Chapter 7: Integration Techniques, L'Hôpital's Rule, and Improper Integrals

7.1 BASIC INTEGRATION FORMULAS

OBJECTIVE: Evaluate indefinite integrals by reducing the integrands to basic forms through algebra and substitution.

1. $\int x(a+x)^{1/3}dx$, where a is constant.

Let $u = a + x$. Then $du =$ ____________ and $x =$ ____________. Thus,

$\int x(a+x)^{1/3}dx = \int$ ____________ $du = \int u^{4/3}du - \int$ ____________ $du =$ ____________ $-\frac{3}{4}au^{4/3} + C$

$=$ ________________________ $+ C$.

2. $\int \frac{dx}{x\sqrt{x-5}}$

Let $u = \sqrt{x-5}$. The $du =$ ____________ and $x =$ ____________.

$\int \frac{dx}{x\sqrt{x-5}} = \int$ ____________ $du =$ ____________ $\int \frac{du}{\left(\frac{u}{\sqrt{5}}\right)^2 + 1} =$ ____________ $\tan^{-1}\frac{u}{\sqrt{5}} + C$

$=$ ________________________ $+ C$.

3. $\int \frac{\cos x\, dx}{\sqrt{1-\sin x}}$

Let $u = 1 - \sin x$ then $du =$ ____________ so that

$\int \frac{\cos x\, dx}{\sqrt{1-\sin x}} = \int$ ____________ $du = \int -u$____ $du =$ ____________ $+ C =$ ________________________.

4. $\int \frac{3^{\tan x}dx}{\cos^2 x}$

Let $u = \tan x$. Then $du =$ ____________ and the integral is

$\int \frac{3^{\tan x}dx}{\cos^2 x} = \int (\sec^2 x)3^{\tan x}dx = \int$ ____________ $du =$ ____________ $+ C =$ ____________.

5. $\int \frac{dx}{x + x(\ln x)^2}$

Let $u = \ln x$. Then $du =$ ____________ so that

$\int \frac{dx}{x+x(\ln x)^2} = \int \frac{dx}{x(\_\_\_\_\_\_\_\_\_\_\_\_)} = \int \frac{du}{\_\_\_\_\_\_\_\_\_\_\_\_} =$ ____________ $+ C =$ ____________.

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1. dx, $u-a$, $(u-a)u^{1/3}$, $au^{1/3}$, $\frac{3}{7}u^{7/3}$, $3(a+x)^{4/3}\left[\frac{1}{7}(a+x) - \frac{a}{4}\right]$

2. $\frac{dx}{2\sqrt{x-5}}$, u^2+5, $\frac{2}{u^2+5}$, $\frac{2}{5}$, $\frac{2\sqrt{5}}{5}$, $\frac{2}{\sqrt{5}}\tan^{-1}\sqrt{\frac{x}{5}-1}$

3. $-\cos x\, dx$, $\frac{-1}{\sqrt{u}}$, $-\frac{1}{2}$, $-2u^{1/2}$, $-2\sqrt{1-\sin x} + C$

4. $\sec^2 x\, dx$, 3^u, $\frac{1}{\ln 3}3^u$, $\frac{1}{\ln 3}3^{\tan x} + C$

5. $\frac{1}{x}dx$, $1+(\ln x)^2$, $1+u^2$, $\tan^{-1}u$, $\tan^{-1}(\ln x) + C$

6. $\int \frac{(x+1)dx}{x^2+x+5}$

Complete the square in the denominator to write it in the form

$$x^2+x+5=(x^2+x+\_\_\_\_\_\_\_\_)+5-\frac{1}{4}=(\_\_\_\_\_\_\_\_)^2+\frac{19}{4}.$$

Now, substitute $u = x + \frac{1}{2}$ and $du =$ __________ so the integral becomes

$$\int \frac{(x+1)dx}{x^2+x+5} = \int \frac{(x+1)dx}{\left(x+\frac{1}{2}\right)^2+\frac{19}{4}} = \int \frac{(\_\_\_\_\_\_\_\_)\,du}{u^2+\frac{19}{4}}.$$

Next, separate the fraction:

$$\int \frac{\left(u+\frac{1}{2}\right)du}{u^2+\frac{19}{4}} = \int \frac{\_\_\_\_\_\_\_\_}{u^2+\frac{19}{4}} + \frac{1}{2}\int \frac{du}{u^2+\frac{19}{4}}.$$

Finally, substitute $z = u^2 + \frac{19}{4}$ and $dz = 2u\,du$ in the first term on the right and then integrate:

$$\begin{aligned}\int \frac{(x+1)\,dx}{x^2+x+5} &= \int \frac{\_\_\_\_\_\_\_\_}{z} + \frac{1}{2}\int \frac{du}{u^2+\frac{19}{4}} \\ &= \_\_\_\_\_\_\_\_ + \frac{1}{2}\sqrt{\frac{4}{19}}\tan^{-1}\frac{u}{\sqrt{\frac{19}{4}}}+C \\ &= \frac{1}{2}\ln(\_\_\_\_\_\_\_\_)+\frac{1}{\sqrt{19}}\tan^{-1}\frac{2u}{\sqrt{19}}+C \\ &= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.\end{aligned}$$

7. $\int \frac{x^3}{x^2+1}dx$

First divide the denominator into the numerator, getting a quotient plus a remainder that is a proper fraction (where the degree of the numerator is less than the degree of the denominator):

$$\frac{x^3}{x^2+1} = x - \_\_\_\_\_\_\_\_.$$

Then,

$$\begin{aligned}\int \frac{x^3}{x^2+1} &= \int \_\_\_\_\_\_\_\_ - \int \frac{x}{x^2+1}dx \\ &= \_\_\_\_\_\_\_\_ - \int \_\_\_\_\_\_\_\_\, du \qquad (\text{where } u = x^2+1) \\ &= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \\ &= \frac{1}{2}x^2 + \ln\sqrt{x^2+1}+C.\end{aligned}$$

7.2 INTEGRATION BY PARTS

OBJECTIVE: Evaluate integrals by the method of integration by parts when the method applies.

8. The method of integration by parts is based on the product rule for differentiation. In terms of integrals,

$$\int u\,dv = \_\_\_\_\_\_\_\_\_\_\_\_.$$

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6. $\frac{1}{4}$, $x+\frac{1}{2}$, dx, $u+\frac{1}{2}$, $u\,du$, $\frac{1}{2}dz$, $\frac{1}{2}\ln|z|$, $u^2+\frac{19}{4}$, $\frac{1}{2}\ln(x^2+x+5)+\frac{1}{\sqrt{19}}\tan^{-1}\frac{2x+1}{\sqrt{19}}+C$

7. $\frac{x}{x^2+1}$, $x\,dx$, $\frac{1}{2}x^2$, $\frac{\frac{1}{2}}{u}$, $\frac{1}{2}x^2+\frac{1}{2}\ln|u|+C$

8. $uv - \int v\,du$

9. To employ the method of integration by parts successfully, we must be able to integrate the part ____________ immediately in order to obtain ____________. Also, it is desired that the new integral $\int v\,du$ be simpler than the original integral $\int u\,dv$.

10. $\displaystyle\int \frac{xe^x\,dx}{(1+x)^2}$

Let $u = xe^x$, $du =$ ____________, $dv = \dfrac{dx}{(1+x)^2}$, $v =$ ____________. Then,

$$\int \frac{xe^x\,dx}{(1+x)^2} = \frac{-xe^x}{1+x} + \int \underline{\hspace{3cm}}\,dx$$

$$\int \frac{xe^x\,dx}{(1+x)^2} = e^x\left(\frac{-x}{1+x} + \underline{\hspace{2.5cm}}\right) + C = \underline{\hspace{3cm}}.$$

11. $\displaystyle\int \tan^{-1}\sqrt{x}\,dx$

Let $u = \tan^{-1}\sqrt{x}$, $du =$ ____________, $dv =$ ____________, $v =$ ____________. Then,

$$\int \tan^{-1}\sqrt{x}\,dx = \underline{\hspace{3cm}} - \int \frac{\sqrt{x}\,dx}{2(1+x)}.$$

Let $w = \sqrt{x}$ and $dw =$ ____________, and the latter integral becomes

$$\int \frac{\sqrt{x}\,dx}{2(1+x)} = \int \frac{w^2\,dw}{\underline{\hspace{1.5cm}}} = \int (1 - \underline{\hspace{3cm}})dw = w - \underline{\hspace{3cm}} + C = \underline{\hspace{4cm}}.$$

Putting this together with our previous result, we find $\int \tan^{-1}\sqrt{x}\,dx =$ ____________.

