25ο Φυλ Ασκ

4) Έχουμε ότι

$$\int \frac{dx}{\sin^{-1}x\sqrt{1-x^2}}$$

$$\left[\begin{array}{l} \text{Θέτω } \omega=\sin^{-1}x \\ \omega=\sin^{-1}x \Rightarrow d\omega=\frac{1}{\sqrt{1-x^2}}dx \end{array}\right]$$

$$=\int \frac{1}{\omega}d\omega$$

$$=\ln|\omega|+c$$

$$=\ln|\sin^{-1}x|+c$$

26ο Φυλλάδιο Ασκήσεων

1) Έχουμε ότι

$$\int \frac{\ln^3 x}{x+2x\ln^4 x}dx=$$

$$\int \frac{\ln^3 x}{x(1+2\ln^4 x)}dx=$$

$$\left[\begin{array}{l} \text{Θέτω } u=\ln x \\ u=\ln x \Rightarrow du=\frac{1}{x}dx \end{array}\right]$$

$$=\int \frac{u^3}{1+2u^4}du$$

$$\left[\begin{array}{l} \text{Θέτω } \omega=1+2u^4 \\ \omega=1+2u^4 \Rightarrow d\omega=8u^3du \rightarrow \frac{1}{8}d\omega=u^3du \end{array}\right]$$

$$=\frac{1}{8}\int\frac{d\omega}{\omega}$$

$$=\frac{1}{8}\ln|\omega|+c$$

$$=\frac{1}{8}\ln|1+2u^4|+c=\frac{1}{8}\ln(1+2u^4)+c=\frac{1}{8}\ln(1+2\ln^4 x)+c$$

2ος τρόπος

Θέτω $u = 1 + 2\ln^2 x$

(Αποδ.)

2) Έχουμε ότι

$$\int \frac{d\theta}{\sec\theta + \tan\theta} =$$

$$= \int \frac{d\theta}{\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}}$$

$$= \int \frac{\cos\theta}{1 + \sin\theta}\, d\theta$$

$$\left[\begin{array}{l} \text{Θέτω } w = 1 + \sin\theta \\ w = 1 + \sin\theta \Rightarrow dw = \cos\theta\, d\theta \end{array} \right]$$

$$= \int \frac{dw}{w}$$

$$= \ln|w| + c$$

$$= \ln|1 + \sin\theta| + c$$

$$= \ln(1 + \sin\theta) + c$$

2ος τρόπος

Έχουμε ότι

$$\int \frac{d\theta}{\sec\theta + \tan\theta} = \int \frac{1}{\sec\theta + \tan\theta} \cdot \frac{\sec\theta - \tan\theta}{\sec\theta - \tan\theta}\, d\theta$$

$$= \int \frac{\sec\theta - \tan\theta}{\sec^2\theta - \tan^2\theta}\, d\theta =$$

$$= \int \frac{\sec\theta - \tan\theta}{1}\, d\theta$$

$$= \int \sec\theta\, d\theta - \int \tan\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| - \ln|\sec\theta| + c$$

$= \ln \frac{|\sec\theta + \tan\theta|}{|\sec\theta|} + c$

$= \ln \left| \frac{\sec\theta + \tan\theta}{\sec\theta} \right| + c$

$= \ln |1 + \sin\theta| + c$

$= \ln(1 + \sin\theta) + c$

3) Έχουμε ότι

$\int_0^{2\pi} \sqrt{\frac{1 - \cos x}{2}}\, dx =$

Γνωρίζουμε ότι

$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

Άρα

$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$

Επομένως

$\frac{1 - \cos x}{2} = \sin^2\left(\frac{x}{2}\right)$

$= \int_0^{2\pi} \sqrt{\sin^2\left(\frac{x}{2}\right)}\, dx$

$= \int_0^{2\pi} \left|\sin\left(\frac{x}{2}\right)\right| dx$

Θέτω $y = \frac{x}{2}$

$y = \frac{x}{2} \Rightarrow dy = \frac{1}{2}dx \Rightarrow 2dy = dx$

$y(0) = \frac{0}{2} = 0$

$y(2\pi) = \frac{2\pi}{2} = \pi$

$= 2\int_0^{\pi} |\sin y|\, dy$

$= 2\int_0^{\pi} \sin y\, dy$ $(0 \leq y \leq \pi \Rightarrow \sin y \geq 0 \Rightarrow |\sin y| = \sin y)$

