists worked on the problem of spectral distribution, both from the experimental and the theoretical aspects every one of them directed his efforts solely toward exhibiting the dependence of the intensity of the radiation on the temperature. On the other hand, I suspected that the fundamental connection lies in the *dependence of entropy upon energy* [my italics]. As the significance of entropy had not yet come to be fully appreciated, nobody paid any attention to the method adopted by me, and I could work out my calculations completely at my leisure, with absolute thoroughness, without fear of interference or competition. Since for the irreversibility of the exchange of energy between an oscillator and the radiation activating it, the second differential quotient of its entropy with respect to its energy is of characteristic significance, I calculated the value of this function on the assumption that Wien's law of the Spectral Energy Distribution is valid—a law which was then in the focus of general interest; I got the remarkable result that on this assumption the reciprocal of that value, which I shall call here R, is proportional to the energy."

On theoretical grounds, Wien had proposed[26] that the spectral energy distribution for black-body radiation with frequency ν at temperature T had the scaling form

$$\rho(\nu, T) = \nu^3 f(\nu/T), \tag{1}$$

where f is a function of a single variable, the ratio of frequency ν, and temperature T. This form satisfies the Stefan-Boltzmann relation that the total black-body energy is proportional to the fourth power of the temperature T. Originally, this dependence was found experimentally by Josef Stefan, and later a theoretical derivation was provided in 1884 by his former student Boltzmann.[27] Boltzmann's method was succinct: applying Maxwell's relation between the energy per unit volume E, and the pressure p of isotropic radiation, $p = E/3$ leads to a relation for the entropy S_R per unit volume of this radiation

$$S_R = \frac{4}{3}\frac{E}{T}. \tag{2}$$

Substituting for the temperature T in this relation the thermodynamic condition

$$\frac{1}{T} = \frac{dS_R}{dE}, \tag{3}$$

and integrating the resulting differential equation yields $S_R = c'E^{3/4}$. Eliminating S_R by applying again Eq. (2), one obtains the relation $E = \sigma T^4$, known as the Stefan-Boltzmann law, where $\sigma = (3c'/4)^4$ is a universal constant.

According to Wien's spectral distribution in Eq. (1), integrating the spectrum over all frequencies and setting $z = \nu/T$ as the variable of integration, one recovers the Stefan-Boltzmann relation

$$E = \int_0^\infty \rho(\nu, T)d\nu = \sigma T^4, \tag{4}$$

where σ is now determined by

$$\sigma = \int_0^\infty z^3 f(z) dz. \tag{5}$$

Probably stimulated by earlier phenomenological work by Paschen, Wien assumed for the function $f(z)$ the exponential form

$$f(z) = \frac{8\pi a}{c^3} e^{-bz}, \tag{6}$$

where a and b are constants that could be obtained by fitting his theoretical distribution of Eq. (1) to the black-body radiation experiments. The constant a has the dimensions of energy time and later it will be seen to correspond to Planck's constant h. For this form of f, according to Eq. (5), $\sigma = 48\pi a / b^4 c^3$.

Subsequently, in a series of five papers written between 1897 and 1899, Planck discussed the thermal equilibrium between the radiation in a cavity and an ensemble of Hertzian electromagnetic oscillators,[28,29] based on Maxwell's theory of electromagnetism. His main result was a relation between the spectral distribution $\rho(\nu, T)$ and the mean energy $U(\nu, T)$ of the oscillators

$$\rho(\nu, T) = \frac{8\pi}{c^3} \nu^2 U(\nu, T). \tag{7}$$

Combining this result with Wien's relation, Eqs. (1) and (6), implies that

$$U(\nu, T) = a\nu e^{-b\nu/T}. \tag{8}$$

In this expression, the constant a (which, like Wien's corresponding constant a) has the dimensions of energy *times* time and turns out to be equal to Planck's constant h. By fitting the data on blackbody radiation obtained in the experiments by Otto Lummer and Ernst Pringscheim[30] on radiation emitted from a small hole in a heated cavity (see Fig. 1), Planck obtained $h = 6.885 10^{-27}$ erg s, in remarkable correspondence to the modern value $h = 6.626 10^{-27}$ ergs, a tribute to the accuracy of the black body radiation experiments at that time. Neither Wien, Planck, nor anyone else seemed to notice, however, until it was pointed out by Lord Rayleigh several years later,[17] that the Wien exponential law, Eq. (6), implied the implausible result that as the temperature T increases the magnitude of the spectral distribution at a fixed frequency ν approaches a constant value $\rho(\nu, T) = 8\pi h \nu^3 / c^3$, and $U = h\nu$, independent of T.

Planck's next step was to consider the dependence of the entropy $S(\nu, U)$ of his oscillators on the energy U. Given the relation between the energy U and the temperature T [of Eq. (8)], he obtained this dependence from the thermodynamic relation

$$\frac{1}{T} = \frac{dS}{dU}. \tag{9}$$

Inverting Eq. (8) to obtain T as a function of U, and substituting the result in Eq. (9), gives a first-order differential equation for S:

$$\frac{dS}{dU} = -\frac{1}{b\nu} \ln\left(\frac{U}{a\nu}\right). \tag{10}$$

Integrating this equation with the boundary condition that S vanishes when $U = 0$ gives

$$S = -\frac{U}{b\nu} \left[\ln\left(\frac{U}{a\nu}\right) - 1 \right]. \tag{11}$$

In the last of a series of five papers by Planck on irreversible radiation processes,[29] this expression appears, without any justification, as a *definition* for the entropy of his oscillators. But as has been shown, it is clear that Planck obtained it in a straightforward fashion from Wien's relation of Eqs. (1) and (6).[31] Taking the second derivative of S with respect to U, he found that its reciprocal depends linearly on U or

$$R = \left(\frac{d^2 S}{dU^2}\right)^{-1} = -b\nu U. \tag{12}$$

While Planck obtained this simple linear dependence of R on U from Wien's relation, he attached to it a special significance claiming to have demonstrated that it was *unique*, leading to a *derivation* of the scaling dependence of $U(\nu, T)$ on ν and T, Eq. (8). Integrating this equation gives

$$\frac{dS}{dU} = \frac{1}{b\nu} \ln\left(\frac{U}{\xi(\nu)}\right), \tag{13}$$

where $\xi(\nu)$ is an undetermined function of ν. Hence, the fact that Wien's relation indicates that $\xi(\nu)$ depends linearly on ν was not justified.