12. $\displaystyle\int_1^e (\ln x)^2\,dx$

Let $u = (\ln x)^2$, $du =$ ____________, $dv =$ ____________, $v =$ ____________. Then,

$$\int_1^e (\ln x)^2\,dx = x(\ln x)^2]_1^e - 2\int_1^e \underline{\hspace{3cm}} = (e - \underline{\hspace{2.5cm}}) - 2[\underline{\hspace{3cm}}]_1^e$$

↑ Example 4 in this section of the text

$$= e - 2(\underline{\hspace{2cm}}) = \underline{\hspace{3cm}}.$$

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9. dv, v

10. $e^x(1+x)\,dx$, $-\dfrac{1}{1+x}$, e^x, 1, $\dfrac{e^x}{1+x} + C$

11. $\dfrac{dx}{2\sqrt{x}(1+x)}$, dx, x, $x\tan^{-1}\sqrt{x}$, $\dfrac{dx}{2\sqrt{x}}$, $1+w^2$, $\dfrac{1}{1+w^2}$, $\tan^{-1} w$, $\sqrt{x} - \tan^{-1}\sqrt{x}$, $(x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$

12. $\dfrac{2}{x}\ln x\,dx$, dx, x, $\ln x\,dx$, 0, $x\ln x - x$, $(e - e + 1)$, $e - 2$

13. $\int x^2 \sin x\, dx$

Let $u = x^2$, $du =$ ______________, $dv =$ ______________, $v =$ ______________. Then,
$\int x^2 \sin x\, dx =$ ______________ $+ 2\int x \cos x\, dx$. To determine the latter integral we integrate by parts again:
$U = x$, $dU =$ ______________, $dV =$ ______________, $V =$ ______________. Then,
$\int x \cos x\, dx = x \sin x - \int$ ______________ $=$ ______________ $+ C$. Therefore, putting these results together,
we find $\int x^2 \sin x\, dx =$ ______________.

14. $\int \frac{\tan^{-1} x\, dx}{x^2}$

Let $u = \tan^{-1} x$, $du =$ ______________, $dv =$ ______________, $v =$ ______________. Then,

$$\int \frac{\tan^{-1} x\, dx}{x^2} = \underline{\hspace{3cm}} + \int \frac{dx}{x(1+x^2)} = \underline{\hspace{3cm}} + \int \frac{dx}{x} + \int \frac{\underline{\hspace{1.5cm}}}{1+x^2}$$

$$= \underline{\hspace{4cm}}.$$

15. Evaluate $\int e^{2x} \sin 3x\, dx$ by tabular integration.

Solution.

Let $u = f(x) = e^{2x}$ and $dv = g(x)\, dx = \sin 3x\, dx$. Then complete the following table:

$f(x)$ and its derivatives	signs	$g(x)$ and its integrals
e^{2x}	(+)	$\sin 3x$
_____	(−)	_____
_____	(+)	_____

The "stopping rule" for the last row in the table is that you can integrate the product of the functions in the last row, or the product is a *constant* times the product of the functions in the first row. Of course, if you obtain a zero in the $f(x)$ column the table is finished because the integral of zero is zero. In this problem, the product of the last row is a constant times the first row. The table reads:

$$e^{2x}\left(-\frac{1}{3}\cos 3x\right) - 2e^{2x}\left(-\frac{1}{9}\sin 3x\right) + \int 4e^{2x}\left(-\frac{1}{9}\right)\sin 3x\, dx$$

Therefore, $\int e^{2x} \sin 3x\, dx = e^{2x}\left(-\frac{1}{3}\cos 3x + \frac{2}{9}\sin 3x\right) - \frac{4}{9}\int e^{2x} \sin 3x\, dx + C$. Combining the two integrals

gives $\int e^{2x} \sin 3x\, dx =$ ________________________________ or, simplifying arithmetically,

$\int e^{2x} \sin 3x\, dx = \frac{1}{13} e^{2x}(2 \sin 3x - 3 \cos 3x) + C_1$.

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13. $2x\, dx$, $\sin x\, dx$, $-\cos x$, $-x^2 \cos x$, dx, $\cos x\, dx$, $\sin x$, $\sin x\, dx$, $x \sin x + \cos x$, $(2 - x^2)\cos x + 2x \sin x + C$

14. $\frac{1}{1+x^2} dx$, $\frac{dx}{x^2}$, $-\frac{1}{x}$, $-\frac{\tan^{-1} x}{x}$, $-\frac{\tan^{-1} x}{x}$, $-x\, dx$,
$-\frac{\tan^{-1} x}{x} + \ln|x| - \frac{1}{2}\ln(1+x^2) + C$

15.

$f(x)$ and its derivatives	signs	$g(x)$ and its integrals
e^{2x}	(+)	$\sin 3x$
$2e^{2x}$	(−)	$-\frac{1}{3}\cos 3x$
$4e^{2x}$	(+)	$-\frac{1}{9}\sin 3x$

$$\frac{9}{13} e^{2x}\left[\frac{2}{9}\sin 3x - \frac{1}{3}\cos 3x\right] + C_1 \left(\text{where } C_1 = \frac{9}{13} C\right)$$

SECTION 7.3 PARTIAL FRACTIONS

OBJECTIVE: Find indefinite integrals of rational functions by the method of partial fractions expansion.

16. The success of separating a rational function $\dfrac{f(x)}{g(x)}$ into a sum of partial fractions hinges upon two things:

(1) The degree of $f(x)$ must be ______________________________. If this is not the case, one must first perform ____________________, then work with the ____________________ term.

(2) The factors of ____________________ must be known. In practice it may be difficult to perform the factorization.

17. To find A and B in the partial fractions expansion of

$$\frac{3x+1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

by the method of undetermined coefficients, first clear the equation of fractions:

$$3x+1 = \underline{\hspace{6cm}}.$$

Next collect like powers of x on the right side:

$$3x+1 = (\underline{\hspace{4cm}})x + (2A+B).$$

This will be an identity in x if and only if

$$A+B = \underline{\hspace{4cm}} \text{ and } \underline{\hspace{4cm}} = 1.$$

These equations determine the values of A and B to be $A =$ ____________________, $B = 5$.

18. From Problem 17,

$$\int \frac{3x+1}{(x+1)(x+2)}\,dx = \int \frac{-2}{x+1}\,dx + \int \underline{\hspace{4cm}} = \underline{\hspace{6cm}}.$$

19. To find A, B, and C in the partial fractions expansion

$$\frac{2x^2-x+1}{(x+1)(x-3)(x+2)} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{x+2}$$

by the method of undetermined coefficients, first clear the equation of fractions:

$$2x^2-x+1 = A(x-3)(x+2) + B(x+1)(x+2) + C(\underline{\hspace{2cm}})(\underline{\hspace{2cm}}).$$

Next expand the products on the right and collect like powers of x:

$$2x^2-x+1 = (A+B+C)x^2 + (\underline{\hspace{4cm}})x + (-6A+2B-3C).$$

This equation is an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A+B+C = \underline{\hspace{3cm}}, -A+3B-2C = \underline{\hspace{3cm}}, \underline{\hspace{4cm}} = 1.$$

These equations determine the values of A, B, and C to be $A = -1$, $B = \dfrac{4}{5}$ and $C = \dfrac{11}{5}$. Check that these values satisfy the three equations.

