MECHANICAL ENGINEERING DESIGN
TUTORIAL 4-20: HERTZ CONTACT STRESSES

CHARACTERISTICS OF CONTACT STRESSES

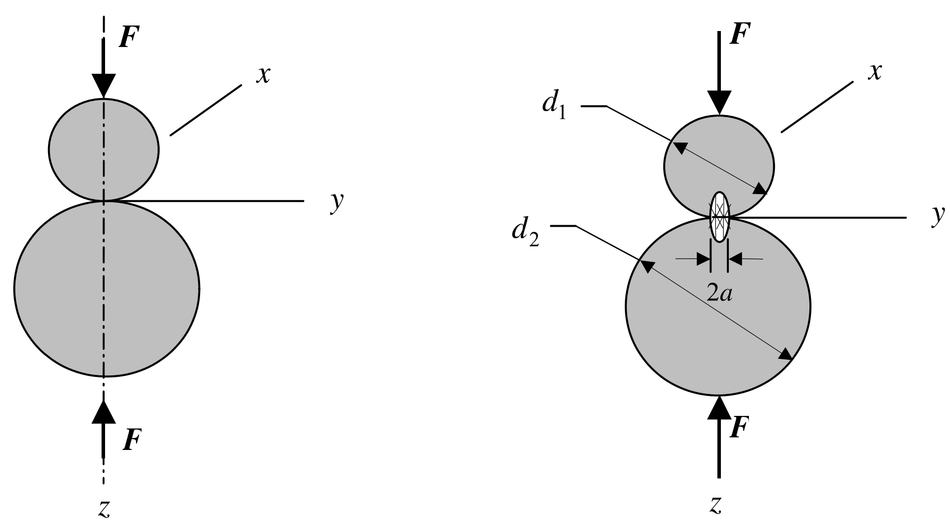
1. Represent *compressive* stresses developed from surface pressures between two curved bodies pressed together;
2. Possess an area of contact. The initial point contact (spheres) or line contact (cylinders) become area contacts, as a result of the force pressing the bodies against each other;
3. Constitute the principal stresses of a triaxial (three dimensional) state of stress;
4. Cause the development of a critical section below the surface of the body;
5. Failure typically results in flaking or pitting on the bodies' surfaces.

TWO DESIGN CASES

Two design cases will be considered,

1. Sphere – Sphere Contact (Point Contact $\Rightarrow$ Circular Contact Area)
2. Cylinder – Cylinder Contact (Line Contact $\Rightarrow$ Rectangular Contact Area)

SPHERE – SPHERE CONTACT



(a) Two spheres held in contact by force *F*.

(b) Contact stress has an elliptical distribution across contact over zone of diameter 2*a*.

TEXT FIGURE 4-42 Two Spheres in Contact

† *Text.* refers to *Mechanical Engineering Design,* 7th edition text by Joseph Edward Shigley, Charles R. Mischke, and Richard G. Budynas; equations and figures with the prefix *T* refer to the present tutorial.

Consider two solid elastic spheres held in contact by a force F such that their point of contact expands into a circular area of radius a, given as:

$$a = K_a \sqrt[3]{F} \qquad \text{(Modified Text Eq. 4-72)}$$

$$\text{where } K_a = \left[\frac{3}{8} \frac{(1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2}{(1/d_1)+(1/d_2)} \right]^{1/3}$$

$F =$ applied force

$\nu_1, \nu_2 =$ Poisson's ratios for spheres 1 and 2

$E_1, E_2 =$ elastic modulii for spheres 1 and 2

$d_1, d_2 =$ diameters of spheres 1 and 2

This general expression for the contact radius can be applied to two additional common cases:

1. Sphere in contact with a plane $(d_2 = \infty)$;
2. Sphere in contact with an internal spherical surface or 'cup' $(d_2 = -d)$.

