# Chapter 4: Integration

## 4.1 INDEFINITE INTEGRALS, DIFFERENTIAL EQUATIONS, AND MODELING

**OBJECTIVE A:** Find an antiderivative for a given function based on your knowledge of derivatives of elementary power and trigonometric functions.

1. If $F'(x) = f(x)$ at every point of an interval $I$, then the function $F(x)$ is known as an ______________ of the function $f(x)$ over the interval $I$.

2. An antiderivative of the function $x^n$, $n \neq 1$, is ______________.

3. An antiderivative of the function $\cos kx$ is ______________.

4. An antiderivative of $3x^4 - x^{-2} + 5$ is ______________.

5. An antiderivative of $x^{3/2} + x^{1/2} + \frac{1}{x^2}$ is ______________.

6. An antiderivative of $\cos\frac{x}{2} + \sec^2 3x$ is ______________.

**OBJECTIVE B:** Evaluate indefinite integrals using 4.1.

7. If the function $f$ is a derivative, then the indefinite integral of $f$ is ______________. It is denoted by the symbol ______________.

8. If $F(x)$ is an antiderivative of $f$, then $\int f(x)dx =$ ______________.

9. $\int (3x^5 - x^2 + \sin \pi x)dx = 3\int x^5 dx -$ ______________ $+ \int \sin \pi x dx$

$= 3(\_\_\_\_\_\_) - \frac{1}{3}x^3 + (\_\_\_\_\_\_) + C.$

10. $\int (x^{3/2} - 2x^{2/3} + 5)\sqrt{x}dx = \int (x^2 -$ ______ $+ 5x^{1/2})dx$

$= (\_\_\_\_\_\_) - 2(\_\_\_\_\_\_) + 5(\_\_\_\_\_\_) + C$

$=$ ______________.

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1. antiderivative
2. $\frac{x^{n+1}}{n+1}$
3. $\frac{1}{k}\sin kx$
4. $\frac{3}{5}x^5 + x^{-1} + 5x$
5. $\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} - x^{-1}$
6. $2\sin\frac{x}{2} + \frac{1}{3}\tan 3x$
7. the set of antiderivatives of $f$, $\int f(x)dx$
8. $F(x) + C$
9. $\int x^2 dx$, $\frac{x^6}{6}$, $-\frac{1}{\pi}\cos \pi x$
10. $2x^{7/6}$, $\frac{x^3}{3}$, $\frac{x^{13/6}}{\frac{13}{6}}$, $\frac{x^{3/2}}{\frac{3}{2}}$, $\frac{1}{3}x^3 - \frac{12}{13}x^{13/6} + \frac{10}{3}x^{3/2} + C$

**OBJECTIVE C:** Solve elementary initial value problems.

**11.** Suppose that $\frac{dy}{dx} = \frac{1}{x^3} + 2x$, $x > 0$ and that $y = \frac{5}{2}$ when $x = 1$. To solve for $y$ we first solve the differential equation by finding an antiderivative of $\frac{1}{x^3 + 2x}$. The indefinite integral is $y(x) =$ ________________. Then we substitute $x = 1$ and $y = \frac{5}{2}$ to find the constant $C$: $\frac{5}{2} = -\frac{1}{2}(1)^{-2} +$ ______________ $+ C$ or solving for $C$, $C =$ ______________. The function we want is then $y = -\frac{1}{2}x^{-2} + x^2 + 2$.

**12.** Solve the following initial value problem for $y$ as a function of $x$:

Differential equation: $\frac{d^2y}{dx^2} = 2x - 1$

Initial conditions: $\frac{dy}{dx} = 1$ and $y = 0$ and when $x = -1$

*Solution.* We integrate the differential equation with respect to $x$ to find $\frac{dy}{dx}$:

$$\int \frac{d^2y}{dx^2}dx = \int (2x - 1)dx$$

$$\frac{dy}{dx} = \text{\_\_\_\_\_\_\_\_\_\_\_\_} + C_1$$

We apply the initial condition $\frac{dy}{dx} = 1$ when $x = -1$ to find $C_1$: $1 =$ ______________ $+ C_1$ or $C_1 = -1$. Thus, $\frac{dy}{dx} =$ ______________. We integrate $\frac{dy}{dx}$ with respect to $x$ to find $y$:

$$\int \frac{dy}{dx}dx = \int \left(x^2 - x - 1\right)dx$$

$$\text{\_\_\_\_\_\_} = \frac{1}{3}x^3 - \frac{1}{2}x^2 - x + C_2.$$

We apply the initial condition $y = 0$ when $x = -1$ to find $C_2$: $0 = \frac{1}{3}(-1)^3 - \frac{1}{2}(-1)^2 - (-1) + C_2$. Thus, $C_2 =$ _________. The formula for $y$ as a function of $x$ is: $y =$ __________________.

**OBJECTIVE D:** Given (a) the velocity and initial position, or (b) the acceleration, initial velocity, and initial position, find the position at any time $t$ of a body moving on a coordinate (straight) line.

**13.** The velocity of a body moving along a coordinate line is known to be $v = t + \cos t$. The initial position is $s = -1$ when $t = 0$. To find the body's position at any time $t$, we first solve the differential equation $\frac{ds}{dt} = t + \cos t$. Thus, $s =$ __________________. To evaluate the constant $C$, we substitute $s = -1$ and $t = 0$: $-1 = \frac{1}{2}(0)^2 +$ __________________ $+ C$ or solving for $C$, $C =$ _______________. The body's position at time $t$ is $s = \frac{1}{2}t^2 + \sin t - 1$.

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**11.** $-\frac{1}{2}x^{-2} + x^2 + C$, $1^2$, $2$

**12.** $x^2 - x$, $(-1)^2 - (-1)$, $x^2 - x - 1$, $y$, $-\frac{1}{6}$, $\frac{1}{3}x^3 - \frac{1}{2}x^2 - x - \frac{1}{6}$

**13.** $\frac{1}{2}t^2 + \sin t + C$, $\sin 0$, $-1$

**14.** Suppose the acceleration of a moving body is given by the equation $a = \sqrt{t}$, and that when $t = 0$ it is known that the body has initial velocity $v = v_0$ and initial position $s = s_0$. Let us find its position at any time. Since $a = \frac{dv}{dt}$, substitution gives $\frac{dv}{dt} = \sqrt{t}$. The indefinite integral is $v =$ ____________ $+ C_1$. Imposing the initial condition $v = v_0$ when $t = 0$ yields $v_0 =$ __________; so $C_1 =$ __________. Hence the velocity equation becomes $v =$ ____________. Now, $v = \frac{ds}{dt}$ so the last equation can be written $\frac{ds}{dt} = \frac{2}{3}t^{3/2} + v_0$. We next solve this differential equation to find $s$: $s =$ ______________ $+ C_2$. From the initial condition $s = s_0$ when $t = 0$ it is readily seen that $C_2 =$ ______________. Therefore, the body's position is $s =$ ____________ valid for all $t \geq 0$.

## 4.2 INTEGRAL RULES; INTEGRATION BY SUBSTITUTION

**OBJECTIVE A:** Find antiderivatives using the rules of algebra for antiderivatives.

**15.** $\int \left( \frac{1}{\sqrt{x}} + x \right) dx = \int x^{-1/2} dx +$ ______________ $=$ ____________ $+ \frac{1}{2}x^2 + C = 2\sqrt{x} + \frac{1}{2}x^2 + C.$

**16.** $\int (\sin^2 x - \sec^2 x) dx = \int \sin^2 x dx - \int \sec^2 x dx =$ ________________.

**OBJECTIVE B:** Evaluate indefinite integrals using the substitution method of integration to reduce the integrals to standard form.

