Chapter 3: Applications of Derivatives

3.1 EXTREME VALUES OF FUNCTIONS

OBJECTIVE A: Define the terms local maximum, local minimum, absolute maximum, and absolute minimum value.

1. A function *f* is said to have a local maximum at $x = c$ if ________________ for all *x* in some ______________ *I* about *c*.

2. A function *f* is said to have an ________________ maximum over its domain at $x = c$ if $f(c) \geq f(x)$ for all *x* belonging to the ________________ of *f*.

3. A function *f* is said to have a ________________ minimum over its domain at $x = c$ if $f(c) \leq f(x)$ for all *x* close to *c*.

4. If $f(c) \leq f(x)$ for all *x* in the domain of *f*, then *f* is said to have a ________________ ________________ at $x = c$.

5. Can a local maximum also be an absolute maximum for a function *f*? Can a local minimum also be an absolute minimum?

OBJECTIVE B: Interpret correctly the Local Extreme Values Theorem relating local extrema at an interior point $x = c$ of the domain $a \leq x \leq b$ of a function *f* and its derivative $f'(c)$.

Answer questions 6 - 9 true or false.

6. If $f'(c) = 0$, then *f* has either a local maximum or a local minimum at the interior point $x = c$. (True or False)

7. If *f* has a local minimum at the interior point $x = c$, then $f'(c) = 0$. (True or False)

8. If *f* has a local maximum or local minimum at an endpoint of the interval of definition of the function, then the left-hand (or right-hand) tangent must have slope zero there. (True or False)

9. If *f* has an absolute maximum at an interior point $x = c$ and $f'(c)$ exists as a finite number then $f'(c)$ is necessarily zero. (True or False)

10. If *f* is continuous over the closed interval $a \leq x \leq b$, then every point where *f* has a (local or absolute) maximum or minimum must be an ____________ of the interval, a point where f' ____________ or an ____________ point where f' equals ____________.

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1. $f(c) \geq f(x)$, open interval
2. absolute, domain
3. local
4. absolute minimum
5. Yes to both questions
6. False; it could have a point of inflection
7. False; the derivative may fail to exist
8. False
9. True
10. endpoint, does not exist, interior, 0

OBJECTIVE C: Given a function $y = f(x)$ continuous over a closed interval $a \le x \le b$, find the critical points of f and for each critical point, determine whether the function has a local maximum or local minimum there, or neither. If possible, find the absolute maximum and minimum values of the function on the closed interval.

11. Consider $y = x^{3/2}(x-8)^{-1/2}$ over $10 \le x \le 16$. Now, $y' =$ ____________ $= \frac{1}{2}x^{1/2}(x-8)^{-3/2}$ ____________.
Thus, $x =$ ____________ is the only critical point in the interval $10 \le x \le 16$. Now when $x = 12, y \approx 20.78$. Since $y(11) \approx 21.06$ and $y(13) \approx 20.96$, we conclude that the function has a local ____________ when $x = 12$. Checking the endpoints of the interval [10,16] we determine that $y(10) \approx 22.36$ and $y(16) \approx 22.63$. Thus the absolute maximum of y occurs at $x =$ ____________ and the absolute minimum of y occurs at $x =$ ____________

3.2 THE MEAN VALUE THEOREM AND DIFFERENTIAL EQUATIONS

OBJECTIVE A: Apply Rolle's Theorem to show that a given equation $f(x) = 0$ has exactly one solution in the specified interval $a \le x \le b$.

12. Suppose $y = f(x)$ and its first derivative $f'(x)$ are continuous over $a \le x \le b$. If $f(a)$ and $f(b)$ have opposite signs, then according to the Intermediate Value Theorem there is at least one point c satisfying $a \le c \le b$ and $f(c) =$ __________.

13. Suppose there is another point d satisfying $a \le d \le b$ and $f(d) = 0$. Then, according to Rolle's Theorem, there is a point between c and d for which ____________ is zero. Thus, if $f'(x)$ is different from zero for all values of x between a and b, there is exactly ____________ solution to the equation $f(x) = 0$ in the interval ____________.

14. Consider the equation $x^3 + 2x^2 + 5x - 6 = 0$ for $0 \le x \le 5$. When $x = 0$ the value of the left side is __________; and when $x = 5$, the value is ____________. These values differ in sign. Calculating the derivative, we have $\frac{d}{dx}\left(x^3 + 2x + 5x - 6\right) =$ ________________, and this is always ____________________ for $0 < x < 5$.
Therefore, we conclude from Problems 12 and 13 that there is exactly one solution to the equation somewhere between $x =$ ____________ and $x =$ ____________. We could in fact use Newton's method of Section 3.8 to locate this solution.