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16. less than the degree of $g(x)$, long division, remainder, $g(x)$

17. $A(x+2) + B(x+1), A+B, 3, 2A+B, -2$

18. $\dfrac{5}{x+2}dx,\ -2\ \ln|x+1| + 5\ \ln|x+2| + C$

19. $x+1, x-3, -A+3B-2C, 2, -1, -6A+2B-3C$

20. To find A, B, and C in the partial fractions expansion $\dfrac{2x^2-x+1}{(x+1)(x-3)(x+2)} = \dfrac{A}{x+1}+\dfrac{B}{x-3}+\dfrac{C}{x+2}$ by the Heaviside technique, the value of A can be found by covering up the factor ______________ in the left side and evaluating the result at $x =$ ____________:

$$A = \frac{2(-1)^2-(-1)+1}{\_\_\_\_\_\_\_\_} = \frac{4}{\_\_\_\_\_\_} = \_\_\_\_\_\_\_\_\_\_\_\_.$$

Similarly, $B = \dfrac{2(3)^2-(3)+1}{\_\_\_\_\_\_\_\_} = \dfrac{16}{\_\_\_\_} =$ ____________, and $C = \dfrac{\_\_\_\_\_\_\_\_\_\_\_\_}{(-2+1)(-2-3)} = \dfrac{\_\_\_\_}{5}$.

21. $\displaystyle\int \frac{(2x^2-x+1)dx}{(x+1)(x-3)(x+2)} = \int\frac{-dx}{\_\_\_\_\_\_} + \frac{4}{5}\int\frac{dx}{\_\_\_\_\_\_} + \frac{11}{5}\int\frac{dx}{\_\_\_\_\_\_} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ + C.$

22. $\displaystyle\int \frac{4\,dx}{x^3+4x}$

To expand $\dfrac{4}{x^3+4x}$ into a sum of partial fractions, first write,

$$\frac{4}{x^3+4x} = \frac{4}{x(x^2+4)} = \frac{A}{x} + \frac{\_\_\_\_\_\_}{x^2+4}.$$

Then, $4 = A(x^2+4) +$ ____________ $= (\_\_\_\_\_\_)x^2 + Cx + 4A$. Thus, equating coefficients of like powers of x, $A + B =$ ____________, $C =$ ____________, and $4A =$ ____________. Solving these equations gives $A =$ ____________, $B =$ ____________, and $C =$ ____________. Hence,

$$\int\frac{4\,dx}{x^3+4x} = \int\frac{dx}{\_\_\_\_\_\_} - \int\frac{x\,dx}{\_\_\_\_\_\_} = \ln|x| - \_\_\_\_\_\_\_\_\_\_\_\_ + C' = \_\_\_\_\_\_\_\_\_\_\_\_.$$

23. $\displaystyle\int\frac{dx}{\sin x\cos x}$

Write $\displaystyle\int\frac{dx}{\sin x\cos x} = \int\frac{\sin x\,dx}{\sin^2 x\,\cos x} = \int\frac{\sin x\,dx}{(1-\cos^2 x)\cos x}$, and let $u = \cos x$, $du =$ ____________ so that $\displaystyle\int\frac{dx}{\sin x\,\cos x} = \int\frac{-du}{\_\_\_\_\_\_\_\_\_\_}$. Next, let $\dfrac{1}{(1-u^2)u} = \dfrac{1}{(1-u)(1+u)u} = \dfrac{A}{u}+\dfrac{B}{1-u}+\dfrac{C}{1+u}$.

By the Heaviside technique, $A =$ ____________, $B =$ ____________, and $C =$ ____________. Thus,

$$\int\frac{-du}{(1-u^2)u} = -\int\frac{du}{\_\_\_\_\_\_} - \frac{1}{2}\int\frac{du}{\_\_\_\_\_\_} + \frac{1}{2}\int\frac{du}{\_\_\_\_\_\_} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ + C'$$

$$= \ln\frac{1}{|u|}[\_\_\_\_\_\_\_\_\_\_\_\_] + C' = \ln\frac{\sqrt{1-\cos^2 x}}{\_\_\_\_\_\_\_\_} + C' = \_\_\_\_\_\_\_\_\_\_\_\_.$$

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20. $(x+1)$, -1, $(-1-3)(-1+2)$, -4, -1, $(3+1)(3+2)$, 20, $\dfrac{4}{5}$, $2(-2)^2-(-2)+1$, 11

21. $x+1$, $x-3$, $x+2$, $-\ln|x+1|+\dfrac{4}{5}\ln|x-3|+\dfrac{11}{5}\ln|x+2|$

22. $Bx+C$, $x(Bx+C)$, $A+B$, 0, 0, 4, 1, -1, 0, x, x^2+4, $\dfrac{1}{2}\ln(x^2+4)$, $\ln\dfrac{|x|}{\sqrt{x^2+4}}+C'$

23. $-\sin x\,dx$, $(1-u^2)u$, 1, $\dfrac{1}{2}$, $-\dfrac{1}{2}$, u, $1-u$, $1+u$, $-\ln|u|+\dfrac{1}{2}\ln|1-u|+\dfrac{1}{2}\ln|1+u|$, $(1-u^2)^{1/2}$, $|\cos x|$, $\ln|\tan x|+C'$

24. $\int \frac{(2x^2+x+2)\,dx}{x(x-1)^2}$

By partial fractions,

$$\frac{2x^2+x+2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \underline{\qquad\qquad}.$$

Clearing fractions and equating numerators gives,

$$2x^2+x+2 = A(x-1)^2 + Bx(x-1) + \underline{\qquad\qquad}$$
$$= (A+B)x^2 + (\underline{\qquad\qquad})x + A.$$

Equating coefficients of like powers of x gives the equations $A + B = 2$, $\underline{\qquad\qquad} = 1$, and $\underline{\qquad\qquad} = 2$. Solving, $A = \underline{\qquad\qquad}$, $B = \underline{\qquad\qquad}$, and $C = 5$. Hence,

$$\int \frac{(2x^2+x+2)\,dx}{x(x-1)^2} = 2\int \frac{dx}{\underline{\qquad}} + 5\int \frac{dx}{\underline{\qquad}} = \underline{\qquad\qquad}.$$

25. $\int \frac{(x^3-x+2)\,dx}{x^2-x}$

Here the degree of the numerator fails to be less than the degree of the denominator so we must use long division and work with the remainder.

By long division, $\frac{x^3-x+2}{x^2-x} = x+1+\underline{\qquad\qquad}$. Therefore, $\int \frac{(x^3-x+2)\,dx}{x^2-x} = \int (x+1)\,dx + 2\int \frac{dx}{\underline{\qquad}}$.

By partial fractions, $\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$. Clearing fractions and equating numerators gives

$$1 = A(x-1) + B\underline{\qquad\qquad} = (\underline{\qquad\qquad})x - A.$$

This will be an identity in x if and only if

$$A + B = \underline{\qquad\qquad} \text{ and } \underline{\qquad\qquad} = 1.$$

These equations determine the values of A and B to be $A = \underline{\qquad\qquad}$ and $B = \underline{\qquad\qquad}$.

Therefore,

$$\int \frac{(x^3-x+2)\,dx}{x^2-x} = \underline{\qquad\qquad} + 2\int \frac{dx}{\underline{\qquad}} - 2\int \frac{dx}{\underline{\qquad}}$$
$$= \frac{x^2}{2} + x + 2(\underline{\qquad\qquad}) + C$$
$$= \underline{\qquad\qquad}.$$

SECTION 7.4 TRIGONOMETRIC SUBSTITUTIONS

OBJECTIVE A: Find indefinite integrals of integrands involving the radicals $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, and $\sqrt{u^2-a^2}$, and the binomials a^2+u^2, a^2-u^2.