**17.** To evaluate $\int (3x-1)^2 dx$ substitutes $u =$ __________ and $du = 3dx$. Then,
$\int (3x-1)^2 dx = \frac{1}{3}\int (3x-1)^2 \cdot 3dx = \frac{1}{3}\int$ ____________ $=$ ____________ $= \frac{1}{9}(3x-1)^3 + C$. (Replace $u$ by ________________.)

**18.** To evaluate $\int 5x\cos(x^2+1)dx$ substitute $u =$ __________ and $du =$ ____________. Then,
$\int 5x\cos(x^2+1)dx = \frac{5}{2}\int \cos(x^2+1)\cdot 2x\,dx = \frac{5}{2}\int$ ____________ $=$ ____________ $= \frac{5}{2}\sin(x^2+1) + C$.
(Replace $u$ by ____________.)

**19.** $\int \sin(3-2x)dx$. Let $u = 3-2x$. Then $du =$ __________ so $dx = -\frac{1}{2}du$. Thus the integral becomes
$\int \sin(3-2x)dx = \int$ ____________ $= \frac{1}{2}\int$ ____________ $= \frac{1}{2}$ __________ $+ C =$ ____________.

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**14.** $\frac{2}{3}t^{3/2}$, $\frac{2}{3}(0)^{3/2} + C_1$, $v_0$, $\frac{2}{3}t^{3/2} + v_0$, $\frac{4}{15}t^{5/2} + v_0 t$, $s_0$, $\frac{4}{15}t^{5/2} + v_0 t + s_0$

**15.** $\int x\,dx$, $\frac{x^{1/2}}{\frac{1}{2}}$

**16.** $\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) - \tan x + C$

**17.** $3x-1$, $u^2 du$, $\frac{1}{9}u^3 + C$, $3x-1$

**18.** $x^2+1$, $2x\,dx$, $\cos u\,du$, $\frac{5}{2}\sin u + C$, $x^2+1$

**19.** $-2dx$, $\sin u \cdot \left(-\frac{1}{2}\right)du$, $-\sin u\,du$, $\cos u$, $\frac{1}{2}\cos(3-2x) + C$

20. $\int x^2 \cos(4x^3)dx$. Let $u = 4x^3$. Then $du =$ __________ so $x^2 dx =$ __________. Thus the integral becomes $\int x^2 \cos(4x^3)dx = \int \cos(4x^3) \cdot x^2 dx = \int$ __________ $du = \frac{1}{12}$ __________ $+ C =$ __________.

21. $\int (3 - \sin 2t)^{1/3} \cos 2t\, dt$. Let $u = 3 - \sin 2t$. Then $du =$ __________. Substitution into the integral gives $\int (3 - \sin 2t)^{1/3} \cos 2t\, dt = \int$ __________ $du =$ __________ $+ C =$ __________.

22. $\int \sec^{5/2} x \tan x\, dx$. Let $u = \sec x$. Then $du =$ __________. Now, $\sec^{5/2} x \tan x\, dx = \sec^{3/2} x \cdot$ __________.
Hence, substitution into the integral gives,
$\int \sec^{5/2} x \tan x\, dx = \int$ __________ $du =$ __________ $+ C =$ __________.

23. $\int x\sqrt{3x^2 + 1}\, dx$. Let $u = 3x^2 + 1$. Then $du =$ __________ so $x\,dx = \frac{1}{6} du$. Thus the integral becomes $\int x\sqrt{3x^2 + 1} dx = \int$ __________ $= \frac{1}{6}$ __________ $+ C =$ __________.

24. $\int \frac{1}{z^2 - 6z + 9} dz$. Let $u = z - 3$ so $u^2 =$ __________ and $du = dz$. Thus the integral becomes $\int \frac{1}{z^2 - 6z + 9} dz = \int$ __________ $du =$ __________ $+ C = \frac{1}{3 - z} + C$.

## 4.3 ESTIMATING WITH FINITE SUMS

**OBJECTIVE:** Estimate a final result (e.g., distance traveled, area under a curve, volume of a solid, average value of a function) by summing a finite number of close estimates made with a standard formula.

25. Use the following table to estimate the distance traveled by a car moving down a highway for one hour, Use the left-end beginning velocity values for each subinterval.

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20. $12x^2 dx$, $\frac{1}{12} du$, $\frac{1}{12} \cos u$, $\sin u$, $\frac{1}{12} \sin(4x^3) + C$

21. $-2\cos 2t\, dt$, $-\frac{1}{2} u^{1/3} du$, $-\frac{3}{8} u^{4/3}$, $-\frac{3}{8}(3 - \sin 2t)^{4/3} + C$

22. $\sec x \tan x\, dx$, $\sec x \tan x$, $u^{3/2}$, $\frac{2}{5} u^{5/2}$, $\frac{2}{5} \sec^{5/2} x + C$

23. $6x\, dx$, $\frac{1}{6} \sqrt{u}\, du$, $\frac{2}{3} u^{3/2}$, $\frac{1}{9}(3x^2 + 1)^{3/2} + C$

24. $z^2 - 6z + 9$, $u^{-2}$, $-u^{-1}$

| Time (min) | Velocity (mi/min) | Time (min) | Velocity (mi/min) |
|---|---|---|---|
| 0 | 0.2 | 35 | 0.8 |
| 5 | 1.0 | 40 | 0.7 |
| 10 | 1.1 | 45 | 0.9 |
| 15 | 1.2 | 50 | 1.0 |
| 20 | 1.0 | 55 | 0.6 |
| 25 | 1.1 | 60 | 0.5 |
| 30 | 0.9 | | |

*Solution.* We use the standard formula distance = (rate)(time). The total distance $S$ traveled from $t = 0$ to $t = 60$ minutes is given by

$S \approx (0.2)(5) + (1.0)(5) + (1.1)(5) + (1.2)(5) + (1.0)(5) +$ ______________________________ $= 52.5$ miles.

**26.** Estimate the average value of the function $f(x) = 3x + 2$ over the interval $1 \le x \le 4$ by partitioning the interval into 6 subintervals of equal length and evaluating $f$ at the subinterval midpoints.

*Solution.* The partition consists of the subintervals

1 1.5 2 2.5 3 3.5 4

The midpoints of the subintervals are 1.25, 1.75, ________, 2.75, 3.25, ________. The subintervals are each of length $\Delta x = \frac{1}{2}$ and the length of the entire interval is 4 – 1 = 3. Thus

$$av(f) \approx \frac{f(1.25) + f(1.75) + f(2.25) + f(2.75) + f(3.25) + f(3.75)}{2 \cdot 3}$$

$$= \frac{5.75 + \_\_\_\_\_ + \_\_\_\_\_ + 10.25 + 11.75 + \_\_\_\_\_}{6} = 9.5.$$

We find where $f$ assumes this value by solving the equation $f(x) = 9.5$ for $x$: $3x + 2 = 9.5$ implies $x \approx$ __________.

## 4.4 RIEMANN SUMS AND DEFINITE INTEGRALS

**OBJECTIVE A:** Interpret and utilize the sigma notation to express or write out sums, and determine (if possible) the value of a sum expressed in sigma notation.

**27.** To write out the sum $\sum_{k=3}^{7} 2^{k-2}$,

Replace the $k$ in $2^{k-2}$ by 3 and obtain __________.
Replace the $k$ in $2^{k-2}$ by 4 and obtain __________.
Replace the $k$ in $2^{k-2}$ by 5 and obtain __________.
Replace the $k$ in $2^{k-2}$ by 6 and obtain __________.
Replace the $k$ in $2^{k-2}$ by 7 and obtain __________.

