Chapter 6: Transcendental Functions and Differential Equations

6.1 LOGARITHMS

1. The natural logarithm is defined by $\ln x =$ ______________ for x satisfying ______________.

2. By the first part of the Fundamental Theorem of Calculus, $\frac{d}{dx}\ln x =$ ______________, so the natural logarithm is a continuous function for $x > 0$ because it is ______________.

OBJECTIVE A: Use the three laws of logarithms to rewrite a logarithmic expression as a sum, difference, or multiple of logarithms.

3. $\ln ax =$ ______________ for $a > 0$ and $x > 0$.

4. $\ln \frac{a}{x} =$ ______________ for $a > 0$ and $x > 0$.

5. $\ln x^n =$ ______________ for $x > 0$ and n rational.

6. $\ln \sqrt[3]{\frac{x^2}{a^4}} = \ln\left(\frac{x^2}{a^4}\right)^{\text{—}} = (\text{\_\_\_\_\_\_\_\_})\ln\left(\frac{x^2}{a^4}\right) = \frac{1}{3}(\ln x^2 - \text{\_\_\_\_\_\_\_\_}) = \frac{2}{3}\ln x -$ ________.

7. $\ln(b^3 \cdot \sqrt{x}) = \ln b^3 +$ ________ $= 3$________ $+ \ln x^{1/2} =$ ______________.

8. $\ln(x^2 + 2x + 1) = \ln(x+1)^{\text{—}} =$ ______________ for $x > -1$.

OBJECTIVE B: Summarize the characteristics of the graph of $y = \ln x$, and graph the functions involving the natural logarithm.

9. The domain of $y = \ln x$ is the set ______________, and its range is the set ______________.

10. The graph of $y = \ln x$ is increasing ______________________________. It is concave down ______________________________.

11. Since $y = \ln x$ is differentiable for $x > 0$, it is a ______________ function of x.

12. $\lim_{x\to\infty} \ln x =$ ______________ and $\lim_{x\to 0^+} \ln x =$ ______________.

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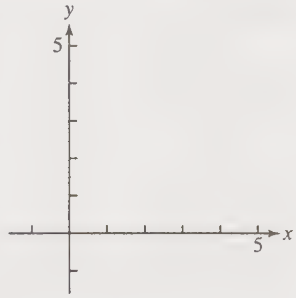
1. $\int_1^x \frac{1}{t}\,dt$, $x > 0$
2. $\frac{1}{x}$, differentiable
3. $\ln a + \ln x$
4. $\ln a - \ln x$
5. $n \ln x$
6. $\frac{1}{3}, \frac{1}{3}, \ln a^4, \frac{4}{3}\ln a$
7. $\ln\sqrt{x}$, $\ln b$, $3\ln b + \frac{1}{2}\ln x$
8. 2, $2\ln(x+1)$
9. $x > 0$, $-\infty < y < +\infty$
10. for all x in the domain, for all $x > 0$
11. continuous
12. $+\infty$, $-\infty$

13. Consider the curve $y = x - \ln x$. The derivative $\frac{dy}{dx} =$ ____________ so that $\frac{dy}{dx} = 0$ implies $\frac{1}{x} =$ ________ or $x =$ ________. Notice that the domain of y is the set ________. The second derivative $\frac{d^2y}{dx^2} =$ ________ is always positive. Therefore, the critical point $x = 1$ gives a relative ____________ value of $y(1) =$ ________. As $x \to 0$, $y \to$ ________. To examine the curve as x increases, notice that $\frac{dy}{dx} = 1 - \frac{1}{x} > 0$ whenever $x > 1$. Thus the graph is everywhere increasing. Sketch a graph of y below.



OBJECTIVE C: Differentiate functions whose expressions involve the natural logarithmic function.

14. $$\frac{d}{dx}\ln(5+2x^3)^4 = \frac{1}{(5+2x^3)^4}\frac{d}{dx}(\_\_\_\_\_\_\_\_)$$
$$= \frac{1}{(5+2x^3)^4}[\_\_\_\_\_\_\_\_]\frac{d}{dx}(5+2x^3)$$
$$= \frac{\_\_\_\_\_\_}{(5+2x^3)^4} = \_\_\_\_\_\_\_\_.$$

15. $$\frac{d}{dx}[\ln(\sin x)]^2 = \_\_\_\_\_\_\_\_ \frac{d}{dx}\ln(\sin x)$$
$$= 2\ln(\sin x)\cdot \_\_\_\_\_\_ \cdot \frac{d}{dx}(\sin x)$$
$$= 2\csc x \ln(\sin x)\cdot \_\_\_\_\_\_ = \_\_\_\_\_\_\_\_.$$

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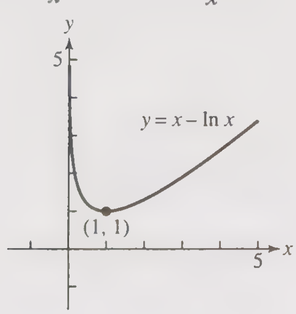
13. $1 - \frac{1}{x}$, 1, 1, $x > 0$, $\frac{1}{x^2}$, minimum, 1, $+\infty$



14. $(5+2x^3)^4$, $4(5+2x^3)^3$, $24x^2(5+2x^3)^3$, $\frac{24x^2}{5+2x^3}$

15. $2\ln(\sin x)$, $\frac{1}{\sin x}$, $\cos x$, $2\cot x \ln(\sin x)$

16. $\frac{d}{dx}x^2 \ln\sqrt{x} = 2x\ln\sqrt{x} + \_\_\_\_\_\_\_\_ \frac{d}{dx}\ln\sqrt{x}$

$= 2x\ln\sqrt{x} + \_\_\_\_\_\_\_\_\_\_ \frac{d}{dx}\sqrt{x}$

$= 2x\ln\sqrt{x} + \_\_\_\_\_\_\_\_\_\_\_\_ = \frac{x}{2}(\_\_\_\_\_\_\_\_\_\_).$

OBJECTIVE D: Use the method of logarithmic differentiation to calculate derivatives.

17. Find $\frac{dy}{dx}$ if $y = x^{\sin x}$, $x > 0$.

Solution. $\ln y = \ln(x^{\sin x}) = \_\_\_\_\_\_\_\_\_\_$ so that

$\frac{1}{y}\frac{dy}{dx} = \cos x \ln x + \_\_\_\_\_\_\_\_\_\_$, or

$\frac{dy}{dx} = x^{\sin x}(\_\_\_\_\_\_\_\_\_\_).$

18. Find $\frac{dy}{dx}$ if $y = \sqrt{\frac{1-x}{1+x}}$, $-1 < x < 1$.

Solution. $\ln y = \ln\sqrt{\frac{1-x}{1+x}} = \frac{1}{2}\ln(1-x) - \_\_\_\_\_\_\_\_\_\_$ so that

$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{2(1-x)} - \_\_\_\_\_\_\_\_ = \frac{-(1+x)-(\_\_\_\_\_\_)}{2(1-x)(1+x)}$

$\frac{dy}{dx} = -y(1-x)^{-1}\cdot \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_.$

19. Find $\frac{dy}{dx}$ if $y = x^{2x^7}$, $x > 0$.

Solution. $\ln y = \ln x^{2x^7} = \_\_\_\_\_\_\_\_\_\_.$

$\frac{1}{y}\frac{dy}{dx} = 14x^6 \ln x + \_\_\_\_\_\_\_\_ = 2x^6(\_\_\_\_\_\_\_\_)$ so that

$\frac{dy}{dx} = \_\_\_\_\_\_\_\_\_\_.$

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16. x^2, $x^2 \cdot \frac{1}{\sqrt{x}}$, $x^2 \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$, $2\ln x + 1$

17. $\sin x \cdot \ln x$, $\sin x \cdot \frac{1}{x}$, $\cos x \ln x + \frac{1}{x}\sin x$

18. $\frac{1}{2}\ln(1+x)$, $\frac{1}{2(1+x)}$, $1-x$, $(1+x)^{-1}$, $-(1-x)^{-1/2}(1+x)^{-3/2}$

19. $2x^7 \ln x$, $2x^7 \cdot \frac{1}{x}$, $7\ln x + 1$, $2x^{2x^7+6}(7\ln x + 1)$

OBJECTIVE E: Differentiate functions involving a logarithmic function $\log_a u$.

20. $\frac{d}{dx}\log_a u = \frac{d}{dx}\frac{\_\_\_\_\_\_}{\ln a} = \_\_\_\_\_\_\_\_ \cdot \frac{du}{dx}$.

21. $\frac{d}{dx}\log_{10}(x^2 - e^x) = \frac{1}{\_\_\_\_\_\_}\frac{d}{dx}(x^2 - e^x) = \_\_\_\_\_\_\_\_$.

22. Find $\frac{dy}{dx}$ if $y = (1+\sqrt{x})^{\log_2 x}$.

Solution. $\ln y = \ln(1+\sqrt{x})^{\log_2 x} = \_\_\_\_\_\_\_\_$.

$\frac{1}{y}\frac{dy}{dx} = \_\_\_\_\_\_ \ln(1+\sqrt{x}) + \log_2 x \cdot \frac{1}{1+\sqrt{x}} \cdot \_\_\_\_\_\_$.

$\frac{dy}{dx} = (1+\sqrt{x})^{\log_2 x}[\_\_\_\_\_\_\_\_]$.

OBJECTIVE F: Integrate functions whose antiderivatives involve the natural logarithm function.