$= 2\left[-\cos y\right]_0^{\pi}$

$= -2(\cos\pi - \cos 0)$

$= -2(-1-1)$

$= -2(-2)$

$= 4$

4) Έχουμε ότι

$$\int \frac{dx}{1-\sec x} = \int \frac{dx}{1 - \frac{1}{\cos x}}$$

$$= \int \frac{\cos x}{\cos x - 1}\, dx$$

$$= \int \frac{\cos x}{\cos x - 1} \cdot \frac{\cos x + 1}{\cos x + 1}\, dx$$

$$= \int \frac{\cos^2 x + \cos x}{\cos^2 x - 1}\, dx$$

$$= \int \frac{\cos^2 x + \cos x}{-\sin^2 x}\, dx$$

$$= \int \left(\frac{\cos^2 x}{-\sin^2 x} - \frac{\cos x}{\sin^2 x}\right) dx$$

$$= \int (-\cot^2 x - \cot x \cdot \csc x)\, dx$$

$$= -\int \cot^2 x\, dx - \int \cot x \csc x\, dx$$

$$= -\int \cot^2 x\, dx - (-\csc x)$$

$$= -\int (\csc^2 x - 1)\, dx + \csc x$$

$$= -\int \csc^2 x\, dx + \int dx + \csc x$$

$$= -(-\cot x) + x + \csc x + c$$

$$= \cot x + x + \csc x + c$$

5) Έχουμε ότι

$$\int \frac{dx}{1+\sin x} =$$

$$= \int \frac{1}{1+\sin x} \cdot \frac{1-\sin x}{1-\sin x}\, dx$$

$$= \int \frac{1-\sin x}{1-\sin^2 x}\, dx$$

$$= \int \frac{1-\sin x}{\cos^2 x}\, dx$$

$$= \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx$$

$$= \int (\sec^2 x - \tan x \sec x)\, dx$$

$$= \int \sec^2 x\, dx - \int \tan x \sec x\, dx$$

$$= \tan x - \sec x + c$$

6) Έχουμε ότι

$$\int \frac{3+5x}{2+9x^2}\, dx =$$

$$= 3 \int \frac{dx}{2+9x^2} + 5 \int \frac{x}{2+9x^2}\, dx$$

Έχουμε ότι

$$\int \frac{dx}{2+9x^2} =$$

$$= \frac{1}{9} \int \frac{dx}{\frac{2}{9} + x^2}$$

$$= \frac{1}{9} \int \frac{dx}{\left(\frac{\sqrt{2}}{3}\right)^2 + x^2}$$

$$= \frac{1}{9} \cdot \frac{1}{\frac{\sqrt{2}}{3}} \cdot \tan^{-1}\left(\frac{x}{\frac{\sqrt{2}}{3}} \right) + c$$

$$= \frac{1}{3\sqrt{2}} \cdot \tan^{-1}\left(\frac{3x}{\sqrt{2}} \right) + c$$

Έχουμε ότι

$$\int \frac{x}{2+9x^2}\,dx =$$

$$\left[\begin{array}{l} \text{Θέτουμε } u = 2+9x^2 \\ u = 2+9x^2 \Rightarrow du = 18x\,dx \Rightarrow \frac{1}{18}du = x\,dx \end{array}\right]$$

$$= \frac{1}{18}\int \frac{1}{u}\,du$$

$$= \frac{1}{18}\ln|u| + c$$

$$= \frac{1}{18}\ln|2+9x^2| + c$$

$$= \frac{1}{18}\cdot \ln(2+9x^2) + c$$

Επομένως

$$\int \frac{3+5x}{2+9x^2}\,dx = 3\cdot\frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{3x}{\sqrt{2}}\right) + 5\frac{1}{18}\ln(2+9x^2) + c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x}{\sqrt{2}}\right) + \frac{5}{18}\ln(2+9x^2) + c$$

7) Διαιρούμε τον αριθμητή με τον παρονομαστή:

$$\begin{array}{rrr|l} 4x^3 & -x^2 & +16x & x^2+4 \\ \cline{4-4} -4x^3 & & -16x & 4x-1 \\ \cline{1-3} & -x^2 & & \\ & x^2 & +4 & \\ \cline{2-3} & & 4 & \end{array}$$

Άρα

$$4x^3 - x^2 + 16x = (x^2+4)(4x-1) + 4$$

Επομένως

$$\frac{4x^3-x^2+16x}{x^2+4}=\frac{(x^2+4)(4x-1)+4}{x^2+4}=4x-1+\frac{4}{x^2+4}$$

Άρα

$$\int \frac{4x^3-x^2+16x}{x^2+4}\,dx=$$

$$=\int\left(4x-1+\frac{4}{x^2+4}\right)dx$$

$$=4\int x\,dx-\int dx+4\int\frac{dx}{x^2+4}$$

$$=4\,\frac{x^2}{2}-x+4\int\frac{dx}{x^2+2^2}$$

$$=2x^2-x+4\,\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+c$$

$$=2x^2-x+2\tan^{-1}\left(\frac{x}{2}\right)+c$$

27º Φυλλάδιο Ασκήσεων

1) Χρησιμοποιώντας ολοκλήρωση κατά παράγοντες παίρνουμε ότι

$$I_n=\int_0^{2\pi} x^n\cos(\alpha x)\,dx$$

$$=\int_0^{2\pi} x^n\left(\frac{\sin(\alpha x)}{\alpha}\right)'dx$$

$$=\left[x^n\,\frac{\sin(\alpha x)}{\alpha}\right]_0^{2\pi}-\int_0^{2\pi}(x^n)'\,\frac{\sin(\alpha x)}{\alpha}\,dx$$

$$=\left((2\pi)^n\cdot\frac{\sin(\alpha\cdot 2\pi)}{\alpha}-0^n\cdot\frac{\sin(\alpha\cdot 0)}{\alpha}\right)-\int_0^{2\pi}(x^n)'\,\frac{\sin(\alpha x)}{\alpha}\,dx$$

$$=\left((2\pi)^n\cdot\frac{0}{\alpha}-0\right)-\int_0^{2\pi}nx^{n-1}\cdot\frac{\sin(\alpha x)}{\alpha}\,dx$$

$$=-\frac{n}{\alpha}\int_0^{2\pi}x^{n-1}\left(-\frac{\cos(\alpha x)'}{\alpha}\right)dx$$