The expanded form is $\sum_{k=3}^{7} 2^{k-2} =$ ______________. This finite sum is equal to ______________.

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**25.** (1.1)(5)+(0.9)(5)+(0.8)(5)+(0.7)(5)+(0.9)(5)+(1.0)(5)+(0.6)(5)

**26.** 2.25, 3.75, 7.25, 8.75, 13.25, 2.5

**27.** $2^1, 2^2, 2^3, 2^4, 2^5, 2^1 + 2^2 + 2^3 + 2^4 + 2^5$, 62

**28.** To express the finite sum $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ in sigma notation, we may observe that the sum can be written as $\left(\frac{1}{2}\right)^0+\left(\frac{1}{2}\right)^1+\left(\frac{1}{2}\right)^2+(\_\_\_\_)^3+(\_\_\_\_)^4$. The $k^{\text{th}}$ term in this expression is $\left(\frac{1}{2}\right)^k$, and see that $k$ starts at __________ and ends at __________. Therefore, the required sigma notation is ________________.

**29.** To write out the sum $\sum_{k=0}^{3}\frac{(-1)^k}{k!}x^k$, first recall that $k!$ means $1\cdot 2\cdot 3\cdots k$. Also, $0!=1$. Then,

Replace the $k$ in $\frac{(-1)^k}{k!}x^k$ by 0 and obtain ______________ = ______________.

Replace the $k$ in $\frac{(-1)^k}{k!}x^k$ by 1 and obtain ______________ = ______________.

Replace the $k$ in $\frac{(-1)^k}{k!}x^k$ by 2 and obtain ______________ = ______________.

Replace the $k$ in $\frac{(-1)^k}{k!}x^k$ by 3 and obtain ______________ = ______________.

The expanded form is $\sum_{k=0}^{3}\frac{(-1)^k}{k!}x^k =$ ______________________. Substitution of $x=1$ gives the value ______________.

**OBJECTIVE B:** Express a limit of Riemann sums over an interval as a definite integral.

**30.** Let $f(x)$ be a function defined on a closed interval $[a,b]$. The expression $\sum_{k=1}^{n} f(c_k)\Delta x_k$ is called a ________________ on the interval $[a,b]$. In this expression the interval has been partitioned $a=x_0<x_1<x_2<\cdots<x_{n-1}<x_n=b$ into $n$ subintervals. The typical closed subinterval $[x_{k-1},x_k]$ is called the ________________. Its length is ______________________. The number $c_k$ is *any* point lying within the ________________.

**31.** The ____________ of a partition of $[a,b]$ is the length of the ____________________ subinterval.

**32.** The limit (provided it exists) of any Riemann sum for *f* as the norm of the partition tends to zero is called the ____________________ over $[a,b]$. It is denoted by the symbol __________________________________.

**33.** The numbers $a$ and $b$ in the symbol $\int_a^b f(x)dx$ are called the ________________________________________.
The function *f* is the _________________________ of the definite integral.

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**28.** $\frac{1}{2}, \frac{1}{2}$, 0, 4, $\sum_{k=0}^{4}\left(\frac{1}{2}\right)^k$

**29.** $\frac{(-1)^0}{0!}x^0$, 1, $\frac{(-1)^1}{1!}x^1$, $-x$, $\frac{(-1)^2}{2^1}x^2$, $\frac{1}{2}x^2$, $\frac{(-1)^3}{3!}x^3$, $-\frac{1}{6}x^3$, $1-x+\frac{1}{2}x^2-\frac{1}{6}x^3$, $\frac{1}{3}$

**30.** Riemann sum for *f*, $k^{\text{th}}$ subinterval, $\Delta x_k = x_k - x_{k-1}$, $k^{\text{th}}$ subinterval

**31.** norm, longest

**32.** definite integral for *f*, $\int_a^b f(x)dx$

**33.** limits of integration, integrand

34. The limit $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (c_k \sin 2c_k)\Delta x_k$, where $P$ denotes a partition of the interval $[0, 2\pi]$, is the definite integral ____________________.

35. The definite integral $\int_a^b f(x)dx$ is the *area* of the region between the graph of $f$ and the $x$-axis from $a$ to $b$ provided that $y = f(x)$ is integrable and ____________________ on the closed interval $[a, b]$. When $f(x) \le 0$ over $[a, b]$ the area is given by ____________________.

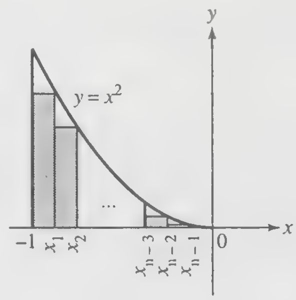
36. For any integrable function, the interpretation of the definite integral in terms of areas is $\int_a^b f(x)dx =$ ____________________.

**OBJECTIVE C:** Find the area bounded by a curve $y = f(x)$ positive over $a \le x \le b$ by finding the limit of the sum of inscribed rectangles, if $f$ is of the form $y = mx^k$ for $k = 0, 1, 2, 3$.

37. Let $y = f(x)$ define a positive continuous function of $x$ on the closed interval $a \le x \le b$. the area under the curve and above the $x$-axis from $x = a$ and $x = b$ is defined to be the ____________________ of the sums of the areas of inscribed ____________________ as their number ____________________ without bound.

38. To find the area under the graph $y = x^2$, $-1 \le x \le 0$ using inscribed rectangles, divide the interval $-1 \le x \le 0$ into $n$ subintervals each of equal length $\Delta x =$ ____________________ by inserting the points $x_1 = -1 + \Delta x$, $x_2 = -1 + 2\Delta x$, ..., $x_{n-1} =$ ____________________. Notice that $y = x^2$ is a ____________________ function of $x$ over the interval $-1 \le x \le 0$. Thus the height of the first inscribed rectangle is ____________________.
The inscribed rectangles have areas

$$f(x_1)\Delta x = (-1 + \Delta x)^2 \cdot \Delta x$$
$$f(x_2)\Delta x = (-1 + 2\Delta x)^2 \cdot \Delta x$$
$$\vdots$$
$$f(x_{n-1})\Delta x = (-1 + (n-1)\Delta x)^2 \cdot \Delta x$$
$$f(\_\_\_\_\_)\Delta x = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_,$$

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34. $\int_0^{2\pi} x \sin 2x \, dx$

35. nonnegative, $-\int_a^b f(x)dx$

36. (area above $x$-axis)$-$(area below $x$-axis)

37. limit, rectangles, increases

38. $\frac{1}{n}$, $-1 + (n-1)\Delta x$, decreasing, $(-1+\Delta x)^2$ or $x_1{}^2$, 0, $0 \cdot \Delta x$ or $(-1 + n\Delta x)^2$, $\frac{1}{n}$, $1 + 2 + 3 + \cdots + (n-1)$, $1^2 + 2^2 + \cdots + (n-1)^2$, $\frac{(n-1)n}{2}$, $\frac{(n-1)n(2(n-1)+1)}{6}$, $\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$, $S_n$, $\frac{1}{3}$

whose sum is $S_n = \left[\left(-1+\frac{1}{n}\right)^2 + \left(-1+\frac{2}{n}\right)^2 + \cdots + \left(-1+\frac{n-1}{n}\right)^2 + 0\right] \cdot$ __________. Expanding each term on the right side and collecting like terms gives,

$$S_n = \left((-1)^2(n-1) + 2(-1)\left(\frac{1}{n}+\frac{2}{n}+\cdots+\frac{n-1}{n}\right) + \left(\frac{1^2}{n^2}+\frac{2^2}{n^2}+\cdots+\frac{(n-1)^2}{n^2}\right)\right)\frac{1}{n}$$

$$= \frac{1}{n}\left[(n-1) - \frac{2}{n}(\_\_\_\_\_) + \frac{1}{n^2}(\_\_\_\_\_)\right] = \frac{1}{n}\left[(n-1) - \frac{2}{n}\cdot(\_\_\_\_\_) + \frac{1}{n^2}\cdot(\_\_\_\_\_)\right] = \_\_\_\_\_\_\_\_\_\_.$$

The area under the graph is defined to be the limit of __________ as $n \to \infty$. This limit equals __________.