23. $\int \frac{x\,dx}{x^2+4}$

Let $u = x^2 + 4$. Then $du = \_\_\_\_\_\_$ so $x\,dx = \_\_\_\_\_\_$. Thus the integral becomes

$\int \frac{x\,dx}{x^2+4} = \int \frac{du}{\_\_\_\_\_\_} = \_\_\_\_\_\_ + C = \_\_\_\_\_\_$.

24. $\int \frac{3x+1}{x}dx = \int (3 + \_\_\_\_\_\_)\,dx = \int 3\,dx + \_\_\_\_\_\_ = 3x + \_\_\_\_\_\_ + C.$

25. $\int \frac{dx}{x\ln\sqrt{x}}$

Let $u = \ln\sqrt{x}$. Then $\frac{du}{dx} = \_\_\_\_\_\_ \frac{d}{dx}(\_\_\_\_\_\_) = \_\_\_\_\_\_$. Hence, $2\,du = \_\_\_\_\_\_\,dx$. Thus the integral becomes $\int \frac{dx}{x\ln\sqrt{x}} = \int \frac{2\,du}{\_\_\_\_\_\_} = \_\_\_\_\_\_ + C = \_\_\_\_\_\_$.

6.2 EXPONENTIAL FUNCTIONS

OBJECTIVE A: Use the derivative rule for inverse functions to calculate the derivatives of the inverse for a specified function.

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20. $\ln u$, $\frac{1}{u\ln a}$

21. $(x^2 - e^x)\ln 10$, $\frac{2x - e^x}{(x^2 - e^x)\ln 10}$

22. $\log_2 x \cdot \ln(1+\sqrt{x})$, $\frac{1}{x\ln 2}$, $\frac{1}{2\sqrt{x}}$, $\frac{1}{x\ln 2}\ln(1+\sqrt{x}) + \frac{\ln x}{2\ln 2\cdot\sqrt{x}(1+\sqrt{x})}$

23. $2x\,dx$, $\frac{1}{2}du$, $2u$, $\frac{1}{2}\ln|u|$, $\frac{1}{2}\ln(x^2+4)+C$

24. $\frac{1}{x}$, $\int\frac{dx}{x}$, $\ln|x|$

25. $\frac{1}{\sqrt{x}}$, $\sqrt{x}$, $\frac{1}{2x}$, $\frac{1}{x}$, u, $2\ln|u|$, $2\ln(\ln\sqrt{x})+C$

26. If f and f^{-1} are inverse functions on suitably restricted domains, then $\frac{df^{-1}}{dx}(b) = \frac{1}{f'(a)}$ where a and b are related by ______________ and ______________.

27. Let $f(x) = -6x + 2$ and let f^{-1} denote the inverse of f. We wish to calculate the derivative $\frac{df^{-1}}{dx}(14)$. First, $-6x + 2 = 14$ implies $x =$ __________. Thus, $f(-2) = 14$ so $a =$ __________ and $b =$ __________ in Problem 26. Then, $f'(x) =$ __________ so that $\frac{df^{-1}}{dx}(14) = \frac{1}{f'(\_\_\_\_\_)} =$ __________.

28. To calculate the inverse of $y = -6x + 2$, interchange the letters x and y obtaining __________. Solving the resultant equation for y yields __________, or $f^{-1}(x) =$ __________ is the inverse function of $f(x) = -6x + 2$. Calculating the derivative $\frac{df^{-1}}{dx}(14)$ directly from the formula for $f^{-1}(x)$ gives $-\frac{1}{6}$ as before.

Remark. The advantage of the derivative formula for the inverse function given by Theorem 1 of the text is that it provides for the calculation of the derivative $\frac{df^{-1}}{dx}$ even though a formula for the inverse function f^{-1} is not known.

29. Let f^{-1} be the inverse of $f(x) = x^2 + 4x - 3$ for $x > -2$. To find $\frac{df^{-1}}{dx}(-6)$, first set $f(x) = x^2 + 4x - 3$ equal to __________ and solve the quadratic equation yielding $x = -3$ or $x =$ __________. We reject $x = -3$ because -3 is outside the allowable interval $x > -2$. Thus, $a =$ __________ and $b = f(a) =$ __________.
Now $\frac{d}{dx}(x^2 + 4x - 3) =$ __________, so $f'(-1) =$ __________. Thus, $\frac{df^{-1}}{dx}(-6) = \frac{1}{f'(\_\_\_\_\_)} =$ __________.
Notice that we did not need a formula for the inverse function f^{-1} itself.

OBJECTIVE B: Use the equivalent equations $y = e^x$ and $x = \ln y$ to simplify logarithms of exponentials and exponentials of logarithms.

30. The equation $y = e^{\ln x}$ is equivalent to $\ln y =$ __________. Since the logarithm is one – one, the last equation is equivalent to __________; that is $e^{\ln x} =$ __________. In other words, the exponential "undoes" the natural logarithm.

31. The equation $y = \ln(e^x)$ is equivalent to $e^y =$ __________. Since the exponential function is one – one, the last equation is equivalent to __________; that is, $\ln(e^x) =$ __________.

32. $e^{-2\ln(x+1)} = e^{\ln \_\_\_\_\_} =$ ______________ by Problem 30.

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26. $b = f(a)$, $a = f^{-1}(b)$

27. $-2, -2, b = 14, -6, -2, -\frac{1}{6}$

28. $x = -6y + 2$, $y = -\frac{1}{6}(x - 2)$, $f^{-1}(x) = -\frac{1}{6}(x - 2)$

29. $-6, -1, -1, -6, 2x + 4, 2, -1, \frac{1}{2}$

30. $\ln x$, $y = x$, x

31. e^x, $y = x$, x

32. $(x+1)^{-2}$, $(x+1)^{-2}$

33. $\ln(\sqrt{x}e^{2-x}) = \ln\sqrt{x} +$ __________ = __________.

OBJECTIVE C: Differentiate functions whose expressions involve exponential functions.

34. $\frac{d}{dx}e^{\sqrt{1-x^2}} = e^{\sqrt{1-x^2}} \cdot \frac{d}{dx}($ __________ $)$

$= e^{\sqrt{1-x^2}} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}($ __________ $) =$ __________.

35. $\frac{d}{dx}e^{x\ln x} =$ __________ $\cdot \frac{d}{dx}(x\ln x) = e^{x\ln x}($ __________ $)$.

36. $\frac{d}{dx}\ln(e^x+1) = \frac{1}{e^x+1} \cdot \frac{d}{dx}($ __________ $) =$ __________.

37. $\frac{d}{dx}\sin\sqrt{e^x} = \cos\sqrt{e^x} \cdot \frac{d}{dx}($ __________ $)$

$= \cos\sqrt{e^x} \cdot \frac{d}{dx}($ __________ $) =$ __________.

OBJECTIVE D: Integrate functions whose antiderivatives involve exponential functions.

38. $\int x^2 e^{-x^3}\,dx$

Let $u = -x^3$. then $du =$ __________ so that $x^2 dx =$ __________. Thus the integral becomes

$\int x^2 e^{-x^3}\,dx = \int$ __________ $du =$ __________ $+ C =$ __________.

39. $\int (e^x-2)^4 e^x dx$

Let $u = e^x - 2$. Then $du =$ __________ and the integral becomes,

$\int (e^x-2)^4 e^x dx = \int$ __________ $du =$ __________ $+ C =$ __________.

40. $\int \frac{e^x}{1+e^{2x}}\,dx$

Let $u = e^x$. Then $du =$ __________ and the integral becomes,

$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{du}{\_\_\_\_\_\_\_\_\_\_} =$ __________ $+ C =$ __________.

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33. $\ln(e^{2-x})$, $\ln\sqrt{x} + (2-x)$

34. $\sqrt{1-x^2}$, $1-x^2$, $\frac{-x}{\sqrt{1-x^2}}e^{\sqrt{1-x^2}}$

35. $e^{x\ln x}$, $\ln x + 1$

36. $e^x + 1$, $\frac{e^x}{e^x+1}$

37. $\sqrt{e^x}$, $e^{x/2}$, $\frac{1}{2}\sqrt{e^x}\cos\sqrt{e^x}$

38. $-3x^2dx$, $-\frac{1}{3}du$, $-\frac{1}{3}e^u$, $-\frac{1}{3}e^{-x^3} + C$

39. $e^x dx$, u^4, $\frac{1}{5}u^5$, $\frac{1}{5}(e^x-2)^5 + C$

40. $e^x dx$, $1+u^2$, $\tan^{-1}u$, $\tan^{-1}(e^x) + C$

OBJECTIVE E: Understand the function $y = a^x$.

41. The function $y = a^x$ is defined by $a^x =$ _______________ and it is well-defined whenever _______________.

42. The definition in Problem 41 is equivalent to saying that $\ln a^x =$ _______________.

43. If $x > 0$, the number x^n is defined for *any* real number n and means $x^n =$ _______________, $x > 0$.

44. If $x > 0$, then $\frac{d}{dx}(x^n) =$ _______________.

OBJECTIVE F: Differentiate functions whose expressions involve an exponential function a^u, where u is a differentiable function of x.

45. The derivative of a^u, where u is a differentiable function of x, is given by $\frac{d}{dx}a^u =$ _______________.

46. $\frac{d}{dx}2^{\sec x} =$ ____________ $\cdot \frac{d}{dx}\sec x =$ ____________________.

47. $\frac{d}{dx}x^2 3^x = 2x \cdot$ ____________ $+ x^2 \cdot$ ____________ $= 3^x($____________$)$.