In a paper presented to the Berlin Academy of Sciences on May 18, 1899, Planck stated that:[32]

> "I believe it must therefore be concluded that the definition given for the entropy of radiation, and also the Wien distribution law for the energy which goes with it, is a necessary consequence of applying the principle of entropy increase to the electromagnetic theory of radiation, and that the limits of validity of this law, should there be any, therefore coincide with those of the second law of thermodynamics. Further experimental test for this law naturally acquires all the greater fundamental interest for this reason."

Later on in his autobiography, Planck recalled that:[22]

> "This relationship is so surprisingly simple that for a while I considered it to possess universal validity, and I endeavored to prove it theoretically. However, this view soon proved to be untenable in the face of later measurements. For although in the case of small energies and correspondingly short waves Wien's Law continued to be confirmed in a satisfactory manner, in the case of large values of the energy and correspondingly long waves, appreciable divergences were found, first by Lummer and Pringsheim; and finally the measurements of H. Rubens and F. Kurlbaum on infrared rays of fluorspar and rock salt revealed a behaviour which, though totally different, is again a simple one, in so far as the function R is proportional not to the energy but to the square of the energy for large values of the energy and the wave-lengths."

By early 1900, the experiments of Otto Lummer and Ernst Pringscheim[30] gave evidence of deviations from Wien's

formula at the longer observed wavelengths of order 10 μm, and at temperature of about 1000 °C.[33] Further data by Heinrich Rubens and Felix Kurlbaum at a wavelength of 51 μm indicated that the black body radiation depends linearly on temperature.[34] These experiments were made possible by a new detection technique developed by Heinrich Rubens and his American collaborator Ernst F. N. Nichols, which enhanced the low intensity longer wavelengths by resonant scattering from a crystal lattice.[35] Planck was informed of these new results by Rubens himself, who visited him with his wife on a Sunday afternoon (Oct. 7, 1900), and he began promptly to reconsider his arguments.

Even before the new data appeared, Lord Rayleigh derived a linear dependence on temperature for the blackbody distribution from the equipartition theorem, applied to classical radiation emitted by charged one-dimensional oscillators in a box in thermal equilibrium.[17] To obtain this dependence on temperature, Planck found that his expression for R of Eq. (12) had to depend quadratically on U.[28] Supposing that

$$U(\nu, T) = \eta T, \tag{14}$$

where η is a constant (named α in Planck's paper and corresponding to k), then according to Eq. (9)

$$\frac{1}{T} = \frac{dS}{dU} = \frac{\eta}{U}, \tag{15}$$

and therefore

$$R = \left(\frac{d^2 S}{dU^2}\right)^{-1} = -\frac{U^2}{\eta}. \tag{16}$$

In Planck's own words,[22]

> "Thus, direct experiments established two simple limits for the function R: for small energies, R is proportional to the energy; for larger energy values R is proportional to the square of the energy. *Obviously, just as every principle of spectra energy distribution yields a certain value for R, so also every formula for R leads to a definite law of the distribution of energy*. The problem was to find such a formula for R which would result in the law of the distribution of energy established by measurement. Therefore, the most obvious step for the general case was to make the values of R equal to the sum of a term proportional to the first power of the energy and another term proportional to the second power of the energy, so that the first term becomes decisive for small values of the energy and the second term for large values. In this way a new radiation formula was obtained, and I submitted it for examination to the Berlin Physical Society, at the meeting on October 19, 1900."

By such phenomenological considerations, Planck generalized his thermodynamic expression for the dependence of the entropy on the oscillator energy to *interpolate* between the short wavelength or Wien regime, and the long wavelength or Rayleigh regime. Setting now

$$R = \left(\frac{d^2 S}{dU^2}\right)^{-1} = -\frac{1}{\eta} U(\eta b \nu + U), \tag{17}$$

he obtained his previous linear dependence of R on U, Eq. (12), for $U \ll \eta b \nu$ and the quadratic dependence on U, Eq. (16), for $U \gg \eta b \nu$. This simple interpolation formula for R turned out, surprisingly, to be valid not only in these two energy regimes, but to be an *exact* relation for all values of U. Integrating this relation by applying the thermodynamic relation between the absolute temperature and the derivative of the entropy with respect to the energy [Eq. (9)], and assuming the boundary condition $U \to \infty$ when $T \to \infty$, one obtains

$$\frac{1}{T} = \frac{1}{b\nu} \ln(1 + \eta b \nu / U), \tag{18}$$

which yields the dependence on temperature T and frequency ν of the mean oscillator energy

$$U(\nu, T) = \frac{\eta b \nu}{\exp(b\nu/T) - 1}. \tag{19}$$

Finally, to recover the relation for $U(\nu, T)$ in the Wien limit when $b\nu \gg T$, Planck obtained a relation for the new constant η:

$$\eta = \frac{h}{b}. \tag{20}$$

Since $b = h/k$, where k is Boltzmann constant, $\eta = k$ in accordance with the equipartition theorem for a one-dimensional harmonic oscillator. Substituting this expression into Eq. (7) for his relation between the spectral distribution $\rho(\nu, T)$ and the mean oscillator energy $U(\nu, T)$, Planck then obtained his blackbody formula which he wrote as a function of the wavelength λ measured in the experiments[28] as

$$\rho(\lambda, T) = \frac{C\lambda^{-5}}{\exp(bc/\lambda T) - 1}, \tag{21}$$

where $C = 8\pi hc$, $\lambda = c/\nu$, and c is the velocity of light. In the limit $bc/\lambda \gg T$, Planck recovered his earlier result for the Wien spectrum, Eq. (6), while for $bc/\lambda \ll T$, he obtained the linear dependence of ρ on T, in accordance with the new experimental results at the PTR. It should be pointed out that it is completely unexpected that by an interpolation procedure to fit experimental data Planck obtained a formula for the spectral distribution of black body radiation that turned out to be *exact* for all temperatures and wavelengths. His procedure was sensible as a phenomenological data fitting approach, but it is purely accidental that he succeeded in this way to obtain the exact formula for black-body radiation. After all, he did not have any arguments to exclude, for example, cubic or higher powers of U in his expansion of R in powers of U [Eq. (17)].

After Rubens checked the new radiation formula against his experiments, Planck described his reaction:[22]

> "The very next morning I received a visit from my colleague Rubens. He came to tell me that after the conclusion of the meeting, he had that very night checked my formula against the results of his measurements and found a satisfactory concordance at every point …Later measurements too confirmed

my radiation formula again and again—the finer the methods of measurement used, the more accurate the formula was found to be. …In this way a new radiation formula was obtained, and I submitted it for examination to the Berlin Physical Society, at the meeting of October 19, 1900.”