**OBJECTIVE D:** Know from memory the rules for working with definite integrals.

**39.** $\int_a^a f(x)dx =$ ____________________.

**40.** $\int_a^b [f(x) \pm g(x)]dx =$ ____________________.

**41.** $\int_a^b kf(x)dx =$ ____________________.

**42.** $\int_b^a f(x)dx =$ ____________________.

**43.** $f(x) \geq g(x)$ on $[a,b]$ implies $\int_a^b f(x)dx$ ____________________.

**44.** $\int_a^b -f(x)dx =$ ____________________.

**45.** $f(x) \geq 0$ on $[a,b]$ implies $\int_a^b f(x)dx =$ ____________________.

**46.** $\int_a^b f(x)dx + \int_b^c f(x)dx =$ ____________________.

**47.** $\min f \cdot (b-a) \leq$ ____________________ $\leq \max f \cdot (b-a)$.

**OBJECTIVE E:** Use the rules in Table 4.5 to find values of definite integrals.

**48.** Suppose $f$ and $g$ are continuous and that $\int_1^3 f(x)dx = -5$, $\int_1^6 f(x)dx = 2$, $\int_1^3 g(x)dx = 4$. Then,

$\int_3^6 f(x)dx = \int_3^1 f(x)dx +$ ____________________ $= -$ ____________________ $+ 2 =$ ____________________.

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**39.** 0

**40.** $\int_a^b f(x)dx \pm \int_a^b g(x)dx$

**41.** $k\int_a^b f(x)dx$

**42.** $-\int_a^b f(x)dx$

**43.** $\geq \int_a^b g(x)dx$

**44.** $-\int_a^b f(x)dx$

**45.** $\geq 0$

**46.** $\int_a^c f(x)dx$

**47.** $\int_a^b f(x)dx$

**48.** $\int_1^6 f(x)dx$, $\int_1^3 f(x)dx$, 7

**49.** Evaluate $\int_0^3 \left(3x^2 - \frac{x}{2} + 5\right) dx$.

*Solution.*

$$\int_0^3 \left(3x^2 - \frac{x}{2} + 5\right) dx = 3\int_0^3 x^2 dx - \frac{1}{2} \underline{\qquad\qquad} + \int_0^3 5\, dx$$

$$= 3(\underline{\qquad\qquad}) - \frac{1}{2}\left(\frac{3^2}{2} - \frac{0^2}{2}\right) + \underline{\qquad\qquad} = 27 - \frac{9}{4} + \underline{\qquad\qquad} = \frac{159}{4}.$$

## 4.5 THE MEAN VALUE AND FUNDAMENTAL THEOREMS

**OBJECTIVE A:** Calculate the average value of a given continuous function $y = f(x)$ over a specified interval $a \le x \le b$.

**50.** Find the average value of $f(x) = x^2 - 4x + 3$ over the interval [0, 2].

*Solution.* $av(f) = \frac{1}{2-0} \underline{\qquad\qquad}$

$$= \frac{1}{2}\left[\int_0^2 x^2 dx - \underline{\qquad\qquad} + \underline{\qquad\qquad}\right]$$

$$= \frac{1}{2}\left[\left(\frac{2^3}{3}\right) - 4(\underline{\qquad\qquad}) + 3(\underline{\qquad\qquad})\right]$$

$$= \frac{1}{2}\left(\frac{8}{3} - \underline{\qquad\qquad} + 6\right) = \frac{1}{3}.$$

**51.** If $f$ is continuous on $[a, b]$, then the Mean Value Theorem for Definite Integrals guarantees that there is a point $c$ in $[a, b]$ such that ____________________.

**52.** To find the point or points in the interval [0, 2] for which the function $f(x) = x^2 - 4x + 3$ in Problem 50 assumes its average value, we must solve the equation ____________________. Using the quadratic formula, the solution is $x = \frac{4 \pm \sqrt{16 - \frac{32}{3}}}{2} = 2 \pm \frac{1}{2}\sqrt{\frac{16}{3}} = 2 \pm \frac{2}{\sqrt{3}} = \frac{6 \pm 2\sqrt{3}}{3}$. The only solution in the interval [0, 2] gives the point $c =$ ____________________ $\approx 0.85$.

**OBJECTIVE B:** Use the Fundamental Theorem, Part 1 to calculate the derivative of an integral $\int_0^{v(x)} f(t)dt$ with respect to $x$. Assume that the integrand $f$ is continuous and that $v$ is a differentiable function of $x$.

**53.** The Fundamental Theorem of Calculus, Part 1 concerns the integral $F(x) = \int_a^x f(t)dt$, where $f$ is continuous on $[a, b]$. This theorem says that $F$ is ________________ at every point $x$ in $[a, b]$ and $F'(x) =$ ________________.

**54.** If $F(x) = \int_1^x \left(t^5 - 2t^3 + 1\right)^4 dt$, then we may find $F'(x)$ by replacing $t$ by $x$ in the integrand. Thus $F'(x) =$ ____________________.

---

**49.** $\int_0^3 x\, dx$, $\frac{3^3}{3} - \frac{0^3}{3}$, $5(3-0)$, 15

**50.** $\int_0^2 \left(x^2 - 4x + 3\right) dx$, $4\int_0^2 x\, dx$, $\int_0^2 3\, dx$, $\frac{2^2}{2}$, 2, 8

**51.** $\frac{1}{b-a}\int_a^b f(x)dx = f(c)$

**52.** $x^2 - 4x + 3 = \frac{1}{3}$ or, equivalently, $x^2 - 4x + \frac{8}{3} = 0$, $\frac{6 - 2\sqrt{3}}{3}$

**53.** differentiable, $f(x)$

**54.** $\left(x^5 - 2x^3 + 1\right)^4$

**55.** Let $F(x)=\int_0^{x^2}\sqrt{1+t}\,dt$. If $u=x^2$, then by the chain rule, $\dfrac{dF}{dx}=\dfrac{dF}{du}\cdot$ ________. Now, $\dfrac{dF}{du}=\dfrac{d}{du}\int_0^{u}\sqrt{1+t}\,dt=$ __________. Thus, $F'(x)=\sqrt{1+u}\cdot$ ________ $=$ __________.

**OBJECTIVE C:** Evaluate definite integrals of elementary continuous functions, using the Fundamental Theorem, Part 2.

**56.** The Fundamental Theorem of Calculus, Part 2 gives a rule for calculating the definite integral $\int_a^b f(x)dx$ of a continuous function. The rule states that you must first find an ____________ $F$ of $f$. That is, the relationship between $F$ and $f$ is __________. Next, calculate the number $F(b)-$ ________.

This computation gives $\int_a^b f(x)dx=$ __________.