48. Find $\frac{dy}{dx}$ if $y = x^{\tan x}$, $x > 0$.

Solution. $\ln y = \ln(x^{\tan x}) =$ _______________ so that

$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln x +$ _______________, or

$\frac{dy}{dx} = x^{\tan x}($____________________$)$.

49. Find $\frac{dy}{dx}$ if $y = (x^r)^x$, $x > 0$, r any real number.

Solution. $\ln y = \ln(x^r)^x =$ _______________ $=$ _______________.

$\frac{1}{y}\frac{dy}{dx} = r\ln x +$ _______________ $= r($_______________$)$, so that

$\frac{dy}{dx} =$ ____________________.

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41. $e^{x\ln a}$, $a > 0$

42. $x \ln a$

43. $e^{n\ln x}$

44. nx^{n-1}

45. $a^u \cdot \frac{du}{dx} \cdot \ln a$

46. $2^{\sec x} \cdot \ln 2$, $(\sec x \tan x)2^{\sec x} \cdot \ln 2$

47. 3^x, $3^x \ln 3$, $2x + x^2 \ln 3$

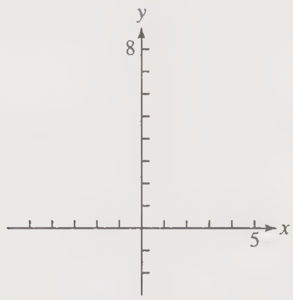
48. $\tan x \cdot \ln x$, $\tan x \cdot \frac{1}{x}$, $\sec^2 x \ln x + \frac{1}{x}\tan x$

49. $x\ln x^r$, $rx\ln x$, $rx \cdot \frac{1}{x}$, $\ln x + 1$, $r(x^r)^x(\ln x + 1)$

OBJECTIVE G: Graph functions whose expressions involve an exponential function a^u.

50. Consider the curve $y = (0.2)^{x+1} + 2$. $\frac{dy}{dx} =$ _______________ $\frac{d}{dx}(x+1) =$ _______________. Therefore, $\frac{dy}{dx}$ is of constant _______________ sign, since $\ln(0.2) < 0$, and the curve is everywhere _______________.

Calculation of the second derivative gives $\frac{d^2y}{dx^2} =$ _______________ which is of constant _______________ sign; hence, the curve is everywhere concave _______________. As $x \to \infty$, $(0.2)^{x+1} \to$ _________ so $y \to$ _________. As $x \to -\infty$, $(0.2)^{x+1} \to$ _________ so $y \to$ _________. Finally, the points (–1, ____), (0, ____), and (1, ____) lie on the curve. You can now sketch the graph of the curve in the coordinate system provided below.



OBJECTIVE H: Use the definition of $\log_a x$ to evaluate simple expressions.

51. If $a > 0$ and $a \neq 1$, then $y = \log_a x$ is defined and equivalent to $a^y =$ _______________ whenever x is positive. If a or x is negative, $\log_a x$ is _______________.

52. In terms of natural logarithms, $\log_a x =$ _______________.

53. $\log_8 4$

If $y = \log_8 4$, then $8^y =$ _________ or $2^{\_\_\_} = 2^2$. Thus, $3y =$ _________ or $y =$ _________.
Therefore, $\log_8 4 =$ _______________.

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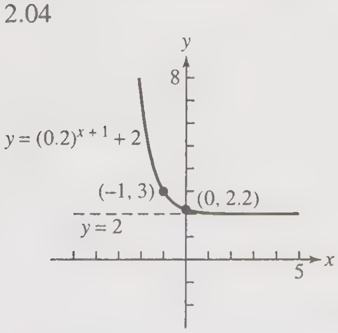
50. $(0.2)^{x+1}\ln(0.2)$, $(0.2)^{x+1}\ln(0.2)$, negative, decreasing, $(0.2)^{x+1}[\ln(0.2)]^2$, positive, up, 0, 2, $+\infty$, $+\infty$, 3, 2.2, 2.04



51. x, undefined

52. $\frac{\ln x}{\ln a}$

53. 4, $3y$, 2, $\frac{2}{3}$, $\frac{2}{3}$

54. $\log_{0.75} \frac{27}{64}$

If $y = \log_{0.75} \frac{27}{64}$, then $\frac{27}{64} =$ ______________. Now, $\left(\frac{3}{4}\right)^3 =$ ______________, so $\log_{0.75} \frac{27}{64} =$ ______________.

OBJECTIVE I: Integrate functions whose antiderivatives involve an exponential function a^u.

55. $\int \frac{dx}{2^x} = \int 2^{-x}\,dx$

Let $u = -x$ so that $du =$ ______________, and the integral becomes

$\int 2^{-x}\,dx = \int$ ____________ $du =$ ____________ $+ C =$ ______________.

56. $\int_1^2 x 10^{x^2-1}\,dx$

Let $u = x^2 - 1$. Then $du =$ ______________ so that $x\,dx =$ ______________.

$\int x 10^{x^2-1}\,dx = \int$ ____________ $du =$ ____________ $+ C$. Hence,

$\int_1^2 x 10^{x^2-1}\,dx =$ ____________ $\Big]_1^2 = \frac{1}{2 \ln 10}($ ____________ $) =$ ______________.

6.3 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS; INTEGRALS

OBJECTIVE A: Differentiate functions whose expressions involve inverse trigonometric functions.

57. $\frac{d}{dx}(\sin^{-1} u) =$ ______________________________.

58. $\frac{d}{dx}(\tan^{-1} u) =$ ______________________________.

59. $\frac{d}{dx}(\sec^{-1} u) =$ ______________________________.

60. $\frac{d}{dx}\left(\sin^{-1}\frac{x}{5}\right)^2 = \left(2\sin^{-1}\frac{x}{5}\right)\left(\frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}}\right)\frac{d}{dx}($ ________ $) = \left(2\sin^{-1}\frac{x}{5}\right)\left(\frac{5}{\sqrt{25-x^2}}\right)($ ________ $) =$ ______________.

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54. $\left(\frac{3}{4}\right)^y, \frac{27}{64}, 3$

55. $-dx, -2^u, -\frac{1}{\ln 2}2^u, -\frac{1}{2^x \ln 2} + C$

56. $2x\,dx, \frac{1}{2}du, \frac{1}{2}10^u, \frac{1}{2\ln 10}10^u, \frac{1}{2\ln 10}10^{x^2-1}, 10^3 - 1, \frac{999}{2\ln 10}$

57. $\frac{\frac{du}{dx}}{\sqrt{1-u^2}}$

58. $\frac{\frac{du}{dx}}{1+u^2}$

59. $\frac{\frac{du}{dx}}{|u|\sqrt{u^2-1}}$

60. $\frac{x}{5}, \frac{1}{5}, \frac{2}{\sqrt{25-x^2}}\sin^{-1}\frac{x}{5}$

61. $\frac{d}{dx}\left(\sec^{-1}\frac{1}{x}\right) = \frac{1}{\frac{1}{|x|}\sqrt{\_\_\_\_\_\_ - 1}} \cdot (\_\_\_\_\_\_) = \frac{-1}{\sqrt{\_\_\_\_\_\_}}$.

62. $\frac{d}{dx}(\tan^{-1}\sqrt{x-1}) = \frac{1}{1+(x-1)}\frac{d}{dx}(\_\_\_\_\_\_) = \_\_\_\_\_\_$.

63. Differentiating $\tan^{-1}\frac{x}{y} = \frac{1}{2}$ implicitly, we find

$$0 = \frac{d}{dx}\left(\tan^{-1}\frac{x}{y}\right) = \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{d}{dx}(\_\_\_\_\_\_) = \frac{y^2}{x^2+y^2}(\_\_\_\_\_\_) = \frac{y - \_\_\_\_\_\_}{x^2+y^2}; \text{ thus } \frac{dy}{dx} = \_\_\_\_\_\_.$$

OBJECTIVE B: Evaluate integrals leading to inverse trigonometric functions.

64. $\int_0^1 \frac{x\,dx}{\sqrt{1-x^4}}$. Let $u = x^2$, then $du =$ ______ so that $x\,dx =$ ______, and the indefinite integral becomes $\int \frac{x\,dx}{\sqrt{1-x^4}} = \frac{1}{2}\int \frac{du}{\_\_\_\_\_\_} = \_\_\_\_\_\_ + C$. Thus,

$$\int_0^1 \frac{x\,dx}{\sqrt{1-x^4}} = \_\_\_\_\_\_\Big]_0^1 = \frac{1}{2}(\sin^{-1}1 - \_\_\_\_\_\_) = \_\_\_\_\_\_.$$

65. $\int_0^{\pi/2} \frac{\cos x\,dx}{1+\sin^2 x}$. Let $u = \sin x$, then $du =$ ______, and the indefinite integral becomes $\int \frac{\cos x\,dx}{1+\sin^2 x} = \int \frac{du}{\_\_\_\_\_\_} = \_\_\_\_\_\_ + C$. Thus,

$$\int_0^{\pi/2} \frac{\cos x\,dx}{1+\sin^2 x} = \_\_\_\_\_\_\Big]_0^{\pi/2} = \tan^{-1}1 - \_\_\_\_\_\_ = \_\_\_\_\_\_ - 0 = \_\_\_\_\_\_.$$

6.4 FIRST-ORDER SEPARABLE DIFFERENTIAL EQUATIONS

66. A first order differential equation is a relation ______.

67. A function $y = y(x)$ is said to be a ______ of a differential equation if the latter is satisfied when ______ and its ______ are replaced throughout by ______ and its corresponding derivative.