OBJECTIVE B: Given a function $y = f(x)$ satisfying the hypotheses of the Mean Value Theorem for $a \le x \le b$, use the theorem to find a number c satisfying the conclusion of the theorem.

15. The hypotheses of the Mean Value Theorem are that f is ____________ over the closed interval $a \le x \le b$ and ______________ over the open interval ______________.

16. The conclusion of the Mean Value Theorem is that there is at least one number c in the open interval __________ satisfying ____________. A geometric interpretation of the conclusion is that the slope of the curve $y = f(x)$ when $x = c$ is the same as the slope of the ____________ joining the endpoints $(a, f(a))$ and ____________ of the curve.

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11. $\frac{3}{2}x^{1/2}(x-8)^{-1/2} - \frac{1}{2}x^{3/2}(x-8)^{-3/2}$, $(2x-24)$, 12, minimum, 16, 12 **12.** 0

13. $f'(x)$, one, $a \le x \le b$

14. -6,194, $3x^2 + 4x + 5$, positive, 0, 5

15. continuous, differentiable, $a < x < b$

16. (a,b), $f(b) - f(a) = f'(c)(b-a)$, chord, $(b, f(b))$

17. Let $f(x) = 3x^2 + 4x - 3$ over $1 \le x \le 3$. Then $f'(x) =$ ____________, so that f and f' satisfy the hypotheses of the Mean Value Theorem. To find a value for c, the equation $f(b) - f(a) = f'(c)(b - a)$ becomes $f(3) - f(1) = f'(c)($____________$)$, or $36 - 4 = 2($____________$)$. Solving for c gives $c =$ ____________.

18. Does the Mean Value Theorem apply to the function $f(x) = |x|$ in the interval $[-2, 1]$?
No, because the derivative $f'(x)$ is not defined for $x =$ ____________ so the function f is not ____________ over the open interval ____________ as required by the hypotheses.

OBJECTIVE C: Know the main consequence of the Mean Value Theorem.

19. If $f'(x) = 0$ for all x in an interval I, then ____________________________________.

20. Functions with the same derivative differ ____________________________________.

OBJECTIVE D: Find all the possible functions with a given derivative. Also find that function whose graph passes through a specified point P.

21. Since $y' = 5x^4 - 2x$ is the derivative of ____________________, we know that $y =$ ______________ for some constant C.

22. If $y = 3$ when $x = -1$ in Problem 21, then the value of C must be ____________________________________.

23. If $y' = \cos 3t$, then $y =$ ____________________________________.

24. If $y' = 3x^4 - x^{-2} + 5$, then $y =$ ____________________________________.

25. If $y' = x^{3/2} + x^{1/2} + \frac{1}{x^2}$, then $y =$ ____________________________________. If the graph of y passes through the point $\left(1, \frac{2}{5}\right)$ then $C =$ ____________________________________.

OBJECTIVE E: Use consequences of the Mean Value Theorem to solve simple differential equations.

26. An object sits at rest at position $x = 0$. Starting at $t = 0$, it is subjected to an acceleration of $\frac{d^2x}{dt^2} = \cos t$. To find x as a function of t, we first note that $\frac{dx}{dt}$ must differ from ____________________ by only a ____________.
So let $\frac{dx}{dt} =$ ____________ $+ C_1$. Since the object is at rest at $t = 0$, $C_1 =$ ______________ and $\frac{dx}{dt} = \sin t$. Then $x(t)$ differs from ____________________ by only a constant, so let $x(t) =$ ____________ $+ C_2$. Since $x(0) =$ ____________, $C_2 =$ ______________ and $x(t) =$ ______________.

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17. $6x + 4$, 2, $6c + 4$, 2

18. 0, differentiable, $(-2, 1)$

19. $f(x)$ is constant for all x in I

20. only by a constant

21. $x^5 - x^2$, $x^5 - x^2 + C$

22. 5

23. $\frac{1}{3}\sin 3t + C$

24. $\frac{3}{5}x^5 + x^{-1} + 5x + C$

25. $\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} - x^{-1} + C$, $\frac{1}{3}$

26. $\sin t$, constant, $\sin t$, 0, $-\cos t$, $-\cos t$, 0, 1, $1 - \cos t$

3.3 THE SHAPE OF A GRAPH

OBJECTIVE A: Use the derivative Test for Increasing and Decreasing to determine the values of x where the graph of y versus x is increasing and where it is decreasing.

27. The function f ______________________ when $f' < 0$.

28. The function f increases when ______________________.

29. The function f increases over a domain D of real numbers if $x_1 < x_2$ implies ______________________.

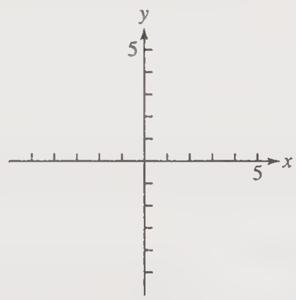
30. Let $y = f(x)$ be a differentiable function of x. When $\frac{dy}{dx}$ has a ______________ value, the graph of y versus x is rising (to the right). In this case it is said that the function f is ______________.