**57.** The notation $F(x)]_d^c$ means ____________.

**58.** Find $\int_{-1}^{2}\left(x^3-2x+5\right)dx$.

$$\int_{-1}^{2}\left(x^3-2x+5\right)dx = \underline{\qquad\qquad\qquad}]_{-1}^{2}$$
$$=\left(\frac{1}{4}\left(2^4\right)-2^2+5\cdot 2\right)-(\underline{\qquad\qquad})$$
$$=10-(\underline{\qquad\qquad})=\underline{\qquad\qquad}.$$

**59.** Find $\int_0^{\pi/4}\left(\sin t+\cos t-\dfrac{t}{2}\right)dt$.

$$\int_0^{\pi/4}\left(\sin t+\cos t-\frac{t}{2}\right)dt=\underline{\qquad\qquad\qquad}]_0^{\pi/4}$$
$$=\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-\frac{\pi^2}{64}\right)-(\underline{\qquad\qquad\qquad})$$
$$=\underline{\qquad\qquad}\approx 0.84579.$$

---

**55.** $\dfrac{du}{dx}$, $\sqrt{1+u}$, $2x$, $2x\sqrt{1+x^2}$

**56.** antiderivative, $F'(x)=f(x)$, $F(a)$, $F(b)-F(a)$

**57.** $F(c)-F(d)$

**58.** $\frac{1}{4}x^4-x^2+5x$, $\frac{1}{4}(-1)^4-(-1)^2+5(-1)$, $-\frac{23}{4}$, $\frac{63}{4}$

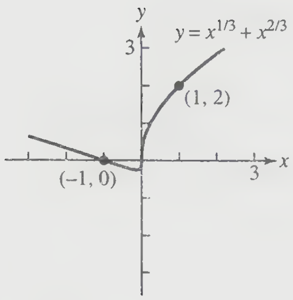
**59.** $-\cos t+\sin t-\dfrac{t^2}{4}$, $-1+0-0$, $1-\dfrac{\pi^2}{64}$

**60.** Find the area of the region between the $x$-axis and the curve $y = x^{1/3} + x^{2/3}$, $-1 \le x \le 1$.

*Solution.* The graph of $y = f(x)$ is shown below. The zeros exist at $x = \underline{\qquad\qquad}$ and $x = \underline{\qquad\qquad}$. Thus, Area $= -\int_{-1}^{0}\left(x^{1/3} + x^{2/3}\right)dx + \underline{\qquad\qquad\qquad}$

$$= -\left(\frac{3}{4}x^{4/3} + \underline{\qquad\qquad}\right]_{-1}^{0} + \left(\frac{3}{4}x^{4/3} + \frac{3}{5}x^{5/3}\right]_{0}^{1}$$

$$= -\left[(0+0) - \left(\frac{3}{4} - \frac{3}{5}\right)\right] + \left[\left(\frac{3}{4} + \frac{3}{5}\right) - (0+0)\right] = \underline{\qquad\qquad}.$$




**61.** To find the average value of $y = \sin x - \cos x$ over $0 \le x \le \frac{\pi}{4}$ we have Av. val. of $y$ on

$$\left[0, \frac{\pi}{4}\right] = \frac{1}{\frac{\pi}{4} - 0}\int_{\underline{\quad}}^{\underline{\quad}}(\underline{\qquad\qquad\qquad})\,dx$$

$$= \frac{4}{\pi}(\underline{\qquad\qquad\qquad})\Big]_{\underline{\quad}}^{\underline{\quad}}$$

$$= \frac{4}{\pi}\left[\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - (\underline{\qquad\qquad}\right)\right] = \frac{4}{\pi}(\underline{\qquad\qquad}) \approx -0.52739.$$

## 4.6 SUBSTITUTION IN DEFINITE INTEGRALS

**OBJECTIVE A:** Evaluate definite integrals using the substitution method of integration.

**62.** Find $\int_{0}^{\pi/4}(3 - \sin 2t)^{1/3}\cos 2t\,dt$.

*Solution.* From Problem 21 of this chapter, the indefinite integral $\int(3 - \sin 2t)^{1/3}\cos 2t\,dt = \underline{\qquad\qquad}$.

Therefore, the definite integral is given by $\int_{0}^{\pi/4}(3 - \sin 2t)^{1/3}\cos 2t\,dt = \underline{\qquad\qquad\qquad}\Big]_{0}^{\pi/4}$

$$= -\frac{3}{8}(\underline{\qquad\qquad})^{4/3} + \frac{3}{8}(\underline{\qquad\qquad})^{4/3}$$

$$= \underline{\qquad\qquad} \approx 0.67759.$$

---

**60.** $-1, 0, \int_{0}^{1}\left(x^{1/3} + x^{2/3}\right)dx, \frac{3}{5}x^{5/3}, \frac{3}{2}$

**61.** $\int_{0}^{\pi/4}(\sin x - \cos x)\,dx, (-\cos x - \sin x)]_{0}^{\pi/4}, (-1-0), \left(1 - \sqrt{2}\right)$

**62.** $-\frac{3}{8}(3 - \sin 2t)^{4/3} + C, -\frac{3}{8}(3 - \sin 2t)^{4/3}, 3-1, 3-0, \frac{3}{8}\left(3^{4/3} - 2^{4/3}\right)$

**63.** If $u = g(x)$ then $du =$ ____________ and $\int_a^b f(g(x)) \cdot g'(x)\,dx =$ __________________ .

**64.** $\int_0^{\pi/6} \frac{\cos x\,dx}{\sqrt{1-\sin x}}$. Let $u = g(x) = \sin x$. Then $du = g'(x)\,dx =$ ____________ . Also, $g(0) = 0$ and

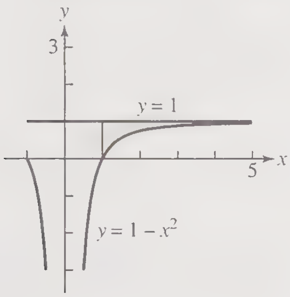
$g\left(\frac{\pi}{6}\right) =$ ____________ . Thus, $\int_0^{\pi/6} \frac{\cos x\,dx}{\sqrt{1-\sin x}} = \int_{\_\_\_}^{\_\_\_} \frac{du}{\sqrt{1-u}} =$ ____________ $= -2($__________$) = 2 - \sqrt{2}$.

**65.** $\int_0^1 \frac{x\,dx}{\sqrt{4-x^2}}$. Let $u = 4 - x^2$. Then, $\int_0^1 \frac{x\,dx}{\sqrt{4-x^2}} = -\frac{1}{2}\int_{\_\_\_}^{\_\_\_} \frac{du}{\sqrt{u}} =$ ____________ $= -\sqrt{3} + 2$.

**OBJECTIVE B:** Find the area bounded by two given continuous curves $y = f(x)$ and $y = g(x)$ over an interval $a \le x \le b$. It may be required to calculate the endpoints $a$ and $b$.

**66.** Find the area between the curves $y = 1$ and $y = 1 - x^{-2}$ for $1 \le x \le 4$.
*Solution.*

STEP 1: The desired region is shown in the figure below.




STEP 2: The limits of integration are already given: $a =$ __________ and $b =$ __________ .

STEP 3: $f(x) - g(x) =$ __________________ $=$ ______________ (simplified).

STEP 4: The area is given by
$A =$ __________________
$=$ ____________ $\Big]_1^4 = -\frac{1}{4} - ($____________$)$
$=$ ______________ .

---

**63.** $g'(x)\,dx$, $\int_{g(a)}^{g(b)} f(u)\,du$

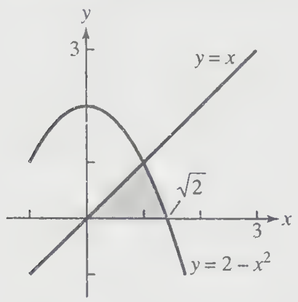
**64.** $\cos x\,dx$, $\frac{1}{2}$, $\int_0^{1/2}$, $-2\sqrt{1-u}\Big]_0^{1/2}$, $\frac{1}{\sqrt{2}} - 1$

**65.** $\int_4^3$, $-\sqrt{u}\Big]_4^3$

**66.** $1, 4, 1-\left(1-x^{-2}\right), x^{-2}, \int_1^4 x^{-2}dx, -x^{-1}, -1, \frac{3}{4}$

**67.** Find the area of the region in the first quadrant bounded above by the curves $y = x$ and $y = 2 - x^2$ and below by the $x$-axis.
*Solution.*

STEP 1: The desired region is shown in the figure at the right. Notice the region has boundaries with changing formulas.