Returning to the sphere-sphere case, the maximum contact pressure, $p_{\max}$, occurs at the center point of the contact area.

$$p_{\max} = \frac{3F}{2\pi a^2} \qquad \text{(Text Eq. 4-73)}$$

State of Stress

- The state of stress is computed based on the following mechanics:

  1. Two planes of symmetry in loading and geometry dictates that $\sigma_x = \sigma_y$;
  2. The dominant stress occurs on the axis of loading: $\sigma_{\max} = \sigma_z$;
  3. The principal stresses are $\sigma_1 = \sigma_2 = \sigma_x = \sigma_y$ and $\sigma_3 = \sigma_z$ given $\sigma_1, \sigma_2 \geq \sigma_3$;
  4. Compressive loading leads to $\sigma_x, \sigma_y,$ and σ_z being compressive stresses.

- Calculation of Principal Stresses

$$\sigma_x = -p_{\max} \left[\left[1 - |\zeta_a| \tan^{-1}\left(\frac{1}{|\zeta_a|} \right) \right] (1+\nu) - \frac{1}{2\left(1+\zeta_a^2\right)} \right] \qquad \text{(Modified Text Eq. 4-74)}$$

$$= \sigma_y = \sigma_1 = \sigma_2$$

$$\sigma_3 = \sigma_z = \frac{-p_{max}}{1+\zeta_a^2} \qquad \text{(Modified Text Eq. 4-75)}$$

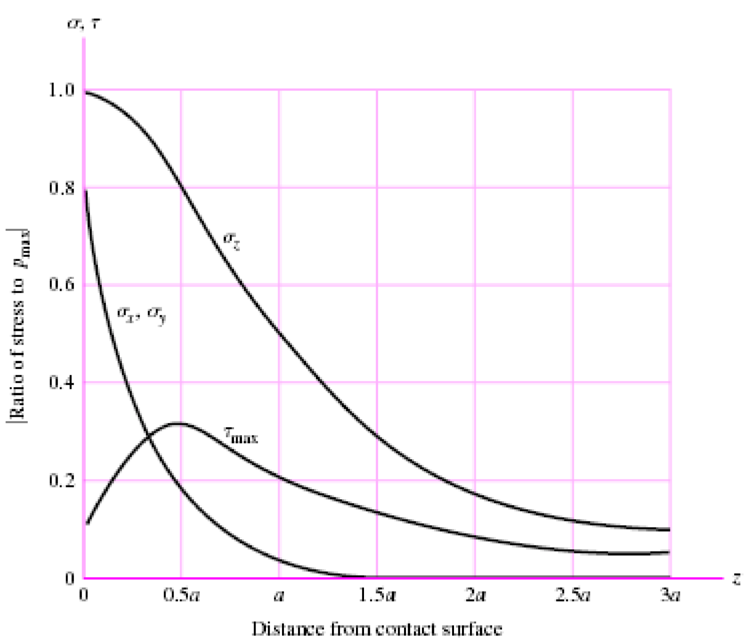
where $\zeta_a = z/a =$ nondimensional depth below the surface
$\nu =$ Poisson's ratio for the sphere examined (1 or 2)

- Mohr's Circle

Plotting the principal stresses on a Mohr's circle plot results in: one circle, defined by $\sigma_1 = \sigma_2$, shrinking to a point; and two circles, defined by σ_1, σ_3 and σ_2, σ_3, plotted on top of each other. The maximum shear stress, τ_{max}, for the plot is calculated as:

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_x - \sigma_z}{2} = \frac{\sigma_y - \sigma_z}{2} \qquad \text{(Modified Text Eq. 4-76)}$$