31. When $\frac{dy}{dx} < 0$ the graph of y versus x is ______________ and the function f is ______________.

32. Let $y = \frac{1}{3}x^3 - x^2 + 2$. Then $y' =$ ______________ $= x($______________$)$. The derivative $\frac{dy}{dx}$ is zero when $x =$ ______________ or $x =$ ______________. Thus, the curve y is increasing when $x < 0$, it is decreasing when x satisfies ______________, and it is increasing again when $x >$ ______________. We construct a table of some values for the curve (complete the table):

x	-2	-1	0	1	2	3	4
y							

Sketch the graph in the coordinate system at the right.



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27. decreases

28. $f' > 0$

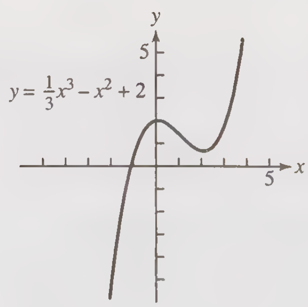
29. $f(x_1) < f(x_2)$

30. positive, increasing

31. falling, decreasing

32. $x^2 - 2x$, $x - 2$, 0, 2, $0 < x < 2$, 2

x	-2	-1	0	1	2	3	4
y	$-\frac{14}{3}$	$\frac{2}{3}$	2	$\frac{4}{3}$	$\frac{2}{3}$	2	$\frac{22}{3}$



OBJECTIVE B: Use the First Derivative Test for Local Extreme Values to identify the local extreme values of a given function $y = f(x)$.

33. Suppose the function $y = f(x)$ has the derivative $f'(x) = x(x+1)(x-2)$. Then the critical points of f are $x = 0$, $x =$ ________, and $x =$ ________. At $x = 0$ the derivative f' changes from __________ to __________ so f has a local maximum value at $x = 0$. At $x = -1$, the derivative f' changes from negative to positive so f has a local __________ value at $x = -1$. At $x = 2$ the function f has a local __________ value because the derivative f' changes from __________ to __________.

34. The function f in Problem 33 is increasing on the intervals __________ and __________, and it is decreasing on the intervals __________ and __________.

35. For the function whose derivative is $f'(x) = x^{-2/3}(2-x)$ the critical points are $x =$ ________ and $x =$ ________. The function f is increasing on the interval __________ and decreasing on the interval __________. Thus the function assumes a local maximum at the critical point $x =$ ________.

OBJECTIVE C: Relate the concavity of a function $y = f(x)$ to the second derivative $\frac{d^2y}{dx^2}$.

36. If the second derivative $\frac{d^2y}{dx^2}$ is positive, the y-curve is concave __________ at that point; if $\frac{d^2y}{dx^2}$ is __________, the curve is concave down at that point.

37. When a curve is concave up at a point, locally the curve lies __________ the tangent line; when it is concave __________, locally the curve lies below the tangent line.

38. A point where the curve changes concavity is called a ____________________, and is characterized by a change in sign of __________.

39. A point of inflection occurs where $\frac{d^2y}{dx^2}$ is _______________ or _______________.

40. Does the condition $\frac{d^2y}{dx^2} = 0$ guarantee a point of inflection? __________

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33. -1, 2, positive, negative, minimum, minimum, negative, positive

34. $(-1,0)$ and $(2,\infty)$, $(-\infty,-1)$ and $(0,2)$

35. 0, 2, $(-\infty,2)$, $(2,\infty)$, 2

36. up, negative

37. above, down

38. point of inflection, $\frac{d^2y}{dx^2}$

39. zero, undefined

40. No, the function $y = x^4$ affords a counterexample at $x = 0$.

OBJECTIVE D: Given a function $y = f(x)$, find the intervals of values of x for which the curve is increasing, decreasing, concave upward, and concave downward. Sketch the curve, showing the points of inflection and the points where the function has local maximum and local minimum values. Use the following procedure to graph $y = f(x)$.

STEP 1: Find y' and y''.

STEP 2: Find where y' is positive, negative, and zero.

STEP 3. Find where y'' is positive, negative, and zero.

STEP 4, Make a summary table, and show the curve's general shape.

STEP 5. Plot specific points and sketch the graph.

41. Consider the function $f(x) = x^4 - 4x^3 + 10$. we follow the five-step strategy in order to sketch the graph of $y = f(x)$.

STEP 1: $\frac{dy}{dx} =$ ________________ and $\frac{d^2y}{dx^2} =$ ________________.