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61. $\frac{1}{x^2}, -\frac{1}{x^2}, 1-x^2$

62. $\sqrt{x-1}, \frac{1}{2x\sqrt{x-1}}$

63. $\frac{x}{y}, \frac{y - x\frac{dy}{dx}}{y^2}, x\frac{dy}{dx}, \frac{y}{x}$

64. $2x\,dx, \frac{1}{2}du, \sqrt{1-u^2}, \frac{1}{2}\sin^{-1}u, \frac{1}{2}\sin^{-1}x^2, \sin^{-1}0, \frac{\pi}{4}$

65. $\cos x\,dx, 1+u^2, \tan^{-1}u, \tan^{-1}(\sin x), \tan^{-1}0, \frac{\pi}{4}, \frac{\pi}{4}$

66. $\frac{dy}{dx} = f(x, y)$

67. solution, y, derivative, $y(x)$

OBJECTIVE A: Show that a given function is a solution to a specified first order differential equation.

68. Consider the differential equation $3xy' - y = \ln x + 1$, $x > 0$, and the function $y = y(x) = Cx^{1/3} - \ln x - 4$, where C is any constant. Then, $\frac{dy}{dx} = y'(x) =$ ____________, and $3xy' =$ ____________. Thus, $3xy' - y = (Cx^{1/3} - 3) - (Cx^{1/3} - \ln x - 4) =$ ____________. Therefore, the function $y = y(x) = Cx^{1/3} - \ln x - 4$, and its derivative, satisfy the differential equation. We have verified that $y = y(x)$ is a solution.

OBJECTIVE B: Solve first order differential equations in which the variables can be separated. If initial conditions are prescribed, determine the value of the constant of integration.

69. Solve the differential equation $(xy - x)\,dx + (xy + y)\,dy = 0$.

Solution. We separate the variables and integrate: $x(y-1)\,dx + y(x+1)\,dy = 0$, $\frac{y\,dy}{y-1} =$ ____________, $y \neq 1$.

Then,

$\left(1 + \frac{1}{y-1}\right)dy =$ __________ and integration gives

$y + \ln|y-1| =$ __________ $+ \ln C$.

We introduce $\ln C$, $C > 0$, as the constant of integration in order to simplify the form of the solution. Thus, by algebra, $x + y = \ln C\left|\frac{x+1}{y-1}\right|$ or $|y-1|e^{x+y} =$ ____________, $y \neq 1$.

70. Solve the differential equation $x^2yy' = e^y$; when $x = 2$, $y = 0$.

Solution. We change to differential form, separate the variables, and integrate: $x^2y\,dy = e^y dx$ or, $ye^{-y}dy =$ ____________. In Chapter 7 you will see that the left side integrates to $-e^{-y}(y+1)$. Thus, $-e^{-y}(y+1) =$ ____________, or simplifying algebraically, $x(y+1) =$ ____________. Using the initial condition $x = 2$ and $y = 0$ gives, $2(0+1) =$ ____________, or $C =$ ____________. Thus, the solution is given by $x(y+1) = \left(1 + \frac{x}{2}\right)e^y$.

OBJECTIVE C: Solve exponential growth and decay initial value problems:

Differential equation: $\frac{dy}{dt} = ky$

Initial condition: $y = y_0$ when $t = 0$.

71. The Law of Exponential Change asserts that the dependent variable y is functionally related to the independent variable t according to the rule: ____________. If $k > 0$ the rule gives exponential ____________; if ____________ the rule gives exponential decay.

72. The *half-life* of a radioactive substance is the length of time it takes for ____________ of a given amount of the substance to disintegrate through radiation. The half-life of the carbon isotope C^{14} is about 5700 years.

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68. $\frac{1}{3}Cx^{-2/3} - \frac{1}{x}$, $Cx^{1/3} - 3$, $\ln x + 1$

69. $-\frac{x\,dx}{x+1}$, $-\left(1 - \frac{1}{x+1}\right)dx$, $-x + \ln|x+1|$, $C|x+1|$

70. $x^{-2}dx$, $-\frac{1}{x} + C$, $(1 - Cx)e^y$, $1 - 2C$, $-\frac{1}{2}$

71. $y = y_0e^{kt}$, growth, $k < 0$

72. half

73. Assume that the amount x of C^{14} present in a dead organism decays exponentially from the time of death. Then, $x = x_0 e^{kt}$, where x_0 is the original amount present. To find the constant k in the case of carbon C^{14}, $\left(\frac{1}{2}\right)x_0 = x_0 e^{\_\_\_\_ k}$ or $\ln\frac{1}{2} =$ ______________. Thus, $k = \frac{-\ln 2}{\_\_\_\_} \approx -1.22 \times 10^{-4}$. Suppose we want to determine the amount of C^{14} present after 10,000 years. Then, the percentage is given by the ratio $\frac{x_0 e^{k \cdot 10^4}}{x_0} \approx e^{\_\_\_\_} \approx 0.2964$, or approximately 29.64 percent of the amount of C^{14} remains after 10,000 years.

74. The "1470" skull found in Kenya by Richard Leakey is reputed to be 2,500,000 years old. The percentage of C^{14} remaining is given by $\frac{x_0 e^{k \cdot 2.5 \times 10^6}}{x_0} \approx e^{\_\_\_\_}$. However, if $x < -21$ then $e^x < 10^{-9}$ so the percentage of C^{14} left in the skull would be negligible. The current reliable limit for C^{14} dating is about 40,000 years, so another method for dating the skull had to be found.

75. Suppose an object is immersed in a fluid having constant temperature T_s. If $T(t)$ is the temperature of the object at time t, then Newton's Law of Cooling asserts that $\frac{dT}{dt} =$ ______________, where $k > 0$ if the object is cooling; if $k < 0$ the object is ______________. If $T = T_0$ is the temperature of the object at time $t = 0$, the solution to the differential equation (also called Newton's Law of Cooling) is ____________________.

76. A thermometer is taken from a room where the air temperature is 70°F to the outside where the temperature is 20°F. Write an initial value problem modeling the situation, assuming $T(t)$ is the temperature of the thermometer at time t.

Differential equation: ____________________

Initial condition: ____________________.

The solution to this initial value problem is ____________________. As $t \to \infty$, the temperature of the thermometer approaches __________.

77. Continuing with Problem 76, to evaluate the constant k we need additional information. Suppose then that after $\frac{2}{3}$ minute the thermometer reads 45°. From the solution in Problem 76, this condition means $\frac{45-20}{50} =$ ______________. Applying the natural logarithm function to both sides to solve for k gives $\ln\frac{1}{2} =$ ______________, or $k =$ ______________ ≈ 1.04. Thus, after 1 minute the temperature of the thermometer will be $T = 20 +$ ______________ ≈ 37.7°F. We used a calculator to evaluate $e^{-\left(\frac{3}{2}\ln 2\right)} \approx 0.354$.

---

73. $5700, 5700k, 5700, -\frac{\ln 2}{57} \times 10^2$

74. -304

75. $-k(T - T_s)$, warming, $T - T_s = (T_0 - T_s)e^{-kt}$

76. $\frac{dT}{dt} = -k(T - 20)$, $T(0) = 70$, $T - 20 = (70 - 20)e^{-kt}$, 20°F

77. $e^{-k(2/3)}, -\frac{2}{3}k, \frac{3}{2}\ln 2, 50e^{-\left(\frac{3}{2}\ln 2\right)1}$

6.5 LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS

OBJECTIVE A: Determine if a differential equation of first order is linear, and if it is, solve it.

78. A differential equation of first order, which is linear in the dependent variable y, can always be put in the standard form ________________, where P and Q are functions of x.

79. Assuming that P and Q are continuous functions of x, we can solve a linear differential equation $y' + Py = Q$ by finding an *integrating factor*, $v(x) =$ ______________, providing a solution equation $v(x) \cdot y =$ ____________.

80. Let us solve the equation $x\frac{dy}{dx} + (x-2)y = 3x^3e^{-x}$. In standard form, $\frac{dy}{dx} + \left(1 - \frac{2}{x}\right)y = 3x^2e^{-x}$. Here $P =$ ____________ and $Q =$ ____________, and the differential equation is linear. An integrating factor is given by $v(x) = e^{\int P\,dx} = e^{\int(1-\frac{2}{x})dx} = e$———— $= x$——e^x. Hence a solution is given by $x^{-2}e^x y = \int$ ____________ $dx + C = \int$ ______________ $dx + C =$ ____________. Thus, $y =$ ____________.

81. The differential equation $y' = x - 4xy$ can be written in standard form as $y' + 4xy = x$, so it is linear. The equation may also be written in the form $\frac{dy}{1-4y} =$ __________, so it is separable in the variables x and y. Thus we have a choice of methods of solution. As a separable equation, we integrate the last equation, and find $-\frac{1}{4}\ln|1-4y| = \frac{1}{2}x^2 + \ln C$, or $|1-4y| = C_1$ ________, where $C_1 = C^{-4}$. If we consider the differential equation as linear, an integrating factor is $v(x) = e^{\int 4x\,dx} =$ ______________, from which we get $ye^{2x^2} = \int$ ____________ $dx + C_2 =$ ____________ $+ C_2$ or $4y =$ ______________. If $1 - 4y < 0$, we choose $4C_2 = C_1$, and if $1 - 4y \geq 0$, we choose $4C_2 = -C_1$. Thus, both solution forms agree.

OBJECTIVE B: Solve application problems involving linear differential equations.