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24. $\frac{C}{(x-1)^2}$, Cx, $-2A-B+C$, $-2A-B+C$, A, 2, 0, x, $(x-1)^2$, $2\ln|x| - \frac{5}{x-1} + C'$

25. $\frac{2}{x^2-x}$, $x(x-1)$, x, $A+B$, $0, -A$, -1, 1, $\frac{1}{2}x^2+x$, $x-1$, x, $\ln|x-1| - \ln|x|$, $\frac{x^2}{2}+x+2\ln\left|\frac{x-1}{x}\right|+C$

26. $\displaystyle\int \frac{x^3\,dx}{\sqrt{4-x^2}}$

Let $x = 2\sin u$ so that $dx =$ ______________, and the integral becomes

$$\int \frac{x^3\,dx}{\sqrt{4-x^2}} = \int \frac{\underline{\qquad}\cdot 2\cos u\,du}{\sqrt{4-4\sin^2 u}} = \int \frac{\underline{\qquad}\,du}{\sqrt{4\cos^2 u}}$$

$$= \pm 8\int \sin^3 u\,du \quad (\pm \text{ depends on sign of } \underline{\qquad\qquad})$$

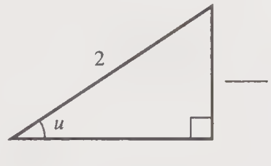
$$= 8\int \sin^3 u\,du, \quad \text{using only the principal value of } u = \sin^{-1}\frac{x}{2}$$

$$= 8\int (1-\cos^2 u)\ \underline{\qquad\qquad\qquad}$$

$$= 8(-\cos u + \underline{\qquad\qquad}) + C.$$

For the substitution $x = 2\sin u$, finish labeling the diagram below. Then, from the diagram,

$\cos u =$ ______________, and substitution gives $\displaystyle\int \frac{x^3\,dx}{\sqrt{4-x^2}} = -4\sqrt{4-x^2} +$ ________________ $+\,C.$



27. $\displaystyle\int \frac{dx}{(x^2-9)^{3/2}}$

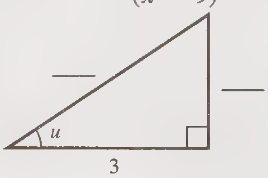
Let $x = 3\sec u$ so that $dx =$ ________________ and the integral becomes

$$\int \frac{dx}{(x^2-9)^{3/2}} = \int \frac{\underline{\qquad\qquad}}{(9\sec^2 u - 9)^{3/2}} = \int \frac{3\sec u\tan u\,du}{27\ \underline{\qquad\qquad}}$$

$$= \frac{1}{9}\int \frac{\sec u\,du}{\underline{\qquad\qquad}} = \frac{1}{9}\int \frac{\cos u\,du}{\underline{\qquad\qquad}} = \underline{\qquad\qquad\qquad} + C.$$

For the substitution $x = 3\sec u$, finish labeling the diagram below. Then, from the diagram,

$\sin u =$ ______________, and substitution gives $\displaystyle\int \frac{dx}{(x^2-9)^{3/2}} =$ __________________________.



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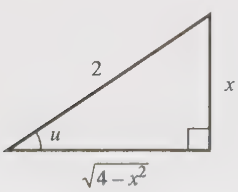
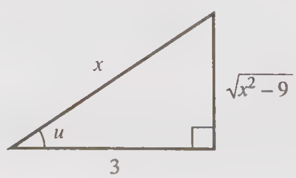
26. $2\cos u\,du$, $8\sin^3 u$, $8\sin^3 u \cdot 2\cos u$, $\cos u$, $\sin u\,du$, $\frac{1}{3}\cos^3 u$, $\frac{1}{2}\sqrt{4-x^2}$, $\frac{1}{3}(4-x^2)^{3/2}$



27. $3\sec u\tan u\,du$, $3\sec u\tan u\,du$, $\tan^3 u$, $\tan^2 u$, $\sin^2 u$, $-\dfrac{1}{9\sin u}$, $\dfrac{\sqrt{x^2-9}}{x}$, $\dfrac{-x}{9\sqrt{x^2-9}} + C$



28. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+16}}$

Let $x = 4\tan u$ so that $dx =$ ______________ and the integral becomes

$$\int \frac{dx}{x^2\sqrt{x^2+16}} = \int \frac{4\sec^2 u\, du}{\underline{\qquad\qquad}} = \frac{1}{16}\int \frac{\sec u\, du}{\underline{\qquad\qquad}}$$

$$= \frac{1}{16}\int \frac{(\cos^2 u)\sec u\, du}{\sin^2 u} = \frac{1}{16}\int \frac{\underline{\qquad\qquad}}{\sin^2 u}\, du.$$

We substitute again: $z = \sin u$ so that $dz =$ ____________________ and the integral becomes

$$\frac{1}{16}\int \frac{\sec u\, du}{\tan^2 u} = \frac{1}{16}\int \frac{\cos u\, du}{\sin^2 u} = \frac{1}{16}\int \underline{\qquad\qquad}\, dz$$

$$= \frac{1}{16}(\underline{\qquad\qquad}) + C$$

$$= \underline{\qquad\qquad\qquad}.$$

We substitute again: $z = \sin u$ so that $dz =$ ______________ and the integral becomes

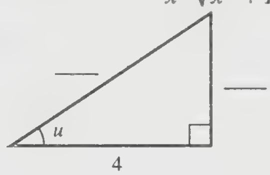
$$\frac{1}{16}\int \frac{\sec u\, du}{\tan^2 u} = \frac{1}{16}\int \frac{\cos u\, du}{\sin^2 u} = \frac{1}{16}\int \underline{\qquad\qquad}\, dz$$

$$= \frac{1}{16}(\underline{\qquad\qquad}) + C$$

$$= \underline{\qquad\qquad\qquad}.$$

For the substitution $x = 4\tan u$, finish labeling the diagram below. Then, from the diagram,

$\csc u =$ ________________, and substitution gives $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+16}} =$ ____________________.



OBJECTIVE B: Calculate definite integrals involving $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, $\sqrt{u^2-a^2}$, a^2+u^2, and a^2-u^2.

29. $\displaystyle\int_4^{4\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+16}}$

As in the previous Problem 28, let $x = 4\tan u$. When $x = 4$, $\tan u = 1$ or $u =$ ____________; when $x = 4\sqrt{3}$, $\tan u = \sqrt{3}$ or $u =$ ______________. Thus, upon substitution,

$$\int_4^{4\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+16}} = \frac{1}{16}\int_{\underline{\quad}}^{\underline{\quad}} \frac{\sec u\, du}{\tan^2 u} \quad \text{(by Problem 28)}$$

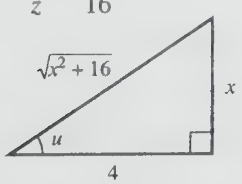
$$= -\frac{1}{16}\csc u\Big]_{\underline{\quad}}^{\underline{\quad}}$$

$$= -\frac{1}{16}(\underline{\qquad\qquad}) \approx 0.016.$$

---

28. $4\sec^2 u\, du$, $16\tan^2 u \cdot 4\sec u$, $\tan^2 u$, $\cos u$, $\cos u\, du$, z^{-2}, **29.** $\dfrac{\pi}{4}$, $\dfrac{\pi}{3}$, $\displaystyle\int_{\pi/4}^{\pi/3}$, $\Big]_{\pi/4}^{\pi/3}$, $\dfrac{2\sqrt{3}}{3} - \sqrt{2}$

$-\dfrac{1}{z}$, $-\dfrac{1}{16}\csc u + C$, $\dfrac{\sqrt{x^2+16}}{x}$, $-\dfrac{\sqrt{x^2+16}}{16x} + C$



SECTION 7.5 INTEGRAL TABLES, COMPUTER ALGEBRA SYSTEMS, AND MONTE CARLO INTEGRATION

OBJECTIVE: Find indefinite integrals with the aid of integral tables.

30. $\int \frac{dx}{x^2\sqrt{4+x^2}}$

We use Formula 27 with $a =$ ________. Thus, $\int \frac{dx}{x^2\sqrt{4+x^2}} =$ ____________ $+ C$.

31. $\int x^2 \ln 3x\, dx$

We use Formula 110 with $n =$ ________ and $a =$ ________. Thus, $\int x^2 \ln 3x\, dx =$ ________________ $+ C$.

SECTION 7.6 L'HÔPITAL'S RULE

OBJECTIVE A: Use l'Hôpital's rule to find limits of indeterminate forms of type 0/0 or ∞/∞.

32. $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x}$ is an indeterminate form of type ____________ so l'Hôpital's rule applies. Thus,

$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty}$ _______ $=$ _______.