STEP 2: In factored terms, $\frac{dy}{dx} = 4x^2(x-3)$, so the curve is decreasing when x belongs to the interval ____________ and increasing when $x >$ ____________. The slope of the curve is zero when $x =$ __________ or $x =$ __________.

STEP 3: In factored form, $\frac{d^2y}{dx^2} =$ ________________. Thus $\frac{d^2y}{dx^2}$ is negative when x belongs to the interval ______________ and consequently the curve is concave ______________ there. The second derivative is positive when x satisfies ______________ or ______________, and the curve is concave ______________. Therefore, the second derivative changes sign when $x =$ __________ or $x =$ __________ so that these are points of inflection of f.

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41. 1. $4x^3 - 12x^2$, $12x^2 - 24x$

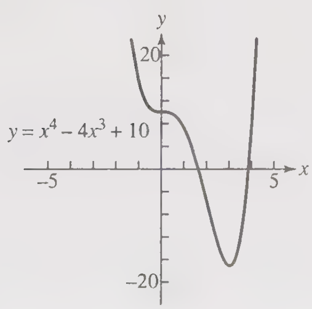
2. $(-\infty, 3)$, 3, 0, 3

3. $12x(x-2)$, $(0, 2)$, down, $x < 0$, $x > 2$, up, 0, 2

4.

x	y	y'	y''	Conclusions
0	10	0	0	point of inflection
1	7	–	–	decreasing, concave down
2	−6	–	0	point of inflection
3	−17	0	+	"Holds water"; min

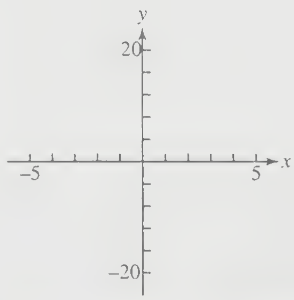
5.



STEP 4: Complete the following table.

x	y	y'	y''	Conclusions
–2	58	–	+	decreasing; concave up
–1	15	–	+	decreasing; concave up
0				
1				
2				
3				
4	10	+	+	increasing; concave up

STEP 5: Sketch a smooth curve of $y = f(x)$ in the given coordinate system to the right.



OBJECTIVE E: Analyze the graph of a given function $y = f(x)$ to investigate the following properties of the curve: (a) symmetry, (b) intercepts, (c) asymptotes, (d) rise and fall, (e) concavity, and (f) end behavior. Using the information you have discovered, sketch the curve.

42. Sketch the graph of $y = x^{-2} + 2x$.
Solution. We follow the five steps as in Problem 41.

1. $y' =$ __________ and $y'' =$ __________ .

2. In fractional form, $y' = \dfrac{\rule{2em}{0.4pt}}{x^3}$, so y' is zero when $x =$ __________ . the curve is decreasing when x belongs to the interval ____________; it is increasing for x satisfying ____________ and ____________.

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42. **1.** $-2x^{-3} + 2,\ 6x^{-4}$ **2.** $2\left(x^3 - 1\right)$, 1, $(0,1)$, $x < 0$, $x > 1$ **3.** positive, up

4. 0, $2x$, $\dfrac{1}{x^2}$

x	y	y'	y''	Conclusions
-1	-1	+	+	increasing; concave up
$-\frac{1}{2}$	3	+	+	increasing, concave up
$\frac{1}{2}$	5	–	+	decreasing; concave up
1	3	0	+	min.; concave up

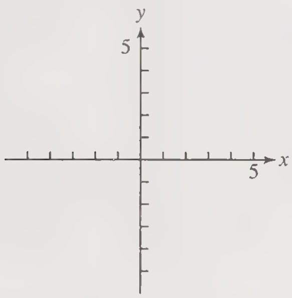
3. $\frac{d^2y}{dx^2}$ is always ______________ so the curve is everywhere concave ______________. Therefore, there are no points of inflection.

4. The curve is discontinuous at $x =$ __________. To identify the end behaviors, note that for large values of $|x|$, the curve is approximately $y \approx$ __________. When x is small, the curve is approximately $y \approx$ __________.

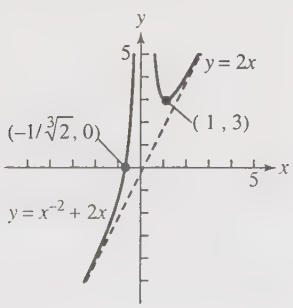
Complete the following table.

x	y	y'	y''	Conclusions
-2	$-\frac{15}{4}$	+	+	increasing; concave up
-1				
$-\frac{1}{2}$				
$\frac{1}{2}$				
1				
2	$\frac{17}{4}$	+	+	increasing; concave up

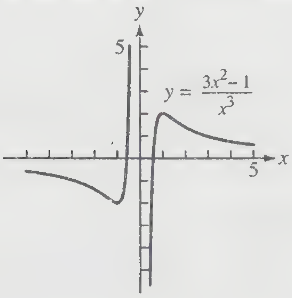
5. Sketch the graph below.