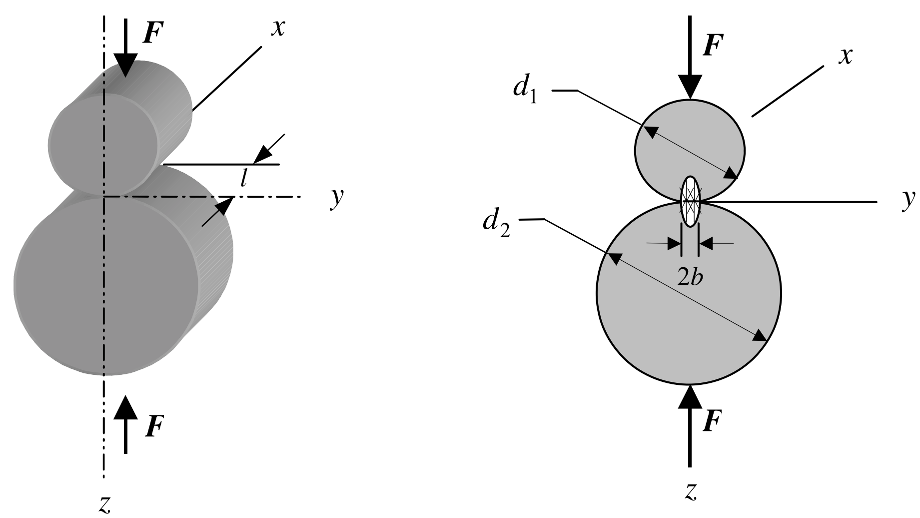
If the maximum shear stress, τ_{max}, and principal stresses, σ_1, σ_2, and σ_3, are plotted as a function of maximum pressure, p_{max}, below the surface contact point, the plot of Fig. 4-43 is generated. This plot, based on a Poisson's ratio of $\nu = 0.3$, reveals that a critical section exists on the load axis, approximately 0.48*a* below the sphere surface. Many authorities theorize that this maximum shear stress is responsible for the surface fatigue failure of such contacting elements; a crack, originating at the point of maximum shear, progresses to the surface where lubricant pressure wedges a chip loose and thus creates surface pitting.



TEXT FIGURE 4-43: Magnitude of the stress components below the surface as a function of maximum pressure of contacting spheres.

CYLINDER–CYLINDER CONTACT

Consider two solid elastic cylinders held in contact by forces F uniformly distributed along the cylinder length l.



(a) Two right circular cylinders held in contact by forces F uniformly distributed along cylinder length l.

(b) Contact stress has an elliptical distribution across contact zone of width $2b$.

TEXT FIGURE 4-44 Two Cylinders in Contact

The resulting pressure causes the line of contact to become a rectangular contact zone of half-width b given as:

$$b = K_b \sqrt{F} \qquad \text{(Modified Text Eq. 4-77)}$$

where $K_b = \left[\dfrac{2}{\pi l} \dfrac{(1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2}{(1/d_1)+(1/d_2)} \right]^{1/2}$

$F =$ applied force

$\nu_1, \nu_2 =$ Poisson's ratios for cylinders 1 and 2

$E_1, E_2 =$ elastic modulii for cylinders 1 and 2

$d_1, d_2 =$ diameters of spheres 1 and 2

$l =$ length of cylinders 1 and 2 ($l_1 = l_2$ assumed)

This expression for the contact half-width, b, is general and can be used for two additional cases which are frequently encountered:

1. Cylinder in contact with a plane, e.g. a rail $(d_2 = \infty)$;
2. Cylinder in contact with an internal cylindrical surface, for example the race of a roller bearing $(d_2 = -d)$.

The maximum contact pressure between the cylinders acts along a longitudinal line at the center of the rectangular contact area, and is computed as:

$$p_{\max} = \frac{2F}{\pi b l} \qquad \text{(Text Eq. 4-78)}$$

State of Stress

- The state of stress is computed based on the following mechanics:

  1. One plane of symmetry in loading and geometry dictates that $\sigma_x \neq \sigma_y$;
  2. The dominant stress occurs along the axis of loading: $\sigma_{\max} = \sigma_z$;
  3. The principal stresses are equal to σ_x, σ_y, and σ_z with $\sigma_3 = \sigma_z$;
  4. Compressive loading leads to σ_x, σ_y, and σ_z being compressive stresses.