Planck, however, did not refer to Rayleigh's result, whose work was apparently motivated by the unphysical dependence of Wien's formula that predicted the spectral energy $\rho(\lambda, T)$ saturates when $\lambda T \gg hc/k$. To this Rayleigh remarked,[17]

“Nevertheless, the [Wien's] law seems rather difficult of acceptance, especially the implication that as the temperature is raised, the radiation of given wavelength approaches a limit…The question is one to be settled by experiment; but in the meantime I venture to suggest a modification of the Wien distribution, which appears to me more probable *apriori. Speculations upon this subject are hampered by the difficulties which attend the Boltzmann-Maxwell doctrine of the partition of energy* [my italics]. According to this doctrine every mode should be alike favoured; and although for some reason not yet explained the doctrine fails in general, it seems plausible that it applies for the graver [longer wavelengths] modes.”

III. PLANCK'S APPLICATION OF BOLTZMANN'S RELATION BETWEEN ENTROPY AND PROBABILITY IN STATISTICAL MECHANICS

This section reviews the critical phase, when in order to find the physical significance for his purely phenomenological formula for black-body radiation of Eq. (21), Planck turned to Boltzmann's 1877 seminal paper on the foundations of statistical mechanics for enlightenment. Later, as Planck described it in his autobiography,[22]

“But even if the absolute precise validity of the radiation formula is taken for granted, so long as it had merely the standing of a law disclosed by lucky intuition, it could not be expected to possess more than a formal significance. For this reason, on the very day when I formulated this law, I began to devote myself to the task of investing it with a true physical meaning. *This quest, as a matter of course, led me to study the interrelation of entropy and probability—in other words to pursue the line of thought inaugurated by Boltzmann* [my italics].”

In his Nobel speech,[23] Planck added the remark,

“After a few weeks of the most strenuous work of my life, the darkness lifted and an unexpected vista began to appear.”

and continued,

“Since the entropy S is an additive magnitude, but the probability W is a multiplicative one, I simply postulated that $S = k \log W$, where k is a universal constant; and I investigated whether the formula

for W, which is obtained when S is replaced by its value corresponding to the above radiation law could be interpreted as a measure of probability.”

By integrating Eq. (18), with the boundary condition that the entropy S vanishes when $U = 0$, Planck obtained

$$S = k[(1 + U/h\nu)\ln(1 + U/h\nu) - (U/h\nu)\ln(U/h\nu)], \tag{22}$$

where $k = h/b$ is Boltzmann's constant. This form of the dependence of S on U and ν satisfies the Wien's scaling relation [Eq. (1)], indicating that S depends only on the ratio U/ν.

Having found a phenomenological expression for the entropy S as a function of the mean energy U, Planck considered the possibility that the function $W = \exp(S/k)$ could be interpreted as a measure of the probability for the configuration of his Hertzian oscillators, corresponding to that of molecular velocities in Boltzmann's 1877 formulation of statistical mechanics.[13] In his Nobel speech,[23] Planck remarked,

“As a result, I found that this was actually possible, and that in this connection k represents the so-called absolute gas constant, referred not to gram molecules or mols, but to the real molecules. Now for the magnitude W, I found that in order to interpret it as a probability, it was necessary to introduce a universal constant, which I called h. Since it had the dimension of action [*energy* × *time*], I gave it the name *elementary quantum of action*. Thus the nature of entropy as a measure of probability, in the sense indicated by Boltzmann, was established in the domain of radiation, too.”

Further evidence for Planck's train of thought can be found in one of the few surviving letters from that period, which he wrote to Otto Lummer on 26 October 1900,[24]

“If the prospect should exist at all of a theoretical derivation of the radiation law, which I naturally assumed, then in my opinion, this can be the case only if it is possible to derive the expression for the probability of a radiant state, and this, you see, is given by the entropy. Probability presumes disorder, and in the theory I have developed, the disorder occurs in the irregularity with which the phase of the oscillations changes even in the most homogeneous light. A resonator, which corresponds to a monochromatic radiation, in resonant oscillations will likewise show irregular changes of its phase [and also of its instantaneous energy, which was more important for Planck's subsequent derivation], and on this the concept and magnitude of its entropy are based. According to my [blackbody radiation] formula communicated on 19 October to the German Academy] the entropy of the resonator should become

$$S = \alpha \ln[(\beta + U)^{\beta+U}/U^U], \tag{23}$$

and this form very much recalls expression occurring in the probability calculus.”

The equation that Planck wrote for the entropy S in this letter corresponds to Eq. (22) for $\beta = h\nu$, apart from an additive constant $\alpha\beta \ln \beta$, where $\alpha\beta = k$. Planck continued,[24]

> "After all, in the thermodynamic of gases, too, the entropy S is the log of a probability magnitude, and Boltzmann has already stressed the close relationship of the function χ^{χ} which enters the theory of combinatorics, with the dynamic entropy. I believe, therefore, that the prospect would certainly exist of arriving at my formula by a theoretical route which would also give us the physical significance of the constants C and c."

To illustrate his statistical principles for thermal equilibrium, Boltzmann had considered a gas of molecules and in order to count the number of configurations or *complexions*, as he called them, he discretized the energy of the molecules in integral multiples of a unit or energy element ϵ. Then each configuration is specified by a set of integers giving the number of energy elements of each molecule subject to the constraint that the mean energy U per molecule satisfies the relation $U = (P/N)\epsilon$, where P is an integer and N is the total number of molecules. It is fairly plausible, as indicated in his letter to Lummer, that Planck started with his empirically successful relation for the entropy S of the oscillators of Eq. (22). Then, working backwards, he proceeded to obtain the quantity $W = \exp(S/k)$ and associated it with the total number of configurations of his oscillators, in analogy with Boltzmann's model for a gas of molecules.[2] In this case, replacing the ratio $U/h\nu$ that appears in Eq. (22) for S by P/N, Planck would have obtained the relation

$$W = \frac{(N + P)^{(N+P)}}{N^N P^P}, \qquad (24)$$

which is Stirling's approximation to the total number of complexions given in Boltzmann's paper.