42. 5.



43. Let $y = \dfrac{3x^2 - 1}{x^3}$. Since $-y = \dfrac{3(-x)^2 - 1}{(-x)^3}$, the curve is symmetric about the ____________. Moreover, y is undefined when $x =$ ________, but the graph is defined for all other real values of x. Now, $y = 0$ when $3x^2 - 1 = 0$. Thus, the x-intercepts occur at ____________. Since $x \neq 0$, there are no y-intercepts. Now, $\lim_{x \to 0^-} \dfrac{3x^2 - 1}{x^3} = \lim_{x \to 0^-} \left(\dfrac{3}{x} - \text{\_\_\_\_\_\_\_\_} \right) =$ ____________, and $\lim_{x \to 0^+} \dfrac{3x^2 - 1}{x^3} =$ ____________. Therefore, the line ____________ is a ____________ asymptote. Also, $\lim_{x \to \pm\infty} \dfrac{3x^2 - 1}{x^3} =$ ____________, so the line ____________ is a ____________ asymptote. Finally, $y' = \dfrac{x^3(6x) - (\text{\_\_\_\_\_\_\_\_})3x^2}{x^6} = \dfrac{3(1 - x^2)}{\text{\_\_\_\_\_\_}}$. Thus, $y' = 0$ when $x =$ ________. The derivative is ______________ for $0 < x < 1$ and negative for ____________ and for ____________. A sketch of the graph is shown below.



3.4 GRAPHICAL SOLUTIONS TO DIFFERENTIAL EQUATIONS

OBJECTIVE A: Sketch a set of solution curves for an autonomous differential equation.

44. The equilibrium values for $\dfrac{dy}{dx} = y \cos y$ are ____________, because those are the values where ____________ or ____________ are equal to ____________.

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43. origin, 0, $x = \dfrac{\pm\sqrt{3}}{3}$, $\dfrac{1}{x^3}$, ∞, $-\infty$, $x = 0$, vertical, 0, $y = 0$, horizontal, $3x^2 - 1$, x^4, ± 1, positive, $x > 1$, $x < -1$

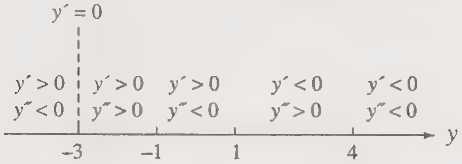
44. 0, $\pm\dfrac{\pi}{2}$, $\pm\dfrac{3\pi}{2}$, $\pm\dfrac{5\pi}{2}$, ..., y, $\cos y$, zero

45. To sketch solution curves for $\frac{dy}{dx} = y^2 - 4$, note first that $\frac{dy}{dx} = y' = 0$ when ____________ or ____________. So these are ____________ values. Next, find y'' using implicit differentiation: $y'' =$ __________ $\cdot\, y' =$ __________. The three values of y for which $y'' = 0$ are __________, __________, and __________. Using this information, draw first the phase line then the set of solution curves.

OBJECTIVE B: Identify stable and unstable equilibria for an autonomous differential equation.

46. If all solution curves except the one through an equilibrium value move away from that value, the equilibrium is considered to be ________________.

47. For a function y of t with the phase line



there is a stable equilibrium at $y =$ _______ because y' is ________________ for values just below ____________ and y' is ________________ for values just above ____________, and there is an unstable equilibrium at $y =$ _______ because $y' = 0$ there, but y' is ________________ for values just above _____.

OBJECTIVE C: Solve application problems involving autonomous differential equations.

48. For a population of wolves in the wild, with population size $P \geq 0$, $\frac{dP}{dt} = -(y-100)(y-200)$. Since $\frac{dP}{dt} = P'$ is positive between ____________ and ____________, and is ____________ elsewhere, any population greater than __________ will eventually settle into a stable equilibrium at ____________, whereas any population less than ____________ will die off. ____________ is an unstable equilibrium.

3.5 MODELING AND OPTIMIZATION

OBJECTIVE: Solve a max−min problem by the following strategy:

STEP 1: Read the problem until you understand it.

STEP 2: Draw a figure, if possible, to illustrate the problem.