- Calculation of Principal Stresses and Maximum Shear Stress

$$\sigma_3 = \sigma_z = -p_{\max} \frac{1}{\sqrt{1+\zeta_b^2}} \qquad \text{(Modified Text Eq. 4-81)}$$

$$\sigma_1 = \begin{cases} \sigma_x & \text{for } 0 \le \zeta_b \le 0.436 \\ \sigma_y & \text{for } 0.436 \le \zeta_b \end{cases}$$

where,

$$\sigma_x = -2\nu p_{\max} \left[\sqrt{1+\zeta_b^2} - |\zeta_b| \right] \qquad \text{(Modified Text Eq. 4-79)}$$

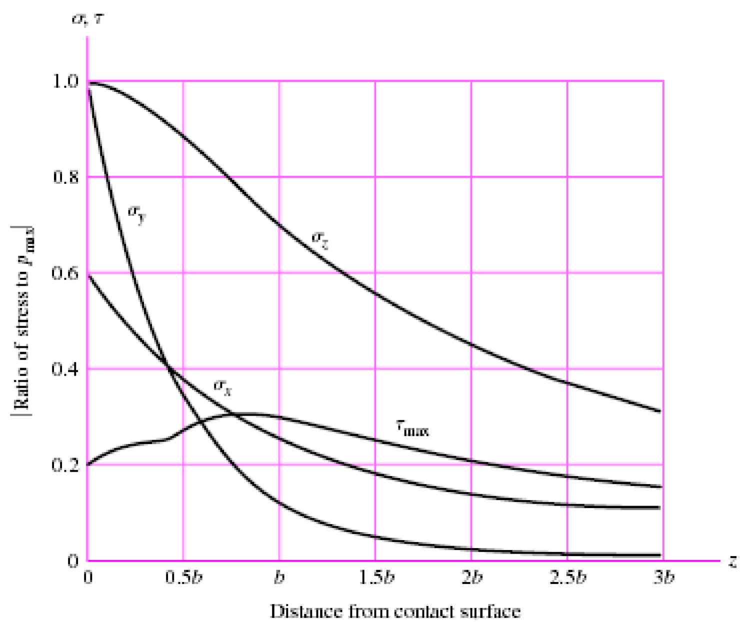
$$\sigma_y = -p_{\max} \left[\left(\frac{1+2\zeta_b^2}{\sqrt{1+\zeta_b^2}} \right) - 2|\zeta_b| \right] \qquad \text{(Modified Text Eq. 4-80)}$$

$$\zeta_b = z/b$$

The maximum shear stress is thus given as:

$$\tau_{max} = \begin{cases} \tau_{1/3} = (\sigma_z - \sigma_x)/2 & \text{for } 0 \le \zeta_b \le 0.436 \\ \tau_{1/3} = (\sigma_z - \sigma_y)/2 & \text{for } 0.436 \le \zeta_b \end{cases}$$

When these equations are plotted as a function of maximum contact pressure up to a distance 3*b* below the surface contact point, the plot of Fig. 4-45 is generated. Based on a Poisson's ratio of 0.3, this plot reveals that τ_{max} attains a maxima for $\zeta_b = z/b = 0.786$ and $0.3p_{max}$.



TEXT FIGURE 4-45: Magnitude of stress components below the surface as a function of maximum pressure for contacting cylinders.

Example T4.20.1:

Problem Statement: A 6-in-diameter cast-iron wheel, 2 in wide, rolls on a flat steel surface carrying a 800 lbf load.