In two articles[19,36] that he presented to the German Physical Society, the first one given on Dec. 14, 1900, Planck now derived his entropy formula by starting with Eq. (24) for the total number W of equally probable complexions of his oscillators. He then obtained the entropy S from the relation $S = k \log W$ by setting $P/N = U/h\nu$, which yields Eq. (22). In this presentation, Planck introduced for the first time the constant $k = h/b$, and pointed out that it is universal and applies to all thermodynamic systems including Boltzmann's model for a gas of molecules. In this case, $k = R_0/N_0$ where R_0 is the gas constant and N_0 is Avogadro's number. Planck also argued that for each frequency ν, there would be corresponding values of N_ν and P_ν such that $P_\nu/N_\nu = U_\nu/h\nu$ and that the total number of configurations W (which is the product of all W_ν) should be maximized for a fixed total energy, but he did not carry out such a calculation.

IV. PLANCK'S 1931 RECOLLECTIONS

In Sec. III, we have made a plausible reconstruction how Planck obtained a theoretical derivation of his black-body formula based on Boltzmann's development of statistical mechanics in 1877. The idea of introducing discrete energy elements of magnitude ϵ originated with Boltzmann, who applied it as a mathematical device to count configurations for a gas of molecules. He subsequently reinstated the classical energy continuum by taking the limit $\epsilon = 0$. Planck followed Boltzmann's approach by assigning integral multiples of these discrete energy elements to his oscillators of frequency ν, but then departed from him in a fundamental way by fixing ϵ to have a constant value, $\epsilon = h\nu$, where h is a new universal constant, instead of letting ϵ become vanishingly small as demanded by classical physics. Why Planck did not take this latter step ($\epsilon \rightarrow 0$) is obvious: the assumption of a discrete energy element of magnitude $h\nu$ gave him precisely the result he was searching for to justify his phenomenological black-body formula. But as would be expected, he did not—and could not—justify such discreteness on *a priori* grounds. It appears that in the short time before Planck submitted his paper to the German Physical Society, he could not have been aware of the broader physical implications of his unprecedented ansatz because he did not offer any comments indicating that he was departing from the canons of classical physics. Planck turned to Boltzmann's approach rather late in his research program, and therefore it seems very likely that in 1900 he had not yet fully mastered the foundation of Boltzmann's statistical mechanics.

In this connection, it is interesting to examine Planck's own account of his discovery some 31 years later. In response to a request from the American physicist Robert Williams Wood, Planck described "the considerations which led him to propose the hypothesis of energy quanta" in a letter he wrote to him on October 7, 1931:[25]

> "Briefly summarized, what I did can be described as simply an act of desperation. By nature I am peacefully inclined and reject all doubtful adventures. But by then I had been wrestling unsuccessfully for six years with the problem of the equilibrium between radiation and matter and I knew that the problem was of fundamental importance to physics; I also knew the formula that expresses the energy distribution in the normal spectrum. A theoretical interpretation therefore had to be found at any cost, no matter how high. *It was clear to me that classical physics could offer no solution to this problem and would have meant that all energy would eventually transfer from matter into radiation* [my italics]. In order to prevent this, a new constant is required to assure that energy does not disintegrate. But the only way to recognize how this can be done is to start from a definite point of view. This approach was open to me by maintaining the two laws of thermodynamics. The two laws, it seems to me, must be upheld under all circumstances. For the rest, I was ready to sacrifice every one of my previous convictions about physical laws. Boltzmann had explained how thermodynamic equilibrium is established by means of statistical equilibrium, and if such an approach is applied to the equilibrium between matter and radiation, one finds that the continuous loss of energy into radiation can be prevented by assuming that energy is forced at the outset to remain together in certain quanta. This was purely a formal assumption and I really did not give it much thought except that no matter what the cost, I must bring about a positive result."

The statements in this letter agree with the description that can be gleaned from the papers and earlier correspondence of Planck, but there are also some oddities in this account. Contrary to Planck's remark that "classical physics could offer no solution to this problem…" he continued to hope, for about a decade after his first application of Boltzmann's discrete statistical approach, that his quantum of action could somehow be justified on purely classical grounds. As he stated in his autobiography,[22]

> "While the significance of the quantum of action for the interrelation between entropy and probability was thus conclusively established, the part played by this new constant in the uniformly regular occurrence of physical processes still remained an open question. I therefore tried immediately to weld the elementary quantum of action h somehow into the framework of classical theory. But in the face of all such attempts, this constant showed itself to be obdurate…"

More likely, originally discreteness in energy for Planck was a "purely formal assumption" that Boltzmann had introduced in 1877 to which Planck "really did not give much thought" because it brought about "a positive result," namely, a theoretical derivation of his black-body formula. Previously, this was obtained from phenomenological considerations by an adaptation of Boltzmann's principles of statistical mechanics to a model of oscillating charges in thermal equilibrium with electromagnetic (black-body) radiation. Planck's good fortune was that these principles could be applied straightforwardly, not only to the systems in thermodynamic equilibrium, which satisfy the laws of classical continuum physics, but also to the systems that are described by discrete energy levels such as Hertzian oscillators in equilibrium with thermal radiation. Planck had been correct "to uphold the two laws of thermodynamics under all circumstances" because, as it turned out, *only* these two laws were not modified by the advent of the new theory of quantum mechanics.

V. SUMMARY AND CONCLUSIONS

The role of Planck in the introduction of the quantum hypothesis has been described in numerous articles in the past,[1–12] and it turns out to be quite controversial (see Appendix B). The account presented here has particularly benefited from the articles by Rosenfeld,[1,2] Klein,[3] and Gearhart.[4] In summary, it has been shown here that:

(a) Assuming that the energy distribution for black body radiation is due to an ensemble of Hertzian oscillators in equilibrium with this radiation at a fixed temperature T, Planck calculated the relation of this distribution to the mean energy of these oscillators, Eq. (7), by applying Maxwell's theory of electromagnetism.

(b) On the basis of Wien's displacement formula for this distribution, Planck then obtained an expression for the *entropy* of his Hertzian oscillators of Eq. (11).

(c) Motivated by new experiments at longer wavelengths that disagreed with Wien's formula, Planck developed a phenomenological interpolation relation, Eq. (22), for his entropy relation that agreed very well with the new and the older experiments.