82. A tank contains 1000 gallons of water containing 60 parts per gallon (ppg) of pollutant. Every second, 50 gallons in the tank receive 100 ppg of additional pollutant from various sources. At the same time, a purifier processes 5 gallons per second, reducing the pollutant level to 15 ppg. Find the long-term (i.e. steady-state) pollutant level in the tank in ppg, and determine when it reaches 600 ppg. Assume that the pollutant has negligible volume and that the tank maintains a uniform concentration through constant stirring.

STEP 1: $y =$ (Total pollutants amount, in "parts") = (Parts per gallon) × ______________

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78. $\frac{dy}{dx} + Py = Q$

79. $e^{\int P(x)dx}$, $\int v(x)Q(x)\,dx + C$

80. $1 - \frac{2}{x}$, $3x^2e^{-x}$, $x - 2\ln x$, -2, $x^{-2}e^x \cdot 3x^2e^{-x}$, 3, $3x + C$, $(3x^3 + Cx^2)e^{-x}$

81. $x\,dx$, e^{-2x^2}, e^{2x^2}, xe^{2x^2}, $\frac{1}{4}e^{2x^2}$, $1 + 4C_2e^{-2x^2}$

82. 1000 gallons, 100, gal, 15, 5, 0, 5000, 1.015 million, 1015 ppg, $\frac{1}{200}$, 5075, $e^{t/200}$, $e^{-t/200}$, $t/200$, $Ce^{-t/200}$, $-955{,}000$, $955{,}000e^{-t/200}$, 167 sec, 2.8

STEP 2: $\frac{dy}{dt}$ = (Total pollutants amount, in parts) – (Pollutant part removed every second)

$= (\_\_\_\_\_\_\_\_ \text{ ppg} \times 50 \_\_\_\_\_\_\_\_\_\_) - \left(\frac{y}{1,000 \text{ gal}} - \_\_\_\_\_\_\_\_ \text{ ppg}\right) \times \_\_\_\_\_\_\_\_ \text{ gal.}$

STEP 3: At steady-state $\frac{dy}{dt} =$ __________, so that $\left(\frac{y}{1,000} - 15\right) \times 5 =$ _____________, which yields $y =$ ____________ parts, and that means a pollutant concentration of ____________________ in steady-state.

STEP 4: To find out when the concentration reaches 600 ppg, solve the differential equation to find when $y = 600,000$ parts. Write it in standard form: $\frac{dy}{dt} + P(t)y = Q(t)$. Here $P(t) =$ _______ and $Q(t) =$ _______.

Then the integrating factor is $v(t) = e^{\int P(t)\,dt} =$ __________ and

$y(t) = \frac{1}{v(t)} \int v(t)Q(t)\,dt = 5,075 \_\_\_\_\_ (200e^{\_\_\_} + C) = 1,015,000 + 5,075 \_\_\_\_\_\_\_$ and since

$y(0) = 60 \times 1,000 = 60,000$ parts, $5075C =$ __________ and $y(t) = 1,015,000 -$ _____________. Solving for $y(t) = 600 \times 1,000 = 600,000$ parts yields $t \approx$ __________, or about ____________ minutes.

6.6 EULER'S METHOD; POPULATION MODELS

OBJECTIVE A: Find the first three approximations y_1, y_2, and y_3, using the Euler approximation for an initial value problem $y' = f(x, y)$, $y(x_0) = y_0$.

83. Consider the initial value problem $y' = f(x, y)$ and $y(x_0) = y_0$. Then Euler's method allows you to approximate the solution stepping along the tangent lines in increments dx according to the formulas $x_n =$ _____________ and $y_n =$ _____________.

84. For the initial value problem $y' = x^2 - y$, $y(1) = 2$ we find using the increment size $dx = 0.1$ that $x_0 =$ _____________ and $y_1 = 2 + ($__________$)(0.1) = 1.9$. Next, $x_1 =$ __________$+ 0.1$ and $y_1 =$ __________$+ [(1.1)^2 - 1.9](0.1) = 1.831$. At the third step we obtain, $x_2 =$ ________ and $y_3 =$ _______________ $= 1.7919$.

OBJECTIVE B: Solve problems involving population models.

85. A population starts at $t = 0$ with a size of $P(0) = 11$. It grows according to the logistic model $\frac{dP}{dt} = 0.00027(120 - P)P$. One can use Euler's method with $dt = 2$ to estimate the population size at $t = 50$. There $P_n = P_{n-1} + \frac{dP}{dt} \cdot dt$ becomes $P_n = P_{n-1} +$ __________ · __________. Starting with $P_1 = P(0) = 11$, $P(50)$ is reached after __________ iterations of the formula: $P(50) \approx$ __________, rounded to the nearest whole number.

86. To find the exact solution for $P(50)$ in Problem 88, rounded to the nearest whole number, use the solution form $P = \frac{M}{1 + Ae^{-rMt}}$. Here $M =$ _______, $r =$ _______, and because $P(0) = 11$, $A =$ _______. Then $P(50) \approx$ _______.

---

83. $x_{n-1} + dx$, $y_{n-1} + f(x_{n-1}, y_{n-1})\,dx$

84. 1, $1^2 - 2$, 1, 1.9, 1.2, $1.831 + [(1.2)^2 - 1.831](0.1)$

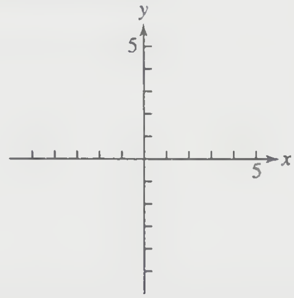
85. $0.00027(120 - P_{n-1})P_{n-1}$, 2, 25, 40

86. 120, 0.00027, $\frac{109}{11} = 9.\overline{90}$, 41

6.7 HYPERBOLIC FUNCTIONS

OBJECTIVE A: Define the six hyperbolic functions and graph them.

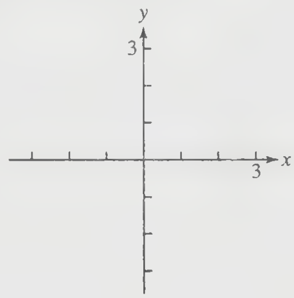
87. $\cosh x =$ ________________. Sketch the graph below.



88. $\sinh x =$ ________________. Sketch the graph above on the same coordinate system as the $\cosh x$.

89. $\tanh x =$ ____________________.

90. $\coth x =$ ____________________. Sketch the graphs below.



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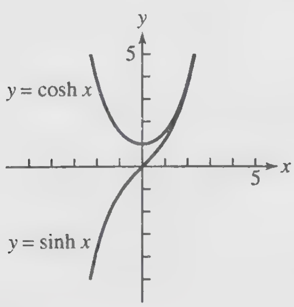
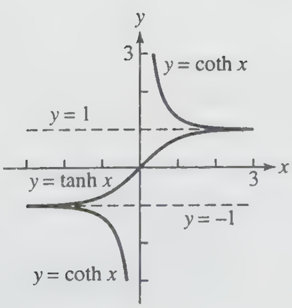
87. $\dfrac{e^x + e^{-x}}{2}$



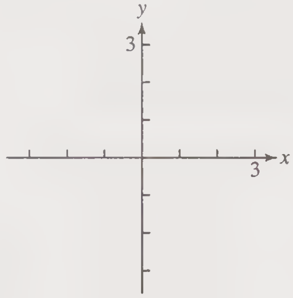
88. $\dfrac{e^x - e^{-x}}{2}$

89. $\dfrac{\sinh x}{\cosh x}$



90. $\dfrac{\cosh x}{\sinh x}$

91. sech x = ________________. Sketch the graph below.



92. csch x = ________________.

OBJECTIVE B: Given the value for one of the six hyperbolic functions at a point, determine the values of the remaining five at that point. Also, use hyperbolic identities.

93. Suppose $\tanh x = -\frac{\sqrt{3}}{2}$. Then, $\operatorname{sech}^2 x = 1 -$ ______________ = ____________ or sech x = ____________.

Thus, $\cosh x = \frac{1}{\operatorname{sech} x}$ = __________.

94. Continuing Problem 96, $\sinh^2 x = \cosh^2 x -$ ______________ = ____________. Since tanh x is negative and cosh x is positive, it follows that sinh x = __________. Then, csch x = ___________ and coth x = __________.

95. $\cosh(-x)$ = ________________.

96. $\sinh(-x)$ = ________________.

97. $\sinh(x + y)$ = ________________.

98. $\cosh(x + y)$ = ________________.

99. $\sinh 2x$ = ________________.

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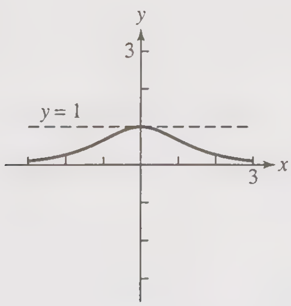
91. $\frac{1}{\cosh x}$



92. $\frac{1}{\sinh x}$

93. $\tanh^2 x, \frac{1}{4}, \frac{1}{2}, 2$

94. $1, 3, -\sqrt{3}, -\frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}$

95. $\cosh x$

96. $-\sinh x$

97. $\sinh x \cosh y + \cosh x \sinh y$

98. $\cosh x \cosh y + \sinh x \sinh y$

99. $2 \sinh x \cosh x$

100. $\cosh 2x =$ ______________________.

101. $\cosh 2x - 1 =$ ______________________.

OBJECTIVE C: Calculate the derivatives of functions expressed in terms of hyperbolic functions.