STEP 2: The two curves $y = x$ and $y = 2 - x^2$ intersect when $x = 2 - x^2$ or $x^2 + x - 2 = 0$. thus $x =$ ________ or $x =$ ________. Since $x \geq 0$ in the first quadrant we must pick $x =$ ________. Thus we partition the region into two subregions and sum the integrations over the intervals [0, 1] and $\left[1, \sqrt{2}\right]$.

STEP 3: For the interval [0,1]: $f(x) - g(x) =$ ____________. For the interval $\left[1, \sqrt{2}\right]$: $f(x) - g(x) =$ ____________.

STEP 4: The desired area is given by

$$A = \int_0^1 \_\_\_\_\_\_ \, dx + \int_1^{\_\_\_} \left(2 - x^2\right) dx$$

$$= \_\_\_\_\_\_ \Big]_0^1 + \_\_\_\_\_\_ \Big]_1^{\sqrt{2}}$$

$$= \frac{1}{2} + \left(2\sqrt{2} - \frac{1}{3}\left(\sqrt{2}\right)^3\right) - (\_\_\_\_\_\_) = \frac{4\sqrt{2}}{3} - \frac{7}{6} \approx 0.719.$$

## 4.7 NUMERICAL INTEGRATION

**OBJECTIVE A:** Approximate a given definite integral by using the trapezoidal rule with a specified number $n$ of subintervals. Estimate the error in this approximation.

**68.** Let $y = f(x)$ be defined and continuous over the interval $a \leq x \leq b$. Divide the interval $[a, b]$ into $n$ subintervals, each of length $h = \frac{(b-a)}{n}$, by inserting the points $x_1 = a + h$, $x_2 = a + 2h$, $\ldots$, $x_{n-1} = a + (n-1)h$. Set $x_0 = a$ and $x_n = b$ for convenience in notation. Define $y_k = f(x_k)$ for each $k = 0, 1, 2, 3, \ldots, n$. Then the trapezoidal approximation for the definite integral is $\int_a^b f(x)\,dx \approx$ ____________. The error estimate is $E_T =$ ____________, where $M$ is any upper bound for the values of ____________ on $[a, b]$.

---

**67.** $1, -2, 1, x - 0, \left(2 - x^2\right) - 0, x, \int_1^{\sqrt{2}}, \frac{1}{2}x^2, 2x - \frac{1}{3}x^3, 2 - \frac{1}{3}$

**68.** $\frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right), \frac{b-a}{12}h^2 M, |f''|$

**69.** Use the trapezoidal rule to approximate $\int_0^1 \sqrt{1+x^2}\,dx$, $n = 5$.

*Solution.* Here, $h = \frac{1-0}{5} =$ ________, $x_0 =$ ________, $x_5 =$ ________. The subdivision points are $x_1 = \frac{1}{5}$, $x_2 =$ ________, $x_3 =$ ________, $x_4 =$ ________. The corresponding function values are computed as, $y_0 = \sqrt{1+0^2} = 1$, $y_1 = \sqrt{1+\left(\frac{1}{5}\right)^2} =$ ________ $\approx$ ________

$y_2 = \sqrt{1+\left(\frac{2}{5}\right)^2} =$ ________ $\approx$ ________, $y_3 \approx$ ________, $y_4 \approx$ ________, and $y_5 \approx$ ________.

Therefore, the trapezoidal approximation is

$T = \frac{1}{2(5)} \cdot (y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + y_5)$, or

$T = \frac{1}{10} \cdot (1 + 2.03960 +$ ________ $+$ ________ $+$ ________ $+$ ________ $) = \frac{1}{10} \cdot ($ ________ $) =$ ________.

Therefore, $\int_0^1 \sqrt{1+x^2}\,dx \approx$ ________.

**70.** To estimate the error in the approximation of Problem 69, let $f(x) = \sqrt{1+x^2}$. Then, $f'(x) =$ ________ and $f''(x) =$ ________. Therefore, for $0 \le x \le 1$, we see that $|f''(x)| = \frac{1}{(1+x^2)^{3/2}} <$ ________. Thus the error $E_T$ satisfies $|E_T| \le \frac{b-a}{12} h^2 M =$ ________ $\cdot$ ________ $\cdot$ ________ $\approx$ ________. Then,

$\int_0^1 \sqrt{1+x^2}\,dx = 1.15015 \pm |E_T|$, or ________ $\le \int_0^1 \sqrt{1+x^2}\,dx \le$ ________. (The value of the integral to five decimal places is 1.14779, so the error is about $\frac{3}{10}$ of one percent.)

**71.** How many subintervals are required to obtain $\int_0^1 \sqrt{1+x^2}\,dx$ to 5 decimal places of accuracy by the trapezoidal rule?

*Solution.* From Problem 70, $|E_T| \le \frac{1}{12} h^2 M \le \frac{1}{12} h^2$. To obtain 5-place accuracy, we need $|E_T| < 5 \cdot 10^{-6}$. Thus, $\frac{1}{12} h^2 < 5 \cdot 10^{-6}$ implies $h^2 <$ ________ or, since $h = \frac{b-a}{n} =$ ________, $n^2 >$ ________. Then $n >$ ________ $\approx$ ________, and choosing $n =$ ________ as the number of subintervals ensures 5-place accuracy. (This is only an upper estimate: fewer subintervals may work, but there are no guarantees.)

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**69.** $\frac{1}{5}$, 0, 1, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{\sqrt{26}}{5}$, 1.01980, $\frac{\sqrt{29}}{5}$, 1.07703, 1.16619, 1.28062, 1.41421, 2.15407, 2.33238, 2.56124, 1.41421, 11.50150, 1.15015, 1.15015

**70.** $x(1+x^2)^{-1/2}$, $(1+x^2)^{-3/2}$, 1, $\frac{1}{12}$, $\frac{1}{25}$, 1, 0.00333, 1.14682, 1.15348

**71.** $60 \cdot 10^{-6}$, $\frac{1}{n}$, $\frac{1}{60} \cdot 10^6$, $\frac{1}{2\sqrt{15}} 10^3$, 129, 130

**OBJECTIVE B:** Approximate a given definite integral by use of Simpson's rule with a specified even number $n$ of subintervals. Estimate the error in this approximation.

**72.** Let $y = f(x)$ be defined and continuous over the interval $a \le x \le b$. Divide the interval $[a, b]$ into $n$ subintervals, where $n$ is an *even* number, each of length $h = \frac{b-a}{n}$, using the points $x_0 = a$, $x_1 = a + h$, $x_2 = 1 + 2h, \ldots, x_n = a + nh = b$. Define $y = f(x_k)$ for each $k = 0, 1, 2, \ldots, n$. Then the Simpson approximation for the definite integral is $\int_a^b f(x)\,dx \approx$ ________________. The error estimate is $|E_S| \le$ __________ where $M$ is any upper bound on the values of __________ on $[a, b]$.