(d) Finally, Planck took his "most essential step," namely, to obtain a theoretical foundation for his new and successful entropy and radiation formula. For this purpose, Planck turned to Boltzmann's 1877 paper on the connection of entropy S with the *maximum* number of complexions W_B of a system in thermal equilibrium ($S \propto \ln W_B$). In particular, this paper contained an expression for the total number of complexions W for an ensemble of one-dimensional molecules having multiples of a discrete energy element ϵ. Of course, since energy in classical mechanics is continuous, at the end of his calculation Boltzmann took the limit $\epsilon \to 0$, corresponding to the result of the equipartition theorem, first published for the distribution of black body radiation by Lord Rayleigh.[17] Planck, however, realized that by setting $\epsilon = h\nu$, he recovered his successful phenomenological relation for the entropy that led to his famous formula for the spectrum of black body radiation.

Klein[3] and subsequently other historians of science have questioned why Planck departed from Boltzmann's derivation by taking the total number of complexions W instead of its maximum number W_B. The explanation is rather straightforward: Planck had obtained a phenomenological formula for the entropy as the logarithm of a function of the variable U/ϵ, with $\epsilon = h\nu$ [Eq. (22)]. Boltzmann's paper, however, did not contain a corresponding expression for the entropy proportional to $\ln W_B$ as a function of this variable, except in the limit $\epsilon \to 0$ (corresponding to classical mechanics). But his paper contained an expression for the total number of complexion W in terms of the number N of molecules, and an integer $P = E/\epsilon$, where E is the total energy of the system. Substituting $P/N = U/h\nu$ in the Stirling approximation for W leads to an expression for $\ln(W)$ equal to N times Planck's phenomenological expression for the entropy S (apart from the constant of proportionality k). It still remained to be shown that $\ln W = \ln W_B$ in the Stirling approximation, which was not carried out in Boltzmann's paper (see Appendix A). In one of the most recent articles on this subject,[4] Clayton Gearhart posed another question similar to that of other historians of science: "Why should Planck's complexions, which represent distributions of energy elements, be equally probable?" The answer is that the assumption of equally probable complexions is at the foundation of Boltzmann's statistical method, and naturally Planck adopted it because it solved his problem. But for Planck's problem, this assumption was not justified because thermal equilibrium is due to the interaction of Planck's oscillators with radiation, which Planck assumed to follow Maxwell's continuous (not discrete) laws. A detailed discussion of the relation between Boltzmann's and Planck's definition and calculation of entropy is given in Appendix A. In particular, it is shown that in the Stirling approximation, $\ln W = \ln W_B$, an essential relation that Planck could have easily demonstrated, but was not given in any of his papers.

The occurrence of energy discreteness caused a great deal of concern to Planck's contemporaries, most notably Einstein, Lorentz, and Ehrenfest, and it took several years before the significance of Planck's quantum of action began to emerge. The acceptance and further development of Planck's hypothesis is a very interesting and important subject in its own right, but it will not be pursued here.

APPENDIX A: THE RELATION BETWEEN BOLTZMANN'S AND PLANCK'S TREATMENT OF ENTROPY

An interesting question that has been raised by several historians of science is to explain why Planck departed from the method introduced by Boltzmann in his 1877 paper on the relation between the second law of thermodynamics and the theory of probability.[13] In this paper, Boltzmann considered the state of thermal equilibrium of an ensemble of molecules having discrete energies that are multiples of an energy unit with finite magnitude ϵ. Therefore, it would appear that by setting $\epsilon = h\nu$, Planck could just have taken over Boltzmann's result for the mean energy U of these molecules, and shown that it corresponded to the result he had obtained previously from his purely phenomenological considerations. It will be shown, however, that the reason why he could not have taken this apparently straightforward step is that Boltzmann had obtained the mean energy U of his model for molecules with discrete energies in term of a parameter x, but he did not determine the dependence of this parameter on temperature except in the classical limit when $\epsilon \rightarrow 0$. He found in this case that $U = kT$, corresponding to the result of the classical equipartition theorem.[37]

Boltzmann related the entropy S to the logarithm of the *maximum* number of configurations W_B of n molecules of fixed total energy $\lambda\epsilon$, where λ is an integer and the energy of each molecule is a multiple of a fixed value ϵ. In Sec. I of his paper, he gave an expression for W_B associated with the distribution of λ discrete energy elements of magnitude ϵ among n molecules such that a number n_j of molecules each have energy $j\epsilon$, where $j = 0, 1, 2, ..., p$, and p is an integer. Then

$$W_B = \frac{n!}{\prod n_j!}, \tag{A1}$$

and in terms of these variables, Boltzmann's fundamental principles of statistical mechanics can be stated as follows:

Equally probable configurations (Boltzmann called them complexions) are characterized by the set integers n_j that are subject to the constraints that the total number of molecules n and the total energy $E = \lambda\epsilon$ are fixed, where

$$n = \sum_{j=0}^{p} n_j, \tag{A2}$$

and

$$\lambda = \sum_{j=0}^{p} j n_j, \tag{A3}$$

where $p \leq \lambda$ because for $j = \lambda$, $n_j = 0$ or 1, and for $j > \lambda$, $n_j = 0$.[38] In this case, the state of thermal equilibrium is obtained by the maximum value of $\ln W_B$, subject to the constraints that $\delta n = 0$ and $\delta\lambda = 0$. Introducing two undetermined constants, α and γ (named h and k by Boltzmann), this condition is satisfied by the requirement that

$$\delta \ln W_B + \alpha \delta n + \gamma \delta \lambda = 0. \tag{A4}$$

In the Stirling approximation for the factorials that appear in the expression for W_B in Eq. (A1), this expression becomes

$$\sum_{j=0}^{p} \delta n_j [\ln(n_j) + \alpha + \gamma j] = 0, \tag{A5}$$

and setting $n_0 = \exp(-\alpha)$ and $x = \exp(-\gamma)$, it is satisfied by

$$n_j = n_0 x^j. \tag{A6}$$

Substituting Eq. (A6) in Eqs. (A2) and (A3), and taking the limit $p \rightarrow \infty$,[39] we have

$$\frac{n_j}{n} = (1-x)x^j, \tag{A7}$$

for the fraction of molecules that have energy $j\epsilon$, and x is determined by the ratio n/λ, or

$$x = \frac{1}{1 + n/\lambda}. \tag{A8}$$

This result was given explicitly by Boltzmann (Ref. 13, last line of p. 180) and leads to an expression for the mean energy $U = \lambda\epsilon/n = \epsilon x/(1-x)$ of the molecules in his ensemble as a function of x.[40] Setting $x = \exp(-b\nu/T)$ and $\epsilon = h\nu$, this relation corresponds to Planck's formula of Eq. (19) for the mean energy of oscillators of frequency ν at the absolute temperature T. But in his paper, Boltzmann did not calculate the dependence of x on T. This dependence is obtained from Boltzmann's identification of $\ln W_B$ with the entropy S of the system, by applying the second law of thermodynamics relation $1/T = dS/dE$. He did apply it *only* in the limit of vanishing ϵ appropriate to classical mechanics.