33. $\lim_{x\to 0} \frac{e^{2x}-1}{x}$ is an indeterminate form of type _________ so l'Hôpital's rule applies. Thus,

$\lim_{x\to 0} \frac{e^{2x}-1}{x} = \lim_{x\to 0} \frac{\_\_\_\_\_}{1} =$ _________.

34. $\lim_{\theta\to\frac{\pi}{2}^-} \frac{\ln(\tan\theta)}{\sec\theta}$

As $\theta \to \frac{\pi}{2}^-$, $\tan\theta \to \infty$ and $\sec\theta \to$ __________. Hence $\ln(\tan\theta) \to$ ________ and this is an indeterminate form of type ____________. Therefore, l'Hôpital's rule applies:

$\lim_{\theta\to\frac{\pi}{2}^-} \frac{\ln(\tan\theta)}{\sec\theta} = \lim_{\theta\to\frac{\pi}{2}^-} \frac{\_\_\_\_\_\_\_\_\_\_}{\sec\theta\tan\theta} = \lim_{\theta\to\frac{\pi}{2}^-} \frac{\_\_\_\_\_}{\sin^2\theta} =$ ___________.

35. $\lim_{x\to 0^+} x^2 \ln x$

As $x \to 0^+$, $x^2 \to 0$ and $\ln x \to$ ________ so this is an indeterminate form $0 \cdot \infty$. Writing the limit as

$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{\_\_\_\_\_\_}$, we see this form is now of the type ∞/∞. Therefore, l'Hôpital's rule applies:

$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x\to 0^+}$ __________ $= \lim_{x\to 0^+}$ __________ $=$ __________. Hence, $\lim_{x\to 0^+} x^2 \ln x =$ ___________.

---

30. $2,\ -\frac{\sqrt{4+x^2}}{4x}$

31. $2,\ 3,\ \frac{x^3}{3}\ln 3x - \frac{x^3}{9}$

32. $\frac{\infty}{\infty},\ \frac{1}{e^x},\ 0$

33. $\frac{0}{0},\ 2e^{2x},\ 2$

34. $\infty,\ \infty,\ \frac{\infty}{\infty},\ \frac{\sec^2\theta}{\tan\theta},\ \cos\theta,\ 0$

35. $-\infty,\ \frac{1}{x^2},\ \frac{\frac{1}{x}}{\frac{-2}{x^3}},\ -\frac{1}{2}x^2,\ 0,\ 0$

OBJECTIVE B: Use l'Hôpital's rule to find limits of indeterminate forms of type 1^∞, 0^0, and ∞^0.

36. $\lim_{x\to 0}(1-x)^{1/\sqrt{x}}$ is the form of __________. We let $f(x)=(1-x)^{1/\sqrt{x}}$ and find $\lim_{x\to 0}\ln f(x)$. Now, $\ln f(x)=$ ________ $\cdot \ln(1-x)$, so l'Hôpital's rule gives

$$\lim_{x\to 0}\ln f(x)=\lim_{x\to 0}\frac{\_\_\_\_\_\_}{\sqrt{x}},\ (0/0 \text{ form})$$

$$=\lim_{x\to 0}\frac{\frac{-1}{1-x}}{\_\_\_\_\_\_}=\lim_{x\to 0}\frac{2\sqrt{x}}{x-1}=\_\_\_\_\_\_.$$

Therefore, $\lim_{x\to 0}f(x)=$ __________.

37. $\lim_{x\to 0^+}x^{\sin x}$ is of the form __________. Let $y=x^{\sin x}$. Then $\ln y=$ __________. Thus $\ln y=\frac{\ln x}{\_\_\_\_}$. Then $\lim_{x\to 0^+}\frac{\ln x}{\csc x}$ is an indeterminate form of type __________ and l'Hôpital's rule applies. Hence,

$$\lim_{x\to 0^+}\frac{\ln x}{\csc x}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\csc x\cot x}=\lim_{x\to 0^+}\frac{\_\_\_\_\_\_}{-x\cos x}$$

$$=\lim_{x\to 0^+}\frac{\_\_\_\_\_\_}{x\sin x-\cos x}=\_\_\_\_\_\_.$$

Since $\ln y\to 0$ as $x\to 0^+$, $y\to$ __________ and the required limit is __________.

SECTION 7.7 IMPROPER INTEGRALS

38. A definite integral $\int_a^b f(x)\,dx$ is termed *improper* if,

(a) either limit of integration is __________ or __________, or
(b) $f(x)$ becomes __________ at some value of $x=c$ satisfying __________.

OBJECTIVE A: Determine whether a given improper integral converges or diverges. If convergent, evaluate it.

39. $\int_1^\infty \frac{\ln x}{x^2}dx=\lim_{b\to\infty}\int_{\_\_}^{\_\_}\frac{\ln x}{x^2}dx$

To calculate the latter integral, we integrate by parts: let $u=\ln x$, $du=$ __________, $dv=$ __________, $v=$ __________. $\int_1^b\frac{\ln x\,dx}{x^2}=-\frac{\ln x}{x}\Big]_1^b+\int_1^b$ __________ $=\frac{-\ln b}{b}+$ __________. Thus,

$$\int_1^\infty\frac{\ln x\,dx}{x^2}=\lim_{b\to\infty}(\_\_\_\_\_\_)=\lim_{b\to\infty}\frac{-\ln b}{b}+\_\_\_\_\_\_$$

$$=\lim_{b\to\infty}\_\_\_\_\_\_+1=\_\_\_\_\_\_.$$

---

36. 1^∞, $\frac{1}{\sqrt{x}}$, $\ln(1-x)$, $\frac{1}{2\sqrt{x}}$, 0, $e^0=1$

37. 0^0, $\sin x\cdot\ln x$, $\csc x$. ∞/∞, $\sin^2 x$, $2\sin x\cos x$, 0, 1, 1

38. $+\infty$, $-\infty$, infinite, $a\le c\le b$

39. $\int_1^b$, $\frac{1}{x}dx$, $\frac{1}{x^2}dx$, $-\frac{1}{x}$, $\frac{dx}{x^2}$, $-\frac{1}{x}\Big]_1^b$, $\frac{-\ln b}{b}-\frac{1}{b}+1$, 1, $\frac{-\frac{1}{b}}{1}$ (l'Hôpital's rule), 1

40. $\int_0^1 \ln x\, dx$

Here, $\ln x \to$ ________ as $x \to 0^+$ making the integral improper. Thus, $\int \ln x\, dx = \lim_{b\to 0^+} \int_{\_\_}^{\_\_} \ln x\, dx$.

To calculate the latter integral, we integrate by parts: let $u = \ln x$, $du =$ ____________, $dv =$ ____________, $v =$ ____________.

$\int_b^1 \ln x\, dx =$ ___________$\Big]_b^1 - \int_b^1 dx =$ ___________. Therefore,

$$\int_0^1 \ln x\, dx = \lim_{b\to 0^+}(-b\ln b - 1 + b) = \lim_{b\to 0^+}\left(\frac{-\ln b}{\frac{1}{b}}\right) - 1$$
$$= \lim_{b\to 0^+}(\_\_\_\_\_\_\_\_) - 1 = \_\_\_\_\_\_\_\_.$$

41. $\int_0^\infty \frac{x\, dx}{\sqrt{1+x^3}}$

Observe that $\frac{x}{\sqrt{1+x^3}} > \frac{1}{x}$ is true whenever $x^2 >$ ____________ or whenever $x^4 >$ ____________, and the last inequality certainly holds if $x \ge 2$. Thus, we consider the improper integral $\int_2^\infty \frac{dx}{x}$. Now, $\int_2^\infty \frac{dx}{x} = \lim_{b\to\infty}[\ln b - \ln 2] = +\infty$. Thus, since the original integral satisfies the inequalities, $\int_0^\infty \frac{x\, dx}{\sqrt{1+x^3}} \ge \int_2^\infty \frac{x\, dx}{\sqrt{1+x^3}} \ge \int_2^\infty \frac{dx}{x}$, it ____________ to $+\infty$.