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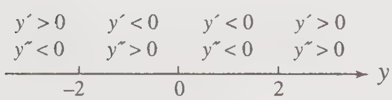
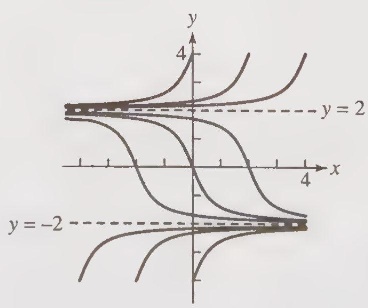
45. $y = 2$, $y = -2$, equilibrium, $2y$, $2y(y^2 - 4)$, $-2, 2, 0$





46. unstable

47. 1, greater than zero, 1, less than zero, 1, −3, greater than zero, −3

48. 100, 200, negative, 100, 200, 100, 100

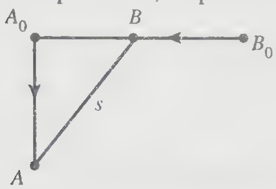
STEP 3: Introduce variables and list every relation in your picture and in the problem as an equation or algebraic expression.

STEP 4: Identify and write an equation for the unknown that is to be a maximum or minimum,

STEP 5: Find and test the critical and endpoints for a possible maximum or minimum. Use what you know about the shape of the function's graph and the physics of the problem.

49. At 9:00 A.M., ship B was 65 miles due east of ship A. Ship B was then sailing due west at 10 miles per hour, and ship A was sailing due south at 15 miles per hour. If they continue to follow their respective courses, when will they be nearest one another and how near?

Solution. Let A_0 and B_0 denote the original positions of the ships at 9:00 A.M., and let A and B denote their new positions, respectively, at t hours later. This is pictured in the figure below.



Let s denote the distance between A and B. The problem is to minimize ____________ and to find the time when its minimum occurs. Since (rate)(time) = distance, the distance covered by ship A in t hours is ____________ miles, and by ship B ____________ miles. The original distance between ships A and B is given as 65 miles, so the distance between the original position A_0 and ship B after t hours is ____________.

Fill this information into the figure and then calculate the square of the distance: $s^2 =$ ____________.

Differentiation of both sides of this equation with respect to t gives $2s\frac{ds}{dt} =$ ____________. Thus,

$\frac{ds}{dt} = 0$ when $30(15t) - 20(65 - 10t)$ equals 0, or $t =$ ________ hours. Simplifying $\frac{ds}{dt}$ algebraically, we see that $\frac{ds}{dt} =$ ____________. Thus, $\frac{ds}{dt}$ is ____________ when $t < 2$ and ____________ when $t > 2$. Therefore, a relative ____________ distance occurs for s at $t = 2$ hours. Solving for the distance s after two hours, we find $s^2 = (30)^2 + (\_\_\_\_\_\_)^2$ or $s =$ ________ miles, the distance the ships are apart at 11:00 A.M. when they are nearest each other.

50. A company's cost function is $C(x) = 10x + 3$, and its revenue function is $R(x) = 50x - 0.5x^2$, both in the thousands of dollars per thousand items. Find the company's maximum profit.

Solution. If $P(x)$ denotes the profit function, then $P(x) = R(x) -$ ____________ $=$ ____________. The maximum profit occurs when $P'(x) =$ ____________, so $P'(x) =$ ____________ $= 0$. Thus, $x =$ ____________ thousand items. Since $P''(x) =$ ____________ is always negative, this yields a maximum profit of $P(40) =$ ____________ thousand dollars.

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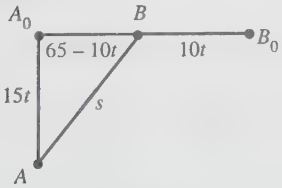
49. s, $15t$, $10t$, $65 - 10t$, $(15t)^2 + (65 - 10t)^2$, $30(15t) - 20(65 - 10t)$, 2, $\frac{325t - 650}{s}$, negative, positive, minimum, 45, $15\sqrt{13}$



50. $C(x)$, $40x - 0.5x^2 - 3$, 0, $40 - x$, 40, -1, 797

51. It is known that the population P for the fur-bearing snowshoe hare in the Hudson Bay area will grow to $f(P) = -0.025P^2 + 4P$ in one year. If they "harvest" the amount $f(P) - P$ so the initial population is not depleted, then the harvest is said to be "sustained." Find the population at which the maximum sustainable harvest occurs, and find the maximum sustainable harvest for the snowshoe hare. Assume P is measured in thousands.

Solution. The harvest function $H(P) = f(P) - P =$ ______________. The maximum sustainable harvest occurs when $H'(P) =$ ____________, so $H'(P) =$ ____________ $= 0$, or $P =$ ____________ thousand hares. This is the population at which the maximum sustainable harvest occurs, since $H''(P) =$ __________ is always __________. The maximum harvest is $H(60) =$ _________ thousand animals.