102. $\dfrac{d}{dx}(\cosh u) =$ ______________________.

103. $\dfrac{d}{dx}(\sinh u) =$ ______________________.

104. $\dfrac{d}{dx}(\tanh u) =$ ______________________.

105. $y = \sinh^3(3 - 2x^2)$

$\dfrac{dy}{dx} = 3\sinh^2(3-2x^2)\cdot\dfrac{d}{dx}$ ______________________ $= 3\sinh^2(3-2x^2)\cosh(3-2x^2)\dfrac{d}{dx}$ ______________________

$=$ ______________________.

106. $y = e^x \tanh 2x$

$\dfrac{dy}{dx} = e^x\dfrac{d}{dx}($______________________$) + e^x \tanh 2x = e^x$ ____________ $\dfrac{d}{dx}($____________$) + e^x \tanh 2x$

$=$ ______________________.

107. $y = x^{\sinh x},\ x > 0$

$\dfrac{dy}{dx} = \dfrac{d}{dx}(e^{\sinh x\cdot\ln x}) = e^{\sinh x\cdot\ln x}\dfrac{d}{dx}($______________________$) = e^{\sinh x\cdot\ln x}\left(\sinh x\cdot\dfrac{d}{dx}\ln x + \ln x\cdot\dfrac{d}{dx}\right.$ ______________$\left.\right)$

$= e^{\sinh x\cdot\ln x}($______________________$) = x^{\sinh x - 1}($______________________$)$.

108. $e^y = \operatorname{sech} x$

Differentiating implicitly, $\dfrac{dy}{dx}(e^y) = \dfrac{d}{dx}\operatorname{sech} x$, or ______________________ $= -\operatorname{sech} x \tanh x$. Thus,

$\dfrac{dy}{dx} = -e^{-y}($______________________$) =$ ______________.

---

100. $\cosh^2 x + \sinh^2 x$

101. $2\sinh^2 x$

102. $\sinh u\dfrac{du}{dx}$

103. $\cosh u\dfrac{du}{dx}$

104. $\operatorname{sech}^2 u\dfrac{du}{dx}$

105. $\sinh(3-2x^2)$, $3-2x^2$, $-12x\sinh^2(3-2x^2)\cosh(3-2x^2)$

106. $\tanh 2x$, $\operatorname{sech}^2 2x$, $2x$, $e^x(2\operatorname{sech}^2 2x + \tanh 2x)$

107. $\sinh x\cdot\ln x$, $\sinh x$, $\dfrac{\sinh x}{x} + \cosh x\cdot\ln x$, $\sinh x + x\cosh x\ln x$

108. $e^y\dfrac{dy}{dx}$, $\operatorname{sech} x\tanh x$, $-\tanh x$

OBJECTIVE D: Integrate functions whose expressions involve hyperbolic functions.

109. $\int x \cosh(x^2+3)\,dx$

Let $u = x^2+3$, then $du =$ ______________, and the integral becomes

$\int x \cosh(x^2+3)\,dx = \int$ _____ $du =$ ______________ $+ C =$ __________________.

110. $\int \sinh^2 x\,dx$

From the identities $\cosh 2x = \sinh^2 x + \cosh^2 x$ and $\cosh^2 x - \sinh^2 x = 1$, we have $\cosh 2x =$ ___________

or $\sinh^2 x = \frac{1}{2}($______________$)$. thus, $\int \sinh^2 x\,dx =$ ______________ $+ C$.

111. $\int \tanh x \ln(\cosh x)\,dx$

Let $u = \ln(\cosh x)$, so $du =$ ______________ and the integral becomes

$\int \ln(\cosh x)\tanh x\,dx = \int$ __________ $du =$ ______________ $+ C =$ __________________.

OBJECTIVE E: Define and use the six inverse hyperbolic functions. Be able to calculate their derivatives.

112. $y = \sinh^{-1} x$ means ______________. Thus $x = \frac{e^y - \_\_\_\_}{2}$ or $2xe^y =$ ______________ or

$e^{2y} -$ ___________ $-1 = 0$. The solution of this quadratic equation by the quadratic formula gives,

$e^y = \frac{2x \pm \sqrt{\_\_\_\_\_\_}}{2}$. Since $e^y > 0$ we must have $e^y =$ ______________ or $\sinh^{-1} x = y =$ ______________.

113. $\operatorname{sech}^{-1} x =$ ______________________.

114. $\coth^{-1} x =$ ______________________.

115. $\operatorname{csch}^{-1} x =$ ______________________.

116. We can use the formula found in Problem 115 to calculate $\frac{dy}{dx}$ for $y = \sinh^{-1} x$:

$$\frac{d}{dx}\ln(x+\sqrt{x^2+1}) = \frac{1}{x+\sqrt{x^2+1}} \cdot \frac{d}{dx}(\_\_\_\_\_\_\_\_) = \frac{1}{x+\sqrt{x^2+1}} \cdot (\_\_\_\_\_\_\_\_)$$

$$= \frac{1}{x+\sqrt{x^2+1}} \cdot \left(\frac{\_\_\_\_\_\_}{\sqrt{x^2+1}}\right) = \_\_\_\_\_\_\_\_.$$

---

109. $2x\,dx, \frac{1}{2}\cosh u, \frac{1}{2}\sinh u, \frac{1}{2}\sinh(x^2+3)+C$

110. $2\sinh^2 x + 1, \cosh 2x - 1, \frac{1}{4}\sinh 2x - \frac{1}{2}x$

111. $\frac{1}{\cosh x}\cdot \sinh x\,dx, u, \frac{1}{2}u^2, \frac{1}{2}[\ln(\cosh x)]^2 + C$

112. $x = \sinh y, e^{-y}, e^{2y}-1, 2xe^y, 4x^2+4, x+\sqrt{x^2+1}, \ln(x+\sqrt{x^2+1})$

113. $\cosh^{-1}\frac{1}{x}$

114. $\tanh^{-1}\frac{1}{x}$

115. $\sinh^{-1}\frac{1}{x}$

116. $x+\sqrt{x^2+1}, 1+\frac{2x}{2\sqrt{x^2+1}}, \sqrt{x^2+1}+x, \frac{1}{\sqrt{x^2+1}}$

117. An alternative way to calculate the derivative of $y = \sinh^{-1} x$ is as follows: Differentiate $x = \sinh y$ implicitly: $\frac{d}{dx} x = \frac{d}{dx} \sinh y$ or $1 =$ ______________. Thus, $\frac{dy}{dx} = \frac{1}{\_\_\_\_\_\_} = \frac{1}{\sqrt{1 + \_\_\_\_\_\_}} =$ ______________.
The positive square root is taken in the penultimate step because $\cosh y$ is always ______________.

OBJECTIVE F: Calculate the derivatives of functions expressed in terms of inverse hyperbolic functions.

118. $\frac{d}{dx}\left(\tanh^{-1} e^x\right) = \frac{1}{\_\_\_\_\_\_} \cdot \frac{d}{dx} e^x =$ ______________.

119. $\frac{d}{dx} \ln\left(\sinh^{-1} x\right) = \frac{1}{\sinh^{-1} x} \cdot \frac{d}{dx}$ ______________ $=$ ______________.

120. $\frac{d}{dx} \sqrt{\coth^{-1} x} = \frac{1}{2}\left(\coth^{-1} x\right)^{-1/2} \cdot \frac{d}{dx}$ ______________ $=$ ______________.

121. $\frac{d}{dx} \cosh^{-1} \frac{3}{x^2} =$ ______________ $\cdot \frac{-6}{x^3} =$ ______________.

OBJECTIVE G: Evaluate integrals using integration formulas for inverse hyperbolic functions.

122. $\int_{-3}^{-2} \frac{dx}{\sqrt{x^2+1}}$

Since $\int \frac{dx}{\sqrt{x^2+1}} =$ ______________ $+ C = \ln($______________$) + C$ (Problem 115),

$\int_{-3}^{-2} \frac{dx}{\sqrt{x^2+1}} =$ ______________ $\Big]_{-3}^{-2} = \ln(\sqrt{5} - 2) -$ ______________ ≈ 0.375.

123. $\int_{0.5}^{0.9} \frac{dx}{x\sqrt{1-x^2}}$

For $0 < x < 1$, $\int \frac{dx}{x\sqrt{1-x^2}} =$ ______________ $+ C$. Now,

$\operatorname{sech}^{-1} x = \cosh^{-1}$ ______________ $= \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right) = \ln\left(\frac{\_\_\_\_\_\_}{x}\right)$. Thus,

$\int_{0.5}^{0.9} \frac{dx}{x\sqrt{1-x^2}} =$ ______________ $\Big]_{0.5}^{0.9} = -\ln\left(\frac{1+\sqrt{1-.81}}{.9}\right) +$ ______________

$= -\ln\left(\frac{10+\sqrt{19}}{9}\right) +$ ______________ ≈ 0.850.