---

40. $-\infty$, $\int_b^1$, $\frac{dx}{x}$, dx, x, $x\ln x$, $-b\ln b - 1 + b$, $\frac{\frac{1}{b}}{\frac{1}{b^2}}$ (l'Hôpital's rule), -1

41. $\sqrt{1+x^3}$ (since $x > 0$), $1 + x^3$, diverges

42. $\int_0^2 \frac{(x^2-3x+1)\,dx}{x(x-1)^2}$

In this case the integrand becomes infinite when $x =$ ____________ or $x =$ ______________, so the integrand is improper. To simplify the notation momentarily, let $f(x)$ denote the integrand. Then,

$$\int_0^2 f(x)\,dx = \lim_{b\to 0^+}\int_b^{1/2} f(x)\,dx + \lim_{c\to 1^-}\int_{1/2}^{c} f(x)\,dx + \lim_{h\to 1^+}\int_{\_\_\_}^{\_\_\_} \_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$

To find the indefinite integral $\int \frac{(x^2-3x+1)\,dx}{x(x-1)^2}$ we expand the integrand by partial fractions:

$$\frac{x^2-3x+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Therefore,

$$x^2-3x+1 = A(x-1)^2 + Bx(x-1) + \_\_\_\_\_\_\_\_\_\_ = (A+B)x^2 + (\_\_\_\_\_\_\_\_\_\_)x + A.$$

It follows that $A = 1$, $A + B =$ _________, and _________ $= -3$. Solving, $B =$ _________ and $C =$ ________.

Hence,

$$\int \frac{(x^2-3x+1)dx}{x(x-1)^2} = \int \frac{dx}{x} + \int \_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \ln|x| + \_\_\_\_\_\_\_\_\_\_ + C'.$$

Therefore,

$$\int_0^2 \frac{(x^2-3x+1)\,dx}{x(x-1)^2} = \lim_{b\to 0^+}(\_\_\_\_\_\_\_\_\_\_)\Big]_b^{1/2} + \lim_{c\to 1^-}(\_\_\_\_\_\_\_\_\_\_)\Big]_{1/2}^{c} + \lim_{h\to 1^+}(\_\_\_\_\_\_\_\_\_\_)\Big]_h^{2}$$

The first limit is ____________, the second limit is ____________, and the third is ____________. Therefore, the improper integral is ________________.

OBJECTIVE B: Use the Comparison Test, if applicable, to determine the convergence or divergence of an improper integral $\int_a^\infty f(x)\,dx$.

43. The improper integrals in Problems 40 and 41 could be implicitly evaluated with the aid of indefinite integrals. Such is not the case, however, for the improper integral

$$\int_0^\infty \frac{\sin x\,dx}{1+x^2} = \lim_{b\to\infty}\int_{\_\_\_}^{\_\_\_} \frac{\sin x\,dx}{1+x^2}.$$

However, since $-1 \le \sin x \le 1$ we have that

$$\frac{\sin x}{1+x^2} \le \frac{1}{1+x^2} \text{ on } [0, \infty).$$

Thus, evaluating the integral of the function on the right gives

$$\lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \_\_\_\_\_\_\_\_\_\_\_\_\Big]_0^b = \_\_\_\_\_\_\_\_\_\_\_\_ .$$

Therefore, we know that the original integral $\int_0^\infty \frac{\sin x\,dx}{1+x^2}$ is ________________.

OBJECTIVE C: Use the Limit Comparison Test, if applicable, to determine the convergence or divergence of an improper integral $\int_a^\infty f(x)\,dx$.

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42. $0, 1, \int_h^2 f(x)\,dx, Cx, -2A - B + C, 1, -2A - B + C, 0, -1, \frac{-dx}{(x-1)^2}, \frac{1}{x-1}, \ln x + \frac{1}{x-1}, \ln x + \frac{1}{x-1}, \ln x + \frac{1}{x-1},$
$-\infty, -\infty, \infty$, divergent

43. $\int_0^b$, $\tan^{-1} x$, $\frac{\pi}{2}$, convergent

44. Consider the improper integral $\int_1^\infty \frac{1}{x(x^2-1)}dx$. With $f(x) = \frac{1}{x^3 - x}$ and $g(x) = \frac{1}{x^3}$ we have

$\lim_{x\to\infty} \frac{1}{x^3} \cdot \frac{x^3 - x}{1} = \lim_{x\to\infty}$ (______________) = ______________. Since the limit is finite and positive we conclude that $\int_1^\infty \frac{dx}{x(x^2-1)}$ ________________ because $\int_1^\infty \frac{dx}{x^3}$ ________________.

---

44. $1 - \frac{1}{x^2}$, 1, converges, converges (See Example 11 in this section of the text.)

CHAPTER 7 SELF-TEST

Evaluate the integrals in Problems 1–15.

1. $\int \frac{x\,dx}{x^4+1}$

2. $\int \frac{(4x+1)\,dx}{x^2-3x-4}$

3. $\int \frac{\ln x\,dx}{(x+1)^2}$

4. $\int \frac{\sqrt{9-4x^2}\,dx}{x}$

5. $\int \frac{e^{3x/2}\,dx}{1+e^{3x/4}}$

6. $\int \csc^3 x\,dx$ (Tables)

7. $\int \sin^3 x \cos^4 x\,dx$

8. $\int \cot^5 x\,dx$

9. $\int \frac{x\,dx}{x^2+x+1}$

10. $\int \cos^2 x \sin 2x\,dx$

11. $\int_{-\pi/3}^{\pi/4} x \sec^2 x\,dx$

12. $\int_0^1 \frac{(2x^3+x+3)\,dx}{x^2+1}$

13. $\int_5^{5\sqrt{3}} \frac{dx}{x\sqrt{25+x^2}}$

14. $\int_0^a \ln(a^2+x^2)\,dx,\ a>0$

15. $\int \frac{x-2}{(3x-1)^2}\,dx$

In Problems 16–23, evaluate the limits using l'Hôpital's rule.

16. $\lim_{x\to 0} \frac{\sin x - x\cos x}{x^3}$

17. $\lim_{x\to\infty} \frac{x^2-5}{2x^2+3x}$

18. $\lim_{x\to\infty} \frac{\sin\left(\frac{3}{x}\right)}{\frac{2}{x}}$

19. $\lim_{x\to\frac{\pi}{2}^-} \left(x\tan x - \frac{\pi}{2}\sec x\right)$

20. $\lim_{x\to 0} \csc x \sin^{-1} x$

21. $\lim_{x\to 1} x^{1/(1-x)}$

22. $\lim_{x\to 0^+} \frac{e^x - \cos x}{x\sin x}$

23. $\lim_{x\to 0} \frac{4^x - 2^x}{x}$

In Problems 24–27, determine the convergence or divergence of the integral. If the integral is convergent, find its value.

24. $\int_0^{\pi/2} \sec x \tan x\,dx$

25. $\int_{-1}^{1} \frac{dx}{x^{2/3}}$

26. $\int_1^{\infty} \frac{\ln x\,dx}{x}$

27. $\int_0^{\infty} \frac{\sin x\,dx}{e^x}$

SOLUTIONS TO CHAPTER 7 SELF-TEST

1. Let $u=x^2$, $du=2x\,dx$. Then, $\int \frac{x\,dx}{x^4+1} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1} x^2 + C.$

2. $\frac{4x+1}{x^2+3x-4} = \frac{4x+1}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}$
Clearing fractions, $4x+1 = A(x+4) + B(x-1) = (A+B)x + (4A-B)$. This last equation requires that $A+B=4$ and $4A-B=1$. The solution of these equations is $A=1$, $B=3$. Thus,
$$\int \frac{4x+1}{x^2+3x-4}\,dx = \int \frac{dx}{x-1} + 3\int \frac{dx}{x+4} = \ln|x-1| + 3\ln|x+4| + C.$$