52. Determine the point on the ellipse $4x^2 + 9y^2 = 36$ that is nearest the origin.

Solution. The problem is to ____________ the distance s from a point (x, y) on the ellipse to the origin. That is, find the minimum of $s =$ _________ subject to the auxiliary condition that $4x^2 + 9y^2 = 36$. We can just as well minimize $s^2 = S$ since that will also minimize s. Since $S = x^2 + y^2$ is a function of both the variables x and y, we use the equation of the ellipse to eliminate the variable y: $y^2 = 4 - \frac{4}{9}x^2$ so that the substitution gives

$S(x) = x^2 + y^2 =$ _________. The minimum distance occurs when $\frac{dS}{dx} =$ _________, so

$\left(\frac{10}{9}\right)x = 0$ or $x =$ ________. Since $S''(x) > 0$, this value of x yields a ____________. When $x = 0$, $y^2 = 4$ on the ellipse so that $(0,2)$ and $(0,-2)$ are the points on the ellipse that are nearest to the origin.

53. A light house is at a point A, 4 miles offshore from the nearest point O of a straight beach; a store is at point B, 4 miles down the beach from O. If the lighthouse keeper can row 4 miles/hour and walk 5 miles/hour, find the point C on the beach to which the lighthouse keeper should row to get from the light house to the store in the least possible time.

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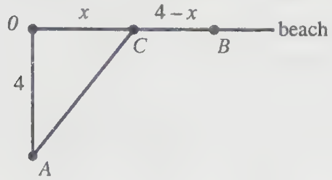
51. $-0.025P^2 + 3P$, 0, $-0.05P + 3$, 60, -0.05, negative, 90

52. minimize, $\sqrt{x^2 + y^2}$, $\frac{5}{9}x^2 + 4$, 0, 0, minimum

53. $\sqrt{16 + x^2}$, $4 - x$, $\frac{1}{8}\left(16 + x^2\right)^{-1/2}(2x) - \frac{1}{5}$, $4\sqrt{16 + x^2}$, $\frac{256}{9}$, $\frac{16}{3}$, endpoints, 1.8, $\sqrt{2}$, 4

Solution. The information is sketched in the figure below. From the diagram and the Pythagorean theorem, the distance from A to C is __________. The total time required to get from A to C to B is $T = \frac{\sqrt{16+x^2}}{4} + \left(\frac{\_\_\_\_\_\_}{5}\right)$, where $0 \le x \le 4$. The minimum time occurs when $\frac{dT}{dx} = 0$, or $0 = T'(x) =$ ________. Simplifying algebraically, $5x =$ ________ or $x^2 =$ ________ or $x =$ ________.
However, $x = \frac{16}{3}$ is outside the allowable range of values $0 \le x \le 4$. Therefore, the minimum must be taken on at one of the __________ of the interval. Checking each point, $T(0) =$ ________ hours and $T(4) =$ ________ hours. The smaller of these values occurs when $x =$ ________ so our conclusion is that the lighthouse keeper should row all the way to get to the store in the least possible time.



54. The cost per hour of driving a ship through the water varies approximately as the cube of its speed in the water. Suppose a ship runs into a current of V miles per hour, measured relative to the ocean bottom. find the total cost for the ship to travel M miles, and find the most economical speed of the ship relative to the ocean bottom.

Solution. Let x denote the speed of the ship relative to the water. Then __________ will be its speed relative to the bottom. The time taken to travel M miles will be __________. The cost per hour in fuel will be kx^3 for some constant of proportionality k, so the total cost function is given by $C(x) =$ ________. To find the most economical speed, minimize the cost. Now, $C'(x) =$ __________. The minimum cost occurs when $\frac{dC}{dx} = 0$, or $kMx^2[3(x-V)-x] = 0$. Thus, $x =$ ________ or $x =$ ________. Since $x = 0$ is ruled out if the ship moves, and since $C(x) \to +\infty$ as $x \to V^+$, we see that $x = 1.5V$ must provide the minimum cost.

3.6 LINEARIZATIONS AND DIFFERENTIALS

OBJECTIVE A: Given a function $y = f(x)$ and a point $x = a$, find the linearization of $f(x)$ at a. Use your linearization to estimate a specified function value.

55. If $y = f(x)$ is differentiable at $x = a$, then the *linearization* of f at a is given by $L(x) =$ __________.

56. To find the linearization to $f(x) = \frac{1}{2}x^2 - 7x + 9$ at $x = 4$, first calculate the derivative $f'(x) =$ __________. The value $f'(4) =$ __________ is the slope of the linearization at the point (4, ________) on the graph of f. Thus, an equation of the linearization is $L(x) = -11 +$ ________$(x-4)$, or $L(x) =$ __________.

57. From the result in Problem 56, an *estimate* to $\frac{1}{2}\left(\frac{1}{6}\right)^2 - \frac{7}{6} + 9$ is ______________.

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54. $x - V, \frac{M}{(x-V)}, \frac{kMx^3}{x-V}, \frac{(x-V)3kMx^2 - kMx^3}{(x-V)^2}$, 0, 1.5$V$

55. $f(a) + f'(a)(x-a)$

56. $x - 7, -3, -11, -3, -3x+1$

57. $-3\left(\frac{1}{6}\right) + 1 = \frac{1}{2}$

OBJECTIVE B: Given $y = f(x)$, find the differential dy.