Now we carry out this straightforward calculation for finite ϵ. It would have been easy for Planck to do it too, but there isn't any evidence that he did it. In the Stirling approximation, the maximum value of $\ln W_B$ as a function of Boltzmann's parameter x is

$$\ln W_B = -n \left[\frac{x}{1-x} \ln(x) + \ln(1-x) \right]. \tag{A9}$$

Substituting for x its dependence on n/λ, we obtain

$$\ln W_B = (n + \lambda)\ln(n + \lambda) - n \ln(n) - \lambda \ln \lambda, \tag{A10}$$

which shows that W_B is the Stirling approximation for the total number of *all* possible complexions W of Boltzmann's ensemble (either of molecules or Planck's oscillators), where

$$W = \frac{(n + \lambda - 1)!}{(n-1)!\lambda!}. \tag{A11}$$

Hence, the only puzzle is why Planck did not carry out this simple calculation that is missing in Boltzmann's paper. Setting U equal to the mean energy/molecule, $U = \lambda\epsilon/n$, we have, according to Eq. (A8)

$$x = \frac{1}{1 + \epsilon/U}, \tag{A12}$$

and substituting this expression into Eq. (A9), one obtains an explicit dependence of $\ln W_B$ on U

$$\ln W_B = n\left[\left(1 + \frac{U}{\epsilon}\right)\ln\left(1 + \frac{U}{\epsilon}\right) - \frac{U}{\epsilon}\ln\left(\frac{U}{\epsilon}\right)\right]. \quad \text{(A13)}$$

Apart from the constant of proportionality k, this expression with $\epsilon = h\nu$ is equal to n times Planck's phenomenological expression, Eq. (22), for the entropy per oscillator $S = k(\ln W_P)$ of an ensemble of linear oscillators of frequency ν.

Finally, according to the second law of thermodynamics, the absolute temperature T for thermal equilibrium is given by

$$\frac{1}{T} = \frac{1}{n}\frac{dS}{dU} = \frac{k}{\epsilon}\ln\left(1 + \frac{\epsilon}{U}\right), \quad \text{(A14)}$$

which was calculated by Planck for $\epsilon = h\nu$ in Eqs. (18)–(20). Hence

$$U = \frac{\epsilon}{\exp(\epsilon/kT) - 1}, \quad \text{(A15)}$$

and according to Eqs. (A12) and (A15), the dependence of Boltzmann's parameter x on the temperature T is

$$x = \exp(-\epsilon/kT). \quad \text{(A16)}$$

Substituting this expression into Eqs. (A6) and (A7) yields

$$n_j = n[1 - \exp(-\epsilon/kT)]\exp(-j\epsilon/kT). \quad \text{(A17)}$$

But in the first section of his paper, Boltzmann did not calculate this relation for n_j except in the limit that $\epsilon \ll U$, and obtained

$$n_j \approx \frac{n\epsilon}{U}\exp\left(-\frac{j\epsilon}{U}\right), \quad \text{(A18)}$$

without giving the equipartition result $U = kT$ of classical thermodynamics (see Ref. 13, p. 186, and Ref. 38, p. 49).

Setting $j\epsilon = E$, $\epsilon = dE$, and $n_j/n = dp$, then $p(E)$ is the Maxwell-Boltzmann classical probability distribution for Boltzmann's molecular ensemble having energy in the interval between E and $E + dE$

$$\frac{dp}{dE} = \frac{1}{U}\exp(-E/U), \quad \text{(A19)}$$

with $U = \int_0^\infty dE\, p(E)E$, the mean energy/molecule.[41] Boltzmann pointed out, however, that this relation, with $E = (1/2)mv^2$, where m is the mass and v is the velocity of a molecule, is valid only in two spatial dimensions.[42]

APPENDIX B: HISTORIAN'S DISAGREEMENT ON PLANCK'S ROLE IN THE INTRODUCTION OF THE QUANTUM

Since the publication of Kuhn's 1978 book, *Black-Body Theory and the Quantum Discontinuity*,[5] the nature of Planck's discovery has become the subject of controversies among historians of science. Recently, some of them have claimed that contrary to what physicists have always been led to believe, Planck did not introduce the concept of energy discreteness or quantization into physics.[5,7] This surprising claim originated with the appearance of Kuhn's book and in his later article[5] on this subject where he disputed the conventional view held by most physicists and earlier historians of science. For example, in his book, Kuhn concludes that (Ref. 5, p. 126)

> "With a single misleading exception, nothing in Planck's published papers, known manuscripts, or autobiographical fragments suggest that the idea of restricting resonator energies to a discrete set of values had even occurred to him as a possibility until others forced it upon him during 1906 and the years following."

Although some historians of science have criticized Kuhn's interpretation of Planck's discovery, others have reached similar conclusions. While Kuhn agreed with many details in the traditional account of Planck's work, he argued that crucial aspects had been profoundly misinterpreted, and concluded that energy quantization was actually first introduced by Einstein in 1906.

Kuhn's book was reviewed by the historians of science Klein,[43] Peter Galison,[44] Allan Needell,[45] and by the physicist Res Jost,[6] who were all critical of Kuhn's thesis, but conclusions similar to Kuhn's were reached by Oliver Darrigol,[7] who even quoted Planck for "irrefutable evidence" that he had not meant to introduce energy discontinuities into physics, and later Greeenberg *et al*. also reached such conclusions.[9] In 1984, Kuhn responded to some of his critics in an article defending his interpretation of Planck's discovery of the quantum hypotheses by stating that[5]

> "Part of the appeal of the standard account of Planck's discovery is, I think, the closeness with which it matches a still cherished view of the nature of science and its developments. Although I appreciate both the charms and the functions of that view, understanding requires that it be recognized as myth."

Planck was aware that with his procedure he was violating the tenets of continuum physics. In his Dec 19, 1900 paper, presented at a meeting of the German Physical Society, he wrote:[19]

> "If E [the total energy] is considered to be a continuous divisible quantity this distribution is possible in infinitely many ways. We consider, however—this is the most essential point of the whole calculation—E to be composed of a well-defined number of equal parts [of magnitude ϵ] and use thereto the constant of nature $h = 6.55 \times 10^{-27}$ erg $\times$ sec [setting $\epsilon = h\nu$]."