3. Let $u = \ln x$, $du = \dfrac{dx}{x}$, $dv = \dfrac{dx}{(x+1)^2}$, $v = \dfrac{-1}{x+1}$; $x > 0$

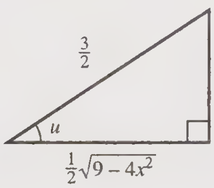
$$\begin{aligned}\int \frac{\ln x\, dx}{(x+1)^2} &= -\frac{\ln x}{x+1} + \int \frac{dx}{x(x+1)}\\ &= -\frac{\ln x}{x+1} + \int \frac{dx}{x} - \int \frac{dx}{x+1}\\ &= -\frac{\ln x}{x+1} + \ln x - \ln(x+1) + C\\ &= -\frac{\ln x}{x+1} + \ln \frac{x}{x+1} + C,\ x > 0.\end{aligned}$$

4. Let $x = \dfrac{3}{2}\sin u$, $dx = \dfrac{3}{2}\cos u\, du$. Then,

$$\begin{aligned}\int \frac{\sqrt{9-4x^2}\, dx}{x} &= \int \frac{\sqrt{9-9\sin^2 u}\cdot \frac{3}{2}\cos u\, du}{\frac{3}{2}\sin u}\\ &= 3\int \frac{\cos^2 u\, du}{\sin u} = 3\int \frac{(1-\sin^2 u)\, du}{\sin u}\\ &= 3\int (\csc u - \sin u)\, du\\ &= 3\ \ln|\csc u - \cot u| + 3\cos u + C.\end{aligned}$$

From the substitution $x = \dfrac{3}{2}\sin u$, and the diagram below, $\csc u = \dfrac{3}{2x}$ and $\cot u = \dfrac{\sqrt{9-4x^2}}{2x}$. Thus,

$$\int \frac{\sqrt{9-4x^2}\, dx}{x} = 3\ \ln\left|\frac{3}{2x} - \frac{\sqrt{9-4x^2}}{2x}\right| + \sqrt{9-4x^2} + C.$$



5. Let $u = e^x$, $du = e^x dx$. Then, $\displaystyle\int \frac{e^{3x/2}\, dx}{1+e^{3x/4}} = \int \frac{e^{x/2}\cdot e^x\, dx}{1+e^{3x/4}} = \int \frac{u^{1/2} du}{1+u^{3/4}}$. Next, let $z = u^{1/4}$, $dz = \dfrac{1}{4}u^{-3/4}du$.

Hence, $z^2 = u^{1/2}$, $z^3 = u^{3/4}$, and $du = 4u^{3/4}\, dz = 4z^3 dz$. Substitution into the last integral gives

$$\begin{aligned}\int \frac{e^{3x/2}\, dx}{1+e^{3x/4}} &= \int \frac{z^2(4z^3 dz)}{1+z^3} = 4\int \frac{z^5\, dz}{z^3+1}\\ &= 4\int \left(z^2 - \frac{z^2}{z^3+1}\right) dz\\ &= \frac{4}{3}z^3 - \frac{4}{3}\ \ln|z^3+1| + C\\ &= \frac{4}{3}u^{3/4} - \frac{4}{3}\ \ln|u^{3/4}+1| + C\\ &= \frac{4}{3}e^{3x/4} - \frac{4}{3}\ \ln(e^{3x/4}+1) + C.\end{aligned}$$

6. $\displaystyle\int \csc^3 x\, dx$

We use the reduction Formula 93 in the integral tables: $\displaystyle\int \csc^3 x\, dx = \frac{-\csc x \cot x}{2} + \frac{1}{2}\int \csc x\, dx$.

The last integral is evaluated using Formula 89 to yield $\displaystyle\int \csc^3 x\, dx = -\frac{1}{2}\csc x \cot x - \frac{1}{2}\ln|\csc x + \cot x| + C$.

7. Let $u = \cos x$, $du = -\sin x\, dx$. Then

$$\begin{aligned}\int \sin^3 x \cos^4 x\, dx &= \int \sin^2 x \cos^4 x \sin x\, dx\\ &= \int (1-\cos^2 x)\cos^4 x \sin x\, dx\\ &= \int -(1-u^2)u^4\, du\\ &= -\frac{1}{5}u^5 + \frac{1}{7}u^7 + C = -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C.\end{aligned}$$

8. Using the trigonometric identity $\cot^2 x = \csc^2 x - 1$

$$\begin{aligned}\int \cot^5 x\, dx &= \int \cot^3 x(\csc^2 x - 1)\, dx\\ &= \int \cot^3 x \csc^2 x\, dx - \int \cot^3 x\, dx\\ &= \int \cot^3 x \csc^2 x\, dx - \int \cot x(\csc^2 x - 1)\, dx\\ &= \int \cot^3 x \csc^2 x\, dx - \int \cot x \csc^2 x\, dx + \int \cot x\, dx\end{aligned}$$

Let $u = \cot x$; $du = -\csc^2 x\, dx$ and we have $\int \cot^5 x\, dx = -\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \ln|\sin x| + C.$

9. Completing the square, $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$. Let $u = x + \frac{1}{2}$, $du = dx$. Then,

$$\begin{aligned}\int \frac{x\, dx}{x^2 + x + 1} &= \int \frac{\left(u - \frac{1}{2}\right) du}{u^2 + \frac{3}{4}} = \int \frac{u\, du}{u^2 + \frac{3}{4}} - \frac{1}{2}\int \frac{du}{u^2 + \frac{3}{4}}\\ &= \frac{1}{2}\ln\left|u^2 + \frac{3}{4}\right| - \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{2u}{\sqrt{3}} + C\\ &= \frac{1}{2}\ln\left|x^2 + x + 1\right| - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C.\end{aligned}$$

10.
$$\begin{aligned}\int \cos^2 x \sin 2x\, dx &= \frac{1}{2}\int (1 + \cos 2x)\sin 2x\, dx\\ &= \frac{1}{2}\int \sin 2x\, dx + \frac{1}{2}\int \sin 2x \cos 2x\, dx\\ &= \frac{1}{2}\int \sin 2x\, dx + \frac{1}{4}\int \sin 4x\, dx\\ &= -\frac{1}{4}\cos 2x - \frac{1}{16}\cos 4x + C.\end{aligned}$$

11. Let $u = x$, $du = dx$, $dv = \sec^2 x\, dx$, $v = \tan x$. Then,

$$\begin{aligned}\int_{-\pi/3}^{\pi/4} x \sec^2 x\, dx &= x \tan x\Big]_{-\pi/3}^{\pi/4} - \int_{-\pi/3}^{\pi/4} \tan x\, dx\\ &= \frac{\pi}{4}\cdot 1 + \frac{\pi}{3}\cdot\sqrt{3} - \ln|\cos x|\Big]_{-\pi/3}^{\pi/4}\\ &= \frac{\pi}{4} + \frac{\pi}{\sqrt{3}} - \ln\frac{\sqrt{2}}{2} + \ln\frac{1}{2}\\ &= \frac{\pi}{4} + \frac{\pi}{\sqrt{3}} - \frac{1}{2}\ln 2.\end{aligned}$$

12. $$\int_0^1 \frac{(2x^3+x+3)\,dx}{x^2+1} = \int_0^1\left(2x+\frac{3-x}{x^2+1}\right)dx$$
$$= \int_0^1 2x\,dx + 3\int_0^1 \frac{dx}{x^2+1} - \int_0^1 \frac{x\,dx}{x^2+1}$$
$$= x^2 + 3\tan^{-1}x - \frac{1}{2}\ln(x^2+1)\Big]_0^1$$
$$= 1+\frac{3\pi}{4}-\frac{1}{2}\ln 2.$$

13. Let $x = 5\tan u$, $dx = 5\sec^2 u\,du$. When $x = 5$, $\tan u = 1$ or $u=\frac{\pi}{4}$; when $x = 5\sqrt{3}$, $\tan u = \sqrt{3}$ or $u = \frac{\pi}{3}$. Thus,
$$\int_5^{5\sqrt{3}} \frac{dx}{x\sqrt{25+x^2}} = \int_{\pi/4}^{\pi/3} \frac{5\sec^2 u\,du}{5\tan u\sqrt{25+25\tan^2 u}}$$
$$= \frac{1}{5}\int_{\pi/4}^{\pi/3}\frac{\sec u\,du}{\tan u} = \frac{1}{5}\int_{\pi/4}^{\pi/3}\csc u\,du$$
$$= \frac{1}{5}\ln\left|\csc u - \cot u\right|\Big]_{\pi/4}^{\pi/3}$$
$$= \frac{1}{5}\left(\ln\left|\frac{2\sqrt{3}}{3}-\frac{\sqrt{3}}{3}\right| - \ln\left|\sqrt{2}-1\right|\right)$$
$$= \frac{1}{5}\ln\left[\frac{\sqrt{3}}{3(\sqrt{2}-1)}\right].$$