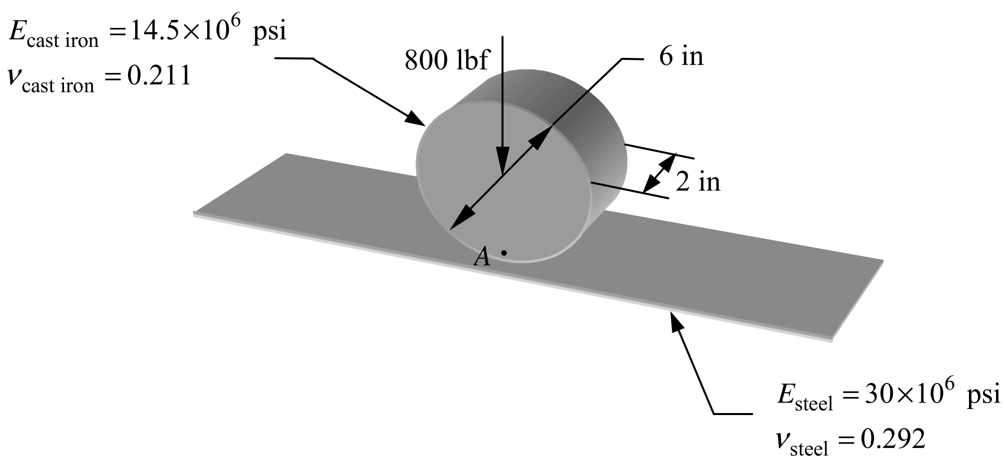
Find:

1. The Hertzian stresses $\sigma_x, \sigma_y, \sigma_z$ and $\tau_{1/3}$ in the cast iron wheel at the critical section;
2. The comparative state of stress and maximum shear stress, arising during a revolution, at point *A* located 0.015 inch below the wheel rim surface.

Solution Methodology:

1. Compute the value of the contact half-width, *b*.
2. Compute the maximum pressure generated by the normal force of the wheel.
3. Use the results of steps (1) and (2) to calculate the contact stresses in the cast iron wheel for the critical section, $z/b = 0.786$.
4. Evaluate the principal stresses based upon the contact stress calculations.
5. Calculate the maximum shear stress.
6. Compare these results with those obtained by using Fig. 4-45.
7. During a single revolution of the wheel, point *A* will experience a cycle of stress values varying from zero (when point *A* lies well outside the contact zone) to a maximum state of stress (when *A* lies within the contact zone and on the line of action of the 800 lbf force.) We expect point *A* to "feel" the effects of a semi-elliptical contact pressure distribution as point *A* moves into and through the contact zone. Thus, we need to calculate the contact stresses for a depth of $z = 0.015$ inch, which we expect to lie within the contact zone.

Schematic:



Solution:

1. Compute contact half-width, *b*

Material Properties: $E_1 = E_{\text{cast iron}} = 14.5\times10^6$ psi; $\nu_1 = \nu_{\text{cast iron}} = 0.211$

$E_2 = E_{\text{steel}} = 30.0\times10^6$ psi; $\nu_2 = \nu_{\text{steel}} = 0.292$

Dimensions: $d_1 = 6.0$ in; $d_2 = \infty$; $l = 2.0$ in

$$b = K_b \sqrt{F} \qquad \text{(Modified Text Eq. 4-72)}$$

$$K_b = \left[\frac{2}{\pi l} \frac{(1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2}{(1/d_1)+(1/d_2)} \right]^{1/2}$$

$$= \left\{ \frac{2}{\pi(2.0)} \frac{\left[1-(0.211)^2\right]/(14.5\times10^6) + \left[1-(0.292)^2\right]/(30.0\times10^6)}{(1/6.0)+(1/\infty)} \right\}^{1/2}$$

$$= 4.291\times10^{-4} \text{ in}/\sqrt{\text{lbf}}$$

$$b = K_b\sqrt{F} = (4.291\times10^{-4} \text{ in}/\sqrt{\text{lbf}})(800 \text{ lbf})^{1/2} = \mathbf{1.214\times10^{-2} \text{ in}}$$

2. Maximum Pressure, $p_{\max}$

$$p_{\max} = \frac{2F}{\pi b l} = \frac{2(800 \text{ lbf})}{\pi(1.214\times10^{-2} \text{ in})(2.0 \text{ in})} = \mathbf{20\ 980 \text{ psi}}$$