In view of this single remark, it is evident that Kuhn's statement quoted earlier that the "idea of restricting resonator energies to a discrete set of values had not even occurred to him [Planck]" is clearly incorrect. But as "irrefutable evidence" that Planck did not have energy discreteness in mind, Darrigol has argued that Planck had remarked afterwards that[19]

> "If the ratio [E/ϵ] thus calculated is not an integer, we take for P an integer in the neighbourhood of this ratio."

The assumption that E/ϵ could not be an integer had to be made by Boltzmann who considered that the magnitude ϵ becomes vanishingly small in the limit that the energy is a continuous quantity, but not by Planck who considered ϵ as a fixed and finite quantity equal to $h\nu$. Moreover, Darrigol's comment is inconsistent with Planck's previous statement that E is "composed of a well defined number of equal parts." Recently, Darrigol summarized this controversy in an article titled, *The Historians' Disagreement over the Meaning of Planck's Quantum.*[7]

It appears that the disagreement among historians of science and physicists is based on a different understanding of the relation between Boltzmann's seminal 1877 article establishing the relation between entropy and statistical physics, and Planck's application of this work to obtain a physical basis his black-body formula, which he had obtained previously by an empirical fit to experimental data based on classical thermodynamics and electrodynamics. Unfortunately, many of Planck's documents and correspondence that could have illuminated the development and early reception of his ideas were destroyed during the bombing of Berlin in 1944. In this connection, it is worthwhile to recall Einstein's observation that "If you want to find out anything from the theoretical physicists about the methods they use, I advise you to stick closely to one principle: do not listen to their words, fix your attention on their deeds."[46]

[1] L. Rosenfeld, "La premiére phase de l'èvolution de la Thèorie de Quanta," Osiris 2, 149–196 (1936).

[2] L. Rosenfeld, "Max Planck et la dèfinition statistique de l'entropie" in *Max Planck Festschrift*, edited by B. Kockel, W. Macke, and A. Papapetrou (Veb Deutscher Verlag Der Wissenschaften, Berlin, 1958), pp. 203–211.

[3] M. Klein, "Max Planck and the beginnings of the quantum theory," Arch. Hist. Exact Sci. 1, 459–479 (1961); "Thermodynamics and quanta in Planck's work," Phys. Today 19(11), 23–32 (1966).

[4] For an excellent historical article on Planck's development of the quantum theory, containing also an extensive list of references, see C. A. Gearhart, "Planck, the quantum, and the historians," Phys. Perspect. 4, 170–215 (2002); See also H. Kragh's account in *Quantum Generations* (Princeton U. P., Princeton, 1999), pp. 58, 63; and his article "Max Planck, the reluctant revolutionary," Phys. World 13, 31–35 (2000); For biographies of Planck, see J. L. Heilbron, *The Dilemmas of an Upright Man: Max Planck as Spokesman for German Science* (University of California Press, Berkeley, CA, 1986); B. R. Brown, *Planck, Driven by Vision, Broken by War* (Oxford U. P., New York, 2015).

[5] T. Kuhn, *Black-Body Theory and the Quantum Discontinuity, 1894–1912* (Oxford U. P., New York, 1978); "Revisiting Planck," Hist. Stud. Phys. Sci. 14(2), 231–252 (1984).

[6] R. Jost, "Planck-Kritik des T. Kuhn," in *Das Märchen von Elfenbeinernen Turm: Reden un Aufsätze*, Lecture Notes in Physics Monographs, German Edition (Springer, 1995), pp. 67–78. In passing, Jost writes: "Im Lichte dieser Feststellung aber bewundere ich Herrn Kuhns Ungehemmtheit, uns seinen Unsinn aufzutischen," which roughly translates "what Kuhn has been dishing us is nonsense."

[7] O. Darrigol, "The historians' disagreement over the meaning of Planck's quantum," Centaurus 43, 219–239 (2001); "The quantum enigma," in *The Cambridge Companion to Einstein*, edited by M. Janssen and C. Lehner (Cambridge U.P., Cambridge, 2014), pp. 117–142.

[8] S. G. Brush, *Statistical Physics and the Atomic Theory of Matter* (Princeton U. P., Princeton, 1983); R. Balian, *From Microphysics to Macrophysics, Methods and Applications of Statistical Physics* (Springer Verlag, Berlin, 1992), Vols. I and II.

[9] D. M. Greenberg, N. Erez, M. O. Scully, A. A. Svidzinsky, and M. S. Zubairy, "Planck, photon statistics, and Bose-Einstein condensation," Prog. Opt. 50, 275–330 (2007).

[10] S. Seth, "Quantum physics," in *Handbook of the History of Physics*, edited by J. Buchwald and R. Fox (Oxford U. P., New York, 2013).

[11] J. Heilbron, "Nascent science. The scientific and psychological background to Bohr's Trilogy," in *Love, Literature and the Quantum Atom* (Oxford U. P., 2013), Chap. 2.2.

[12] M. Badino, "The odd couple: Boltzmann, Planck and the application of statistics to physics (1900–1913)," Ann. Phys. (Berlin) 18, 81–101 (2009); *The Bumpy Road: Max Planck from Radiation Theory to the Quantum, (1896–1906)* (Springer Briefs, 2015).

[13] L. Boltzmann, "On the relation between the second law of thermodynamics and the probability calculations of the principles of thermal equilibrium," Wien. Ber. 76, 373–435 (1877); Reprinted in *Wissenschaftliche Abhandlungen von Ludwig Boltzmann* (Chelsea Publishing Company, New York, 1968), Vol. 2, pp. 164–223.

[14] G. Holton and Stephen G. Brush, *Introduction to Concepts and Theories in Physical Science* (Addison-Wesley, 1973).

[15] A. Pais, *Subtle is the Lord, The Science and the Life of Albert Einstein* (Oxford Press, New York, 1982), pp. 370–371.

[16] R. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures in Physics* (Addison-Wesley, 1963), Vol. 1, pp. 41–46.

[17] Lord Rayleigh, "Remarks upon the law of complete radiation," Philos. Mag. XLIX, 539–540 (1900); "A comparison of two theories of radiation," Nature 72, 293–294 (1905).

[18] In the limit of vanishing ϵ, Boltzmann's discrete model is only applicable to the motion of molecules in two dimension (See Ref. 13, p. 190).