**73.** approximate $\int_0^1 \sqrt{1+x^2}\,dx$ by Simpson's rule with $n = 6$.

*Solution.* Here $h =$ ______, $x_0 =$ ______, and $x_6 =$ ______. The subdivision points are, $x_1 = \frac{1}{6}$, $x_2 =$ ______, $x_3 =$ ______, $x_4 =$ ______, and $x_5 =$ ______. The corresponding function values are $y_0 = 1$, $y_1 = \sqrt{1+\left(\frac{1}{6}\right)^2} =$ ______ $\approx$ ______, $y_2 \approx$ ______, $y_3 \approx$ ______, $y_4 \approx$ ______, $y_5 \approx$ ______, and $y_6 =$ ______ $\approx$ ______. Therefore, the Simpson approximation is,

$$S = \frac{1}{6 \cdot 3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$$
$$= \frac{1}{18}(1 + 4.05518 + \_\_\_ + \_\_\_ + \_\_\_ + \_\_\_ + \_\_\_)$$
$$= \frac{1}{18}(\_\_\_) = \_\_\_.$$

Compare the answer with that found in Problem 69 where we employed the trapezoidal rule.

**74.** To estimate the error in the approximation in Problem 73, let $f(x) = \sqrt{1+x^2}$. Then, from Problem 70, $f''(x) = (1+x^2)^{-3/2}$, so that $f^{(3)}(x) =$ __________ and $f^{(4)}(x) =$ __________. Now,

$\frac{3(4x^2-1)}{(1+x^2)^{7/2}} < \frac{12x^2}{(1+x^2)^{7/2}} < \frac{12x^2}{1+x^2} < 12$ for $0 \le x \le 1$. Thus, the error $E_S$ satisfies

$|E_S| = \frac{b-a}{180} h^4 M =$ ____ $\cdot$ ____ $\cdot$ ____ $\approx$ ______. Therefore, we observe that Simpson's rule, with $n = 6$, provides the value of the integral to at least 3 decimal places of accuracy; with $n = 10$ we will obtain at least 4 decimal place accuracy, according to our error estimates.

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**72.** $\frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$, $\frac{b-a}{180}h^4 M$, $|f^{(4)}|$

**73.** $\frac{1}{6}$, 0, 1, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$, $\frac{\sqrt{37}}{6}$, 1.01379, 1.05409, 1.11803, 1.20185, 1.30171, $\sqrt{2}$, 1.41421, 2.10819, 4.47212, 2.40370, 5.20684, 1.41421, 20.66021, 1.14779

**74.** $-3x(1+x^2)^{-5/2}$, $3(4x^2-1)(1+x^2)^{-7/2}$, $\frac{1}{180}$, $\frac{1}{6^4}$, 12, 0.00005

## CHAPTER 4 SELF-TEST

1. Find an antiderivative of the following functions.

(a) $7-4x+5x^2$ (b) $\sin\frac{x}{3}+\frac{1}{\sqrt{x}}+x^{1/3}$

2. Find the following indefinite integrals.

(a) $\int (x-1)(2+x)\,dx$ (b) $\int \sqrt{2x-1}\,dx$

(c) $\int x^2\left(5-3x^3\right)^{-1/2}dx$ (d) $\int x^{-1/2}\sin\left(\sqrt{x}-3\right)dx$

3. Solve the initial value problems.

(a) $\frac{dy}{dx}=\sec^2 x,\ y=3$ when $x=0$ (b) $\frac{dy}{dx}=\frac{x^3+1}{x^3},\ y=\frac{7}{2}$ when $x=1$

4. Approximate the area under the curve $y=x^2-2x+4$ between $x=1$ and $x=4$ by summing $n=6$ inscribed rectangles of uniform width.

5. Find the numerical values of each of the following.

(a) $\sum_{k=1}^{4}\frac{1}{2k}$ (b) $\sum_{n=1}^{5} n(n-3)$ (c) $\sum_{k=5}^{6}(2k-1)$

6. Find the area under the curve $y=x\sqrt{x^2+1}$, above the $x$-axis, between $x=1$ and $x=4$.

7. Evaluate the definite integrals.

(a) $\int_1^4 \frac{(x-2)^2}{\sqrt{x}}\,dx$ (b) $\int_{-2}^{0} x^2(4-x)\,dx$

8. Find $\frac{d}{dx}\int_0^{1-x^2}\sqrt[3]{t^2+1}\,dt$.

9. Suppose $g$ and $h$ are continuous and that $\int_0^4 g(x)\,dx=3$, $\int_1^4 g(x)\,dx=-5$, $\int_1^0 h(x)\,dx=2$. Find $\int_0^1 [2g(x)-h(x)]\,dx$.

10. Find the area of the planar region bounded by the curves $y=x^3+1$ and $y=x^2+x$.

11. A train leaving a railroad station has an acceleration of $a=0.5+0.02t$ ft / sec$^2$. How far will the train move in the first 20 sec of motion? What is its velocity after 20 seconds?

12. Find the average value of the function $y=x^2-x+1$ over the interval $0\le x\le 2$.

13. Use the trapezoidal rule with $n=4$ to approximate $\int_0^1 \sqrt{1+x^3}\,dx$.

14. Use Simpson's rule with $n=6$ to approximate $\int_1^2 \frac{dx}{x}$. Estimate the error in your approximation.

15. Compute $\int_0^{\pi} f(x)\,dx$ where $f(x) = \begin{cases} \sin x, & 0 \le x < \frac{\pi}{2} \\ \pi x, & \frac{\pi}{2} \le x \le \pi. \end{cases}$

## SOLUTIONS TO CHAPTER 4 SELF-TEST

1. (a) $7x - 2x^2 + \frac{5}{3}x^3$ (b) $-3\cos\frac{x}{3} + 2x^{1/2} + \frac{3}{4}x^{4/3}$

2. (a) $\int (x-1)(2+x)\,dx = \int \left(x^2 + x - 2\right)dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + C$

(b) $\int \sqrt{2x-1}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{3}(2x-1)^{3/2} + C\,(u = 2x-1)$

(c) $\int x^2\left(5-3x^3\right)^{-1/2} dx = -\frac{1}{9}\int u^{-1/2}\,du = -\frac{2}{9}\left(5-3x^3\right)^{1/2} + C\left(u = 5-3x^3\right)$

(d) $\int x^{-1/2}\sin\left(\sqrt{x}-3\right)dx = 2\int \sin u\,du = -2\cos\left(\sqrt{x-3}\right) + C\left(u = \sqrt{x}-3\right)$

3. (a) $\frac{dy}{dx} = \sec^2 x$, so $y = \tan x + C$. Since $y = 3$ when $x = 0$, $3 = \tan(0) + C$ or $C = 3$. Hence, the solution of the initial value problem is $y = \tan x + 3$.

(b) $\frac{dy}{dx} = 1 + \frac{1}{x^3}$ has the general solution $y = x - \frac{1}{2}x^{-2} + C$. Substituting $y = \frac{7}{2}$ and $x = 1$ gives $\frac{7}{2} = 1 - \frac{1}{2} + C$ or $C = 3$. Hence, $y = x - \frac{1}{2x^2} + 3$.