3. Hertz Contact Stresses in Cast Iron Wheel

At the critical section, $\zeta_b = z/b = 0.786$,

$$\sigma_x = -2\nu_1 p_{\max}\left[\sqrt{1+\zeta_b^2} - |\zeta_b|\right]$$

$$= -2(0.211)(20\,980 \text{ psi})\left[\sqrt{1+(0.786)^2} - 0.786\right]$$

$$= \mathbf{-4302 \text{ psi}}$$

$$\sigma_y = -p_{\max}\left[\left(\frac{1+2\zeta_b^2}{\sqrt{1+\zeta_b^2}}\right) - 2|\zeta_b|\right]$$

$$= (-20\,980 \text{ psi})\left\{\left[\frac{1+2(0.786)^2}{\sqrt{1+(0.786)^2}}\right] - 2(0.786)\right\}$$

$$= \mathbf{-3895 \text{ psi}}$$

$$\sigma_z = -p_{\max}\frac{1}{\sqrt{1+\zeta_b^2}} = \frac{-20\,980 \text{ psi}}{\sqrt{1+(0.786)^2}}$$

$$= \mathbf{-16\ 490 \text{ psi}}$$

Note that the small contact area involved in this type of problem gives rise to very high pressure, relative to the applied force, and thus exceptionally high stresses.

4. Since σ_x, σ_y, and σ_z are all principal stresses, we can conclude:

$$\sigma_1 = \sigma_y = -3895 \text{ psi}$$
$$\sigma_2 = \sigma_x = -4302 \text{ psi}$$
$$\sigma_3 = \sigma_z = -16\,490 \text{ psi}$$

5. Maximum Shear Stress

$$\tau_{\max} = \tau_{1/3} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_y - \sigma_z}{2} = \frac{-3895 \text{ psi} - (-16\,490 \text{ psi})}{2}$$
$$= \mathbf{6298\ psi}$$

6. Comparison with results based on Text Figure 4-45:

For $z/b \approx 0.75$,

$$\sigma_x \approx -0.3 p_{\max} = -6294 \text{ psi}$$
$$\sigma_y \approx -0.2 p_{\max} = -4196 \text{ psi}$$
$$\sigma_z \approx -0.8 p_{\max} = -16\,780 \text{ psi}$$
$$\tau_{\max} \approx 0.3 p_{\max} = 6294 \text{ psi}$$

Comparing these results with those calculated using a value of $\nu = 0.211$, we find that only σ_x is a function of ν; σ_y, σ_z, and $\tau_{\max}$ are independent of ν since the graphical estimates of their values are within 3 % of those obtained from the plot which assumes a Poisson's ratio of 0.3.

7. For a depth of 0.015 in below the cylinder surface,

$$\zeta_b = \frac{0.015 \text{ in}}{1.214 \times 10^{-2} \text{ in}} = 1.236$$

Substituting,

$$\sigma_x = -2\nu\, p_{\max}\left[\sqrt{1+\zeta_b^2} - |\zeta_b|\right]$$

$$= -2(0.211)(20\,980 \text{ psi})\left[\sqrt{1+(1.236)^2} - |1.236|\right]$$

$$= \mathbf{-3133\ psi}$$

$$\sigma_y = -p_{\max}\left[\left(\frac{1+2\zeta_b^2}{\sqrt{1+\zeta_b^2}}\right) - 2|\zeta_b|\right]$$

$$= (-20\,980 \text{ psi})\left\{\left[\frac{1+2(1.236)^2}{\sqrt{1+(1.236)^2}}\right] - 2|1.236|\right\}$$

$$= \mathbf{-1652\ psi}$$

$$\sigma_z = -p_{\max}\frac{1}{\sqrt{1+\zeta_b^2}} = \frac{-20\,980 \text{ psi}}{\sqrt{1+(1.236)^2}}$$

$$= \mathbf{-13\ 200\ psi}$$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_y - \sigma_z}{2} = \frac{-1652-(-13\ 200)}{2} = \mathbf{5774\ psi}$$

As expected, at a depth corresponding greater than the critical section ($z/b = 1.236 > 0.786$), the magnitudes of all three principal stresses are smaller than those calculated for $z/b = 0.786$. The difference between the principal stresses is also smaller and consequently, τ_{max} also decreases.