[19] M. Planck, "Zur theorie des gesetzes der energieverteilung im normal spectrum," Verh. Dtsch. Phys. Ges. 2, 237–245 (1900).

[20] See Ref. 15 p. 372.

[21] A. Einstein, "On a heuristic point of view concerning the production and tranformation of light," Ann. Phys. 17, 132–148 (1905).

[22] M. Planck, *Scientific Autobiography*, translated from German by F. Gaynor (Philosophical Library, New York, 1949), pp. 39–41; "Wissenschaftliche selbstbiographie," Acta Hist. Leopold. 19, 16–17 (1990).

[23] M. Planck, "The genesis and present state of development of the quantum theory, Nobel lecture, June 2, 1920," in *Nobel Lectures, Physics 1901–1921* (Elsevier, Amsterdam, 1967), pp. 407–420.

[24] C. Jungnickel and R. McCormmach, *Intellectual Mastery of Nature: Theoretical Physics from Ohm to Einstein* (University of Chicago Press, Chicago, 1986), pp. 261–262.

[25] A. Hermann, *The Genesis of Quantum Theory*, originally published by Physik Verlag, Mosbach/Baden, under the title *Frügeschichte der Quantumtheorie (1899–1913)*, English translation by C. W. Nash (Massachusetts Institute of Technology, Cambridge, MS, 1971), pp. 23–24.

[26] W. Wien, "Über die Energievertheilung im Emissionsspectrum eines schawrzen Körpers," Ann. Phys. 294, 662–669 (1896).

[27] L. Boltzmann, "Ableitung des stefanschen gesetzes, betreffend die abhängigkeit der wärmestrahlung von der temperatur aus der elektromagnetischen lichttheorie," Ann. Phys. 22, 291–294 (1884).

[28] M. Planck, "Ueber eine verbesserung der wien'schen spectralgleichunge," reprinted in Max Planck, *Physikalische Abhandlungen und Vorträge* (Friedr. Wieweg & Sohn, Braunschweig, 1958), Vol. I, pp. 687–689.

[29] M. Planck, "On irreversible radiation processes," reprinted in Max Planck, *Physikalische Abhandlungen und Vorträge* (Friedr. Wieweg & Sohn, Braunschweig, 1958), Vol. I, pp. 493–600.

[30] A. Franklin, *The Neglect of Experiments* (Cambridge U. P., Cambridge, MS, 1986), p. 1. Franklin wrote that one of the greatest anticlimaxes in all of literature occurs at the end of Shakespeare's Hamlet. On a stage strewn with noble and heroic corpses—Hamlet, Laertes, Claudius, and Gertrude—the ambassadors from England arrive and announce that "Rosencrantz and Guildenstern are dead." No one cares. A similar reaction might be produced among a group of physicist, or even among historians and philosophers, were someone to announce that "Lummer and Pringstein are dead."

[31] Historians of science have followed Klein's statement that "it seems most likely that Planck was guided by the form Wien's distribution law," when if fact there was not any other alternative.

[32] See Ref. 29, p. 597.

[33] O. Lummer and E. Pringsheim, "Die verteilung der energie in spectrum des schwarzen körpers," Verh. Dtsch. Phys. Ges. 1, 23–41 (1899).

[34] H. Rubens and F. Kurlbaum, "Über die emission langwelliger wärmestrahlen dur die schwarzen körper bei verschieden termperaturen," Preussiche Akad. Wiss. 929–941 (1900).

[35] H. Rubens and E. F. N. Nichols, "Versusche mit wärmerstraheln von grosser wellenlänge," Ann. Phys. 60, 418–496 (1897).

[36] M. Planck, "Über des gesetzes der energieverteilung in normal spectrum," Ann. Phys. 309(3), 564–566 (1901).

[37]It should be pointed out that it was Planck's very good fortune that Boltzmann's first treatment of his molecular gas model was in two dimensions. For each degree of freedom, the classical mean energy of a molecule is $U = \frac{1}{2}kT$. Hence the physical three dimensional case with $U = \frac{3}{2}kT$ would not have been a useful model for Planck's electrical oscillators.

[38]But this constraint on p does not apply in the Stirling approximation for the factorials in WB. In his book (Ref. 5, p. 49), Kuhn sets $p = \lambda$ and states that "standard variational techniques lead directly to the conclusion that for $p \gg n$," n_j is given by Boltzmann's classical expression, Eq. (A18). But this claim is incorrect, because to obtain the distribution in the classical limit, Boltzmann set $p = \infty$ and $\lambda \gg n$.

[39]Boltzmann also considered p finite, but the case of interest related to Planck's formula corresponds to $p \to \infty$.

[40]In the Stirling approximation the resulting values of n_j, Eq. (A7), are not integers. In this case λ and n are also infinite, but the ratio λ/n is fixed and the ratio n_j/n is finite corresponding to the fraction of molecules with energy $j\epsilon$.

[41]J. C. Maxwell, "Illustrations of the dynamical theory of gases," Philos. Mag. **19**, 19–32 (1860); *ibid.* **20**, 21–37 (1860); reprinted in *The Scientific Papers of James Clerk Maxwell* (Dover Publication, Inc, New York, 1965) pp. 377–409.

[42]See Ref. 13, p. 190.

[43]M. Klein, "Contribution to paradigm lost, a review symposium," ISIS **70**, 429–433 (1979).

[44]P. Galison, "Kuhn and the quantum controversy," Br. J. Philos. Sci. **32**, 71–85 (1981).

[45]A. A. Needell, "Review of "Black-Body Theory and the Quantum Discontinuity," by Thomas Kuhn," ISIS **78**, 604–605 (1987).

[46]A. Einstein, *Ideas and Opinions* (Crown Publishers, New York, 1954), p. 270.

Induction Coil Mechanism

This is the business end of a Ducretet induction coil in the collection of St. Mary's College in Notre Dame, Indiana. At the top is the glass end plate that caps the primary and secondary coils. To prevent eddy currents, the core of the coil consists of a bundle of iron wires. An iron bob on the left-hand side of the horizontal brass strip is attracted to the core when the primary coil is energized, and this breaks the contact attached to the middle of the bar, thus causing the magnetic field to fall to zero. As the strip springs back, the primary is reenergized, thus inducing a time-varying signal in the secondary. The very elaborate reversing switch in the foreground is used from time to time to prevent the core from building up a permanent magnetization. (Notes and picture by Thomas B. Greenslade, Jr., Kenyon College)