58. If $y = x^2 + \sin 3x$, then $\frac{dy}{dx} =$ ________________. Thus, $dy =$ ________________.

59. If $y = f(x)$ is differentiable at $x = a$, and x changes from a to $a + \Delta x$, the error $|\Delta y - dy|$ in the approximation $f'(a)\Delta x$ is given by ________________, where ________________ as $\Delta x \to 0$.

OBJECTIVE C: Estimate the change Δf produced in a function $y = f(x)$ when $x = x_0$ changes by a small amount dx.

60. An estimate of $\Delta f = f(x_0 + \Delta x) - f(x_0)$ is given by the *differential* $df =$ ________________. thus, df denotes the change in the linearization of f that results from the change dx in x.

61. The equation $\frac{df}{dx} = f'(x)$ says we may regard the derivative as a ________________ of differentials.

62. Suppose we wish to estimate the change in $y = x^3$ when x changes by $dx = 0.1$ at $x = 2$. Now $\Delta y \approx dy = \frac{dy}{dx}dx =$ ________________ dx. When $x = 2$ and $dx = 0.1$, $dy = 3($________$)^2($________$) = 1.2$. Therefore, since $f(x + dx) = y + \Delta y \approx y + dy$, $(2 + 0.1)^3 \approx 2^3 +$ __________ $=$ __________. The actual values of $(2.1)^3$ is ________________ giving an error in our estimate of $\varepsilon \cdot dx = \Delta y - dy =$ ____________. The positive sign if $\varepsilon \cdot dx$ indicates that our estimate 9.2 is too small.

63. To estimate the value of $\sqrt{16.56}$, let $y = \sqrt{x}$, $x = 16$, and $dx = 0.56$. Then $dy = ($________$)dx$, so when $x = 16$ and $dx = 0.56$, $dy = ($________$)(0.56) = .07$. Thus, $\sqrt{16.56} = \sqrt{}$______ $+ .07 =$ __________. (The actual value of $\sqrt{16.56}$ is 4.0694 correct to 5 decimal places, so our estimate is accurate.)

3.7 NEWTON'S METHOD

OBJECTIVE : Use Newton's method to estimate the root of an equation $f(x) = 0$ within specified $a \le x \le b$.

64. In using Newton's new method, to go from the n^{th} approximation x_n of the root to the next approximation x_{n+1}, use the formula $x_{n+1} =$ ______________. This formula fails if the derivative $f'(x_n)$ equals ______________.

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58. $2x + 3\cos 3x$, $(2x + 3\cos 3x)dx$ **59.** $|\varepsilon \Delta x|, \varepsilon \to 0$ **60.** $f'(x_0)\Delta x$ or $f'(x_0)dx$

61. quotient **62.** $3x^2$, 2, 0.1, 1.2, 9.2, 9.261, 0.061

63. $\frac{1}{2\sqrt{x}}, \frac{1}{8}$, 16, 4.07 **64.** $x_n - \frac{f(x_n)}{f'(x_n)}$, 0

65. Suppose it is required to find a real root to the equation $f(x) = x^3 + x - 1$. Since $f(0) =$ ____________ and $f(1) =$ ____________ differ in sign, an unknown root lies somewhere in the interval $0 < x < 1$. As a first guess, choose $x_1 = 0.5$. To apply Newton's formula, we calculate the derivative $f'(x) =$ ____________. Then,

$$x_2 = x_1 - \frac{x_1^3 + x_1 - 1}{3x_1^2 + 1} = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right) - 1}{\_\_\_\_\_\_} = \frac{1}{2} + \frac{\_\_\_\_\_\_}{14} = \frac{\_\_\_\_\_\_}{7} \approx 0.71429, \text{ and}$$

$$x_3 = x_2 - \frac{\_\_\_\_\_\_}{3x_2^2 + 1} = \frac{5}{7} - \frac{\_\_\_\_\_\_}{3\left(\frac{5}{7}\right)^2 + 1} = \frac{5}{7} - \frac{\_\_\_\_\_\_}{7 \cdot 124} = \frac{\_\_\_\_\_\_}{7 \cdot 124} \approx 0.68318.$$

With the aid of a calculator, we have computed the following iterations in the same way: $x_4 = 0.68233$ and $x_5 = 0.68233$. Thus, a root to $f(x) = x^3 + x - 1$ is $r = 0.68233$ correct to 5 decimal places. The method is easy, but the arithmetic can be cumbersome without the aid of a calculator.

66. The speed with which Newton's method converges to a root r is expressed by the formula $|r - x_{n+1}| \le$ ____________ in an interval surrounding r. For