---

117. $\cosh y \cdot \frac{dy}{dx}$, $\cosh y$, $\sinh^2 y$, $\frac{1}{\sqrt{1+x^2}}$, positive

118. $1 - e^{2x}$, $\frac{e^x}{1-e^{2x}}$

119. $\sinh^{-1} x$, $\frac{1}{\sinh^{-1} x \cdot \sqrt{1+x^2}}$

120. $\coth^{-1} x$, $\frac{1}{2(1-x^2)\sqrt{\coth^{-1} x}}$

121. $\frac{1}{\sqrt{\frac{9}{x^4} - 1}}$, $\frac{-6}{x\sqrt{9-x^4}}$

122. $\sinh^{-1} x$, $x + \sqrt{x^2+1}$, $\ln(x + \sqrt{x^2+1})$, $\ln(\sqrt{10} - 3)$

123. $-\operatorname{sech}^{-1} x$, $\frac{1}{x}$, $1 + \sqrt{1-x^2}$, $-\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$, $\ln\left(\frac{1+\sqrt{1-.25}}{.5}\right)$, $\ln(2+\sqrt{3})$

124. $\int \frac{dx}{16-x^2} = \frac{1}{16}\int$ ______________

Let $u = \frac{x}{4}$, $du =$ ______________, and the integral becomes

$\int \frac{dx}{16-x^2} = \int \frac{4du}{________} = \frac{1}{4}($______________$) + C =$ ______________ $+ C$.

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124. $\frac{dx}{1-\left(\frac{x^2}{16}\right)}$, $\frac{1}{4}dx$, $16(1-u^2)$, $\frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|$, $\frac{1}{8}\ln\left|\frac{4+x}{4-x}\right|$

CHAPTER 6 SELF-TEST

In Problems 1–14 calculate the derivative $\frac{dy}{dx}$.

1. $y = \tan\left(\cos\frac{2}{x}\right)$

2. $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ x > 0$

3. $y = 5^x \log_5 x$

4. $y = 2^{\sin^{-1} x}$

5. $y = 2x\tan^{-1} 2x - \ln\sqrt{1+4x^2}$

6. $y = \frac{\sec 3\sqrt{x}}{\sqrt{x}}$

7. $y = (x)^{\sqrt{x}},\ x > 0$

8. $e^y + e^x = e^{x+y}$

9. $y = \tanh(\sin x)$

10. $y = \coth^{-1}(\ln x)$

11. $y = \sqrt{\cosh^{-1} x^2}$

12. $y = \ln(\sinh x^3)$

13. $y = x^{-1}\tanh^{-1} x^2$

14. $y = \sinh^{-1}(\tan x)$

In Problems 15–23 calculate the indicated integrals.

15. $\int_{-5/4}^{5/4} \frac{dx}{25+16x^2}$

16. $\int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx$

17. $\int_1^e \frac{2^{\ln x}}{3x}\,dx$

18. $\int \frac{4\,dx}{(e^x - e^{-x})^2}$

19. $\int \frac{\sinh(\ln x)\,dx}{x}$

20. $\int \sqrt{1+\cosh x}\,dx$

21. $\int_3^7 \frac{dx}{\sqrt{x^2-1}}$

22. $\int_0^{1/2} \frac{\cosh x\,dx}{1-\sinh^2 x}$

23. $\int_0^1 \frac{dx}{\sqrt{e^{2x}+1}}$

24. Simplify the expression $\frac{\log_3 243}{\log_2 \sqrt[4]{64} + \log_8 8^{-10}}$.

25. Let $f(x) = \log_a x$, $f(5) = 1.46$, $f(2) = 0.63$, $f(7) = 1.77$. Use the properties of logarithms to find,

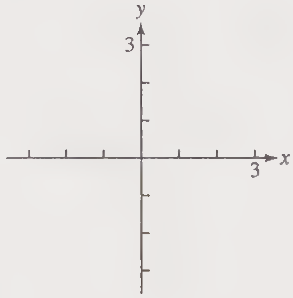
(a) $f(10)$ **(b)** $f(49)$ **(c)** $f\left(\frac{5}{7}\right)$

(d) $f(1.4)$

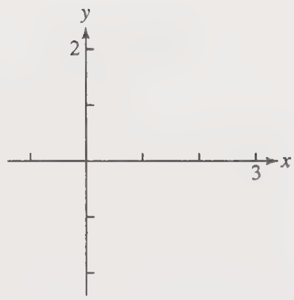
26. Solve the following equations for x.

(a) $3^{-8x+6} = 27^{-x-8}$ **(b)** $\log_5(5x-1) = -2$ **(c)** $e^x = 10^{x+1}$

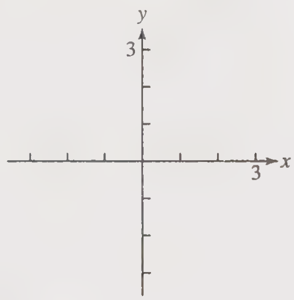
27. Graph the curve $y = \dfrac{2}{1+3e^{-2x}}$.



28. Sketch the graph of $y = \dfrac{\ln x}{x^2}$.



29. Define the hyperbolic function $y = \operatorname{csch} x$, and sketch its graph.



30. Given that $\cosh x = 2$, $x < 0$, find the values of the remaining hyperbolic functions at x.

31. Find the length of the catenary $y = 3\cosh\dfrac{x}{3}$ from $x = 0$ to $x = 3$.

32. Suppose that a dose d mg of a drug is injected into the bloodstream. Assume that the drug leaves the blood and enters the urine at a rate proportional to the amount of the drug present in the blood. Assume that at the end of 1 hour the amount of the drug in the urine is $\dfrac{1}{2}d$. Find the time at which 10 percent of the original dose is in the bloodstream.

33. Suppose that the number of bacteria in a yeast culture grows at a rate proportional to the number present. If the population of a colony of yeast bacteria doubles in one hour, find the number of bacteria present at the end of 3.5 hours.

34. Solve the differential equation $\dfrac{dy}{dx} = x^3e^x + \dfrac{2y}{x} - 1$, $x > 0$.

35. Solve the differential equation $e^x(y-1)\,dx + 2(e^x + 4)\,dy = 0$.

SOLUTIONS TO CHAPTER 6 SELF-TEST

1. $\frac{dy}{dx} = \sec^2\left(\cos\frac{2}{x}\right)\cdot\left(-\sin\frac{2}{x}\right)\cdot\left(-\frac{2}{x^2}\right)$

2. $\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}\cdot\left[\frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}\right] = \frac{-(1+x^2)}{\sqrt{(1+x^2)^2-(1-x^2)^2}}\cdot\frac{-4x}{(1+x^2)^2} = \frac{4x}{(1+x^2)\sqrt{4x^2}}$
$= \frac{2x}{(1+x^2)|x|}$

3. $\frac{dy}{dx} = 5^x \cdot \ln 5 \cdot \log_5 x + \frac{5^x}{x\ln 5} = 5^x\left(\ln x + \frac{1}{x\ln 5}\right)$

4. $\frac{dy}{dx} = 2^{\sin^{-1}x}(\ln 2)\frac{1}{\sqrt{1-x^2}}$

5. $\frac{dy}{dx} = 2\tan^{-1}2x + 2x\cdot\frac{1}{1+4x^2}\cdot 2 - \frac{1}{\sqrt{1+4x^2}}\cdot\frac{8x}{2\sqrt{1+4x^2}} = 2\tan^{-1}2x$

6. $\frac{dy}{dx} = \frac{1}{\sqrt{x}}(\sec 3\sqrt{x}\tan 3\sqrt{x})\cdot\frac{3}{2\sqrt{x}} + \left(-\frac{1}{2}x^{-3/2}\right)\sec 3\sqrt{x} = \frac{\sec 3\sqrt{x}}{2x\sqrt{x}}(3\sqrt{x}\tan 3\sqrt{x} - 1)$

7. $y = (x)^{\sqrt{x}}$ gives $\ln y = \sqrt{x}\,\ln x$
$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\cdot\frac{1}{x} = \frac{1}{2\sqrt{x}}(\ln x + 2)$
$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}(\ln x + 2)(x)^{\sqrt{x}}$

8. $e^y\frac{dy}{dx} + e^x = e^{x+y}\frac{d}{dx}(x+y) = e^{x+y}\left(1+\frac{dy}{dx}\right)$ Hence, $(e^y - e^{x+y})\frac{dy}{dx} = e^{x+y} - e^x$, or from the original expression, $-e^x\frac{dy}{dx} = e^y$. Thus, $\frac{dy}{dx} = -e^{y-x}$.

9. $\frac{dy}{dx} = \text{sech}^2(\sin x)\cdot\cos x$

10. $\frac{dy}{dx} = \frac{1}{[1-(\ln x)^2]}\cdot\frac{1}{x},\ \ln x > 1$

11. $\frac{dy}{dx} = \frac{1}{2}(\cosh^{-1}x^2)^{-1/2}\cdot\frac{d}{dx}(\cosh^{-1}x^2) = \frac{1}{2}(\cosh^{-1}x^2)^{1/2}\cdot\frac{1}{\sqrt{x^4-1}}\cdot\frac{d}{dx}(x^2) = \frac{x}{\sqrt{(x^4-1)\cosh^{-1}x^2}},\ x > 1$

12. $\frac{dy}{dx} = \frac{1}{\sinh x^3}\cdot\frac{d}{dx}(\sinh x^3) = \frac{1}{\sinh x^3}\cdot\cosh x^3\cdot\frac{d}{dx}(x^3) = 3x^2\coth x^3$

13. $\frac{dy}{dx} = -\frac{1}{x^2}\tanh^{-1}x^2 + x^{-1}\cdot\frac{1}{1-x^4}\cdot 2x = -x^{-2}\tanh^{-1}x^2 + \frac{2}{1-x^4},\ x^4 < 1$

14. $\frac{dy}{dx} = \frac{1}{\sqrt{\tan^2 x + 1}}\cdot\sec^2 x = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \sec x$, if $-\frac{\pi}{2} < x < \pi$

15. $\int \frac{dx}{25+16x^2} = \frac{1}{25}\int \frac{dx}{1+\left(\frac{4}{5}x\right)^2} = \frac{1}{25}\cdot\frac{5}{4}\tan^{-1}\frac{4}{5}x + C$

$\int_{-5/4}^{5/4} \frac{dx}{25+16x^2} = \frac{1}{20}(\tan^{-1}1 - \tan^{-1}(-1)) = \frac{\pi}{40}$

16. $u = e^{-x}$, $du = -e^{-x}dx$

$\int \frac{-du}{\sqrt{1-u^2}} = \cos^{-1}(e^{-x}) + C$

17. $u = \ln x$, $du = \frac{1}{x}dx$

$\int \frac{2^{\ln x}}{3x}dx = \frac{1}{3}\int 2^u\,du = \frac{1}{3\ln 2}2^{\ln x} + C$

$\int_1^e \frac{2^{\ln x}}{3x}dx = \frac{1}{3\ln 2}(2^{\ln e} - 1) = \frac{1}{3\ln 2}$

18. $\int \frac{4\,dx}{(e^x - e^{-x})^2} = \int \operatorname{csch}^2 x\,dx = -\coth x + C$

19. Let $u = \ln x$, $du = \frac{1}{x}dx$, and the integral becomes $\int \frac{\sinh(\ln x)\,dx}{x} = \int \sinh u\,du = \cosh u + C = \cosh(\ln x) + C$.