4. The partition points are $x_0 = 1$, $x_1 = \frac{3}{2}$, $x_2 = 2$, $x_3 = \frac{5}{2}$, $x_4 = 3$, $x_5 = \frac{7}{2}$, and $x_6 = 4$. Since $\frac{dy}{dx} > 0$ on the interval $1 \le x \le 4$, the curve is increasing, so the altitude of each rectangle is its left edge. Thus, the areas of the inscribed rectangles for $y = f(x)$ are,

$f(1)\Delta x = 3 \cdot \frac{1}{2} = \frac{3}{2}$

$f\left(\frac{3}{2}\right)\Delta x = \frac{13}{4} \cdot \frac{1}{2} = \frac{13}{8}$

$f(2)\Delta x = 4 \cdot \frac{1}{2} = 2$

$f\left(\frac{5}{2}\right)\Delta x = \frac{21}{4} \cdot \frac{1}{2} = \frac{21}{8}$

$f(3)\Delta x = 7 \cdot \frac{1}{2} = \frac{7}{2}$

$f\left(\frac{7}{2}\right)\Delta x = \frac{37}{4} \cdot \frac{1}{2} = \frac{37}{8}$

Sum $= \frac{127}{8} = 15.875$, approximate area

5. (a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{25}{24}$

(b) $1(-2) + 2(-1) + 3(0) + 4(1) + 5(2) = 10$

(c) $(2 \cdot 5 - 1) + (2 \cdot 6 - 1) = 9 + 11 = 20$

6. $\int_1^4 x\sqrt{x^2+1}\,dx = \frac{1}{3}\left(x^2+1\right)^{3/2}\Big|_1^4 = \frac{1}{3}\left(17^{3/2}-2^{3/2}\right) \approx 22.42146$

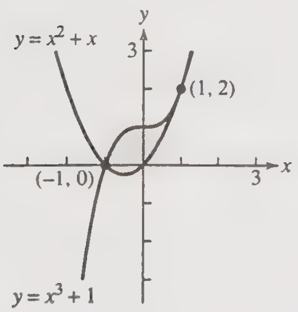
7. (a) $$\int_1^4 \frac{(x-2)^2}{\sqrt{x}}\,dx = \int_1^4 \frac{x^2-4x+4}{\sqrt{x}}\,dx$$
$$= \int_1^4 \left(x^{3/2}-4x^{1/2}+4x^{-1/2}\right)dx$$
$$= \frac{2}{5}x^{5/2}-\frac{8}{3}x^{3/2}+8x^{1/2}\Big|_1^4$$
$$= \left(\frac{2}{5}(32)-\frac{8}{3}(8)+16\right)-\left(\frac{2}{5}-\frac{8}{3}+8\right) \approx 1.73333$$

(b) $\int_{-2}^0 x^2(4-x)\,dx = \int_{-2}^0 \left(4x^2-x^3\right)dx = \frac{4}{3}x^3-\frac{1}{4}x^4\Big|_{-2}^0 = -\left[\frac{4}{3}(-2)^3-\frac{1}{4}(-2)^4\right] = \frac{44}{3} \approx 14.66667$

8. $\frac{d}{dx}\int_0^{1-x^2} \sqrt[3]{t^2+1}\,dt = \sqrt[3]{\left(1-x^2\right)^2+1}\cdot\frac{d}{dx}\left(1-x^2\right) = -2x\sqrt[3]{x^4-2x^2+2}$

9. $$\int_0^1 [2g(x)-h(x)]\,dx = 2\int_0^1 g(x)\,dx - \int_0^1 h(x)\,dx$$
$$= 2\left[\int_0^1 g(x)\,dx + \int_1^4 g(x)\,dx - \int_1^4 g(x)\,dx\right] - \left[-\int_1^0 h(x)\,dx\right]$$
$$= 2\left[\int_0^4 g(x)\,dx - \int_1^4 g(x)\,dx\right] + \int_1^0 h(x)\,dx$$
$$= 2[3-(-5)]+2 = 18$$

10. The graph of $y = x^3+1$ crosses the $x$-axis at $(-1, 0)$ and so does the graph of $y = x^2+x$. (See the figure below.) Solving for the other point of intersection of the two curves, $x^3+1 = x^2+x$ or $x^3-x^2-x+1=0$. Since $x=-1$ is a root, by division $x^3-x^2-x+1 = (x+1)\left(x^2-2x+1\right) = 0$ or $(x+1)(x-1)^2 = 0$. Thus, the other point of intersection is $(1, 2)$. The area between the curves is then given by
$$A = \int_{-1}^1 \left[\left(x^3+1\right)-\left(x^2+x\right)\right]dx = \left(\frac{1}{4}x^4+x-\frac{1}{3}x^3-\frac{1}{2}x^2\right)\Big]_{-1}^1 = \frac{4}{3} \text{ square units.}$$




11. Since $v = \int a\,dt$ we have $v = 0.5t + 0.01t^2 + C_1$. At $t = 0$ the train is at rest, so $v = 0$; hence $C_1 = 0$. Next, $s = \int v\,dt$ or $s = 0.25t^2 + \frac{1}{300}t^3 + C_2$. At $t=0$, $s=0$ so that $C_2 = 0$. Thus, when $t = 20$ seconds, $s = \frac{1}{4}(400) + \frac{1}{300}(8000) = 126\frac{2}{3}$ feet, the distance traveled by the train in the first 20 seconds. Its velocity at that time is $v = (0.5)(20)+(0.01)(400) = 14$ ft/sec.

**12.** The average value is $\frac{1}{2-0}\int_0^2 \left(x^2 - x + 1\right)dx = \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + x\right]_0^2 = \frac{4}{3}$.

**13.** Subdivision points are $x_0 = 0,\ x_1 = \frac{1}{4},\ x_2 = \frac{1}{2},\ x_3 = \frac{3}{4},\ x_4 = 1$, and $h = \frac{(1-0)}{4} = \frac{1}{4}$. Then, $y_0 = \sqrt{1+0} = 1$, $y_1 = \sqrt{1+\left(\frac{1}{64}\right)} \approx 1.00778,\ y_2 = \sqrt{1+\left(\frac{1}{8}\right)} \approx 1.06066,\ y_3 = \sqrt{1+\left(\frac{27}{64}\right)} \approx 1.19242,\ y_4 = \sqrt{1+1} \approx 1.41421$. Thus, $\int_0^1 \sqrt{1+x^3}\,dx \approx T = \frac{1}{8}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right) \approx \frac{1}{8}(8.93593) \approx 1.11699$.

**14.** Subdivision points are $x_0 = 1,\ x_1 = \frac{7}{6},\ x_2 = \frac{4}{3},\ x_3 = \frac{3}{2},\ x_4 = \frac{5}{3},\ x_5 = \frac{11}{6},\ x_6 = 2$, and $h = \frac{1}{6}$. Then, $y_0 = 1$, $y_1 = \frac{6}{7} \approx 0.85714,\ y_2 = \frac{3}{4} = 0.75,\ y_3 = \frac{2}{3} \approx 0.66667,\ y_4 = \frac{3}{5} = 0.6,\ y_5 = \frac{6}{11} \approx 0.54545,\ y_6 = \frac{1}{2} = .5$. Thus, $\int_1^2 \frac{dx}{x} \approx \frac{1}{3\cdot 6}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6\right] \approx \frac{1}{18}(12.47706) \approx 0.69317$. To estimate the error, $f(x) = \frac{1}{x},\ f'(x) = -\frac{1}{x^2},\ f''(x) = \frac{2}{x^3},\ f^{(3)}(x) = \frac{-6}{x^4}$, and $f^{(4)}(x) = \frac{24}{x^5}$. Since $\left|\frac{24}{x^5}\right| \le 24$ on $1 \le x \le 2$, the error satisfies $|E_S| \le \frac{b-a}{180}h^4 M = \frac{1}{180}\cdot\frac{1}{1296}\cdot 24 \approx 0.0001$. Therefore, the approximation is accurate to 3 decimal places.

**15.**

$$\int_0^{\pi} f(x)\,dx = \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^{\pi} \pi x\,dx$$

$$= [-\cos x]_0^{\pi/2} + \left[\frac{1}{2}\pi x^2\right]_{\pi/2}^{\pi} = 1 + \frac{3\pi^3}{8} \approx 12.62735$$

NOTES.