14. Let $u = \ln(a^2+x^2)$, $du = \dfrac{2x\,dx}{a^2+x^2}$, $dv = dx$, and $v = x$. Then,
$$\int_0^a \ln(a^2+x^2)\,dx = x\ln(a^2+x^2)]_0^a - \int_0^a \frac{2x^2\,dx}{a^2+x^2}$$
$$= a\ln 2a^2 - 2\int_0^a\left(1-\frac{a^2}{a^2+x^2}\right)dx$$
$$= a\ln 2a^2 - \left(2x+2a\tan^{-1}\frac{x}{a}\right]_0^a$$
$$= a\ln 2a^2 - 2a + \frac{2\pi a}{4}.$$

15. $$\frac{x-2}{(3x-1)^2} = \frac{A}{3x-1}+\frac{B}{(3x-1)^2}$$
Clearing fractions, $x-2 = A(3x-1)+B = 3Ax+(B-A)$. Thus, $3A = 1$ and $B - A = -2$. The solution of these equations is $A=\frac{1}{3}$ and $B=-\frac{5}{3}$. Thus,
$$\int\frac{x-2}{(3x-1)^2}dx = \frac{1}{3}\int\frac{dx}{3x-1}-\frac{5}{3}\int\frac{dx}{(3x-1)^2} = \frac{1}{9}\ln\left|3x-1\right| + \frac{5}{9}(3x-1)^{-1}+C.$$

16. $$\lim_{x\to 0}\frac{\sin x - x\cos x}{x^3} = \lim_{x\to 0}\frac{\cos x-\cos x + x\sin x}{3x^2} = \lim_{x\to 0}\frac{\sin x}{3x} = \frac{1}{3}\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{3}.$$

17. $\displaystyle\lim_{x\to\infty}\frac{x^2-5}{2x^2+3x}$ is of the form ∞/∞. Applying l'Hôpital's rule,
$$\lim_{x\to\infty}\frac{x^2-5}{2x^2+3x} = \lim_{x\to\infty}\frac{2x}{4x+3}\ \ (\text{still }\infty/\infty) = \lim_{x\to\infty}\frac{2}{4} = \frac{1}{2}.$$

18. $\lim_{x\to\infty} \frac{\sin\left(\frac{3}{x}\right)}{\frac{2}{x}}$ is of the form 0/0. Applying l'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\sin\left(\frac{3}{x}\right)}{\frac{2}{x}} = \lim_{x\to\infty} \frac{\left(-\frac{3}{x^2}\right)\cos\left(\frac{3}{x}\right)}{-\frac{2}{x^2}} = \lim_{x\to\infty} \frac{3}{2}\cos\frac{3}{x} = \frac{3}{2}\cos 0 = \frac{3}{2}.$$

19. $\lim_{x\to\frac{\pi}{2}^-}\left(x\tan x - \frac{\pi}{2}\sec x\right)$ is of the form $\infty-\infty$. However, $\lim_{x\to\frac{\pi}{2}^-}\left(x\tan x - \frac{\pi}{2}\sec x\right) = \lim_{x\to\frac{\pi}{2}^-} \frac{x\sin x - \frac{\pi}{2}}{\cos x}$ is of the form 0/0. Applying l'Hôpital's rule, $\lim_{x\to\frac{\pi}{2}^-} \frac{x\sin x - \frac{\pi}{2}}{\cos x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\sin x + x\cos x}{-\sin x} = \frac{1+\left(\frac{\pi}{2}\right)(0)}{(-1)} = -1.$

20. $\lim_{x\to0} \csc x \sin^{-1} x = \lim_{x\to0} \frac{\sin^{-1} x}{\sin x}(\text{type } 0/0) = \lim_{x\to0} \frac{\frac{1}{\sqrt{1-x^2}}}{\cos x} = \frac{1}{1} = 1.$

21. Let $y = x^{1/(1-x)}$ so $\ln y = \frac{\ln x}{1-x}$, then

$$\lim_{x\to1} \ln y = \lim_{x\to1} \frac{\ln x}{1-x}(\text{type } 0/0) = \lim_{x\to1} \frac{\frac{1}{x}}{-1} = -1.$$

By the continuity of the natural logarithm, $\lim_{x\to1} x^{1/(1-x)} = e^{-1}$.

22. $\lim_{x\to0^+} \frac{e^x - \cos x}{x\sin x}(\text{type } 0/0) = \lim_{x\to0^+} \frac{e^x + \sin x}{x\cos x + \sin x} = +\infty.$

23.
$$\lim_{x\to0} \frac{4^x - 2^x}{x}(\text{type } 0/0) = \lim_{x\to0} \frac{4^x \ln 4 - 2^x \ln 2}{1} = \ln 4 - \ln 2 = \ln 2.$$

24. $\int_0^{\pi/2} \sec x \tan x\, dx = \lim_{b\to\frac{\pi}{2}^-} \int_0^b \sec x \tan x\, dx = \lim_{b\to\frac{\pi}{2}^-} \sec x]_0^b = \lim_{b\to\frac{\pi}{2}^-} (\sec b - 1) = \infty.$

Therefore, the improper integral diverges.

25.
$$\begin{aligned}\int_{-1}^1 \frac{dx}{x^{2/3}} &= \lim_{a\to0^-} \int_{-1}^a \frac{dx}{x^{2/3}} + \lim_{a\to0^+} \int_a^1 \frac{dx}{x^{2/3}} \\ &= \lim_{a\to0^-} 3x^{1/3}]_{-1}^a + \lim_{a\to0^+} 3x^{1/3}]_a^1 \\ &= \lim_{a\to0^-} (3a^{1/3}+3) + \lim_{a\to0^+} (3-3a^{1/3}) = 6.\end{aligned}$$

26. For $x \ge e$, $\ln x \ge 1$. Hence, $\int_1^\infty \frac{\ln x\, dx}{x} \ge \int_e^\infty \frac{\ln x\, dx}{x} \ge \int_e^\infty \frac{dx}{x}$. Now, $\int_e^\infty \frac{dx}{x} = \lim_{b\to\infty} \ln x]_e^b = \lim_{b\to\infty} \ln b - 1 = \infty.$

Therefore, the integral $\int_1^\infty \frac{\ln x\, dx}{x}$ diverges.

27. $\int_0^{\infty} \frac{\sin x\,dx}{e^x} = \lim_{b\to\infty} \int_0^b e^{-x} \sin x\,dx$

Let $u = e^{-x}$, $du = -e^{-x}\,dx$, $dv = \sin x\,dx$, $v = -\cos x$, then

$$\int_0^b e^{-x} \sin x\,dx = -e^{-x} \cos x\Big]_0^b - \int_0^b e^{-x} \cos x\,dx.$$

Let $U = e^{-x}$, $dU = -e^{-x}\,dx$, $dV = \cos x\,dx$, $V = \sin x$, then $\int_0^b e^{-x} \cos x\,dx = e^{-x} \sin x]_0^b + \int_0^b e^{-x} \sin x\,dx.$

Putting these results together,

$$\begin{aligned} 2\int_0^b e^{-x} \sin x\,dx &= -e^{-x} \cos x - e^{-x} \sin x]_0^b \\ &= -e^{-b} \cos b + 1 - e^{-b} \sin b. \end{aligned}$$

Now, $\lim_{b\to\infty} \frac{\cos b}{e^b} = \lim_{b\to\infty} \frac{\sin b}{e^b} = 0$. Thus, $\int_0^{\infty} \frac{\sin x\,dx}{e^x} = \lim_{b\to\infty} \int_0^b e^{-x} \sin x\,dx = \frac{1}{2}$.