20. From the identity $2\cosh^2 x = \cosh 2x + 1$, we find that $\sqrt{2}\cosh\frac{x}{2} = \sqrt{\cosh x + 1}$. Thus,

$\int \sqrt{\cosh x + 1}\,dx = \sqrt{2}\int \cosh\frac{x}{2}dx = 2\sqrt{2}\sinh\frac{x}{2} + C.$

21. $\int_3^7 \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}x\Big]_3^7 = \ln(x+\sqrt{x^2-1})]_3^7 = \ln(7+\sqrt{48}) - \ln(3+\sqrt{8}) = \ln\left(\frac{7+4\sqrt{3}}{3+2\sqrt{2}}\right) \approx 2.39.$

22. Let $u = \sinh x, du = \cosh x\,dx$, so that

$\int_0^{1/2} \frac{\cosh x\,dx}{1-\sinh^2 x} = \tanh^{-1}(\sinh x)]_0^{1/2} = \frac{1}{2}\ln\left[\frac{1+\sinh x}{1-\sinh x}\right]_0^{1/2} = \frac{1}{2}\ln\left[\frac{1+0.5211}{1-0.5211}\right] \approx 0.5778.$

23. $\int_0^1 \frac{dx}{\sqrt{e^{2x}+1}} = \int_0^1 \frac{e^x\,dx}{e^x\sqrt{e^{2x}+1}}$, which is of the for, $\int \frac{dx}{|u|\sqrt{u^2+1}} = -\operatorname{csch}^{-1}u + C$, for $u = e^x \neq 0$. Thus,

$\int_0^1 \frac{dx}{\sqrt{e^{2x}+1}} = -\operatorname{csch}^{-1}e^x]_0^1 = -\ln\left(\frac{1+\sqrt{1+e^{2x}}}{e^x}\right)\Bigg]_0^1 = -\ln\left(\frac{1+\sqrt{1+e^2}}{e}\right) + \ln(1+\sqrt{2}) \approx 0.52.$

24. $\frac{\log_3 243}{\log_2 \sqrt[4]{64} + \log_8 8^{-10}} = \frac{5}{\frac{1}{4}(6) - 10(1)} = -\frac{10}{17}$

25. **(a)** $\log_a 10 = \log_a 2 + \log_a 5 = 2.09$

(b) $\log_a 49 = 2\log_a 7 = 3.54$

(c) $\log_a \frac{5}{7} = \log_a 5 - \log_a 7 = -0.31$

(d) $\log_a 1.4 = \log_a 14 - \log_a 10 = \log_a 2 + \log_a 7 - \log_a 5 - \log_a 2 = 0.31$

26. **(a)** $3^{-8x+6} = 3^{-3x-24}$ or $-8x+6=-3x-24$ or $x=6$.

(b) $5^{-2} = 5x-1$ or $x = \dfrac{26}{125}$.

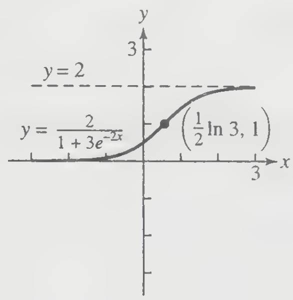
(c) $x=(x+1)\ln 10$ or $x = \dfrac{\ln 10}{1-\ln 10} \approx -1.768$.

27. $f(x) = \dfrac{2}{1+3e^{-2x}}$, $f'(x) = \dfrac{12e^{-2x}}{(1+3e^{-2x})^2}$, $f''(x) = \dfrac{24e^{-2x}(-1+3e^{-2x})}{(1+3e^{-2x})^3}$

$f'>0$ for every x; $f''(x)=0$ when $x = \dfrac{1}{2}\ln 3 \approx 0.55$.

$\lim\limits_{x\to+\infty} f(x) = 2$ and $\lim\limits_{x\to-\infty} f(x) = 0$

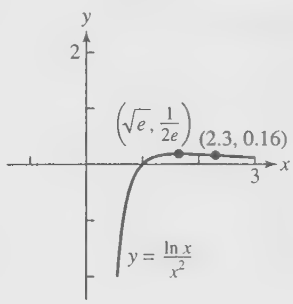
The graph is sketched below.



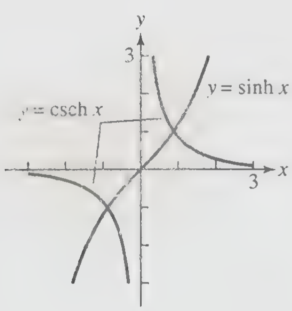
28. $f(x) = \dfrac{\ln x}{x^2}$, $f'(x) = \dfrac{1-2\ln x}{x^3}$ so $f'(x)=0$ implies $\ln x = \dfrac{1}{2}$ or $x=\sqrt{e}$.

$f''(x) = \dfrac{-5+6\ln x}{x^4}$ so $f''(x)=0$ implies $x = e^{5/6} \approx 2.3$.

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{2x} = 0$ and $\lim\limits_{x\to 0^+} f(x) = -\infty$. The graph is sketched below.



29. $y = \operatorname{csch} x = \dfrac{1}{\sinh x}$, where $\sinh x = \dfrac{1}{2}(e^x - e^{-x})$. The graph is sketched below.



30. $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{1}{2}$; $\tanh^2 x = 1 - \operatorname{sech}^2 x = \frac{3}{4}$ so that $\tanh x = \frac{-\sqrt{3}}{2}$. Since $x < 0$;
$\sinh x = \cosh x \tanh x = -\sqrt{3}$; $\coth x = \frac{1}{\tanh x} = -\frac{2}{\sqrt{3}}$ and $\operatorname{csch} x = \frac{1}{\sinh x} = -\frac{1}{\sqrt{3}}$.

31. $L = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^3 \sqrt{1 + \sinh^2 \frac{x}{3}}\, dx = \int_0^3 \cosh \frac{x}{3}\, dx = 3\sinh \frac{x}{3}\Big]_0^3 = 3\sinh 1 \approx 3.53.$

32. Let $y(t)$ denote the amount of the drug in the bloodstream at any time t. Then, $\frac{dy}{dt} = -ky$, $k > 0$ and $y(0) = d$. The solution to this initial value problem is $y(t) = de^{-kt}$. We are given that $y(1) = \frac{1}{2}d$, so $\frac{1}{2}d = de^{-k(1)}$. Then $-k = \ln \frac{1}{2}$ or $k = \ln 2$. Substituting into the original solution, $y(t) = de^{-(\ln 2)t}$. The desired time $t = T$ is such that $y(T) = \frac{1}{10}d$. Thus, $\frac{1}{10}d = de^{-(\ln 2)T}$ or $\ln \frac{1}{10} = (-\ln 2)T$. Solving, $T = \frac{-\ln 10}{-\ln 2} \approx 3.32$ hr.

33. Let x denote the number of bacteria present at any time t. Then $\frac{dy}{dx} = kx$ or $x = Ce^{kt}$, for some constant C. If x_0 is the initial number of bacteria at $t = 0$, then $C = x_0$, so $x = x_0 e^{kt}$. When $t = 1$, $x = 2x_0$ so that $2 = e^k$ or $k = \ln 2$. Therefore, $x = x_0 e^{t \ln 2}$. Finally, when $t = 3.5, x = x_0 e^{3.5 \ln 2} \approx 11.31 x_0$. Thus, there are 11.31 times the initial number of bacteria present at the end of 3.5 hours.

34. $y' - \frac{2}{x}y = x^3 e^x - 1$, $x > 0$ is linear. An integrating factor is $v(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$, $x > 0$. Thus,
$x^{-2}y = \int x^{-2}(x^3 e^x - 1)\, dx = \int xe^x dx - \int x^{-2}\, dx = (x-1)e^x + x^{-1} + C.$
Remark: You will learn $\int xe^x dx = (x-1)e^x + C$ in Chapter 7. Thus, $y = Cx^2 + x + x^2(x-1)e^x$.

35. The variables are separable, and the differential equation can be written as $\frac{e^x}{e^x + 4}dx + \frac{2}{y-1}dy = 0$. Integration gives, $\ln(e^x + 4) + 2\ln|y - 1| = \ln C$, or $(y-1)^2(e^x + 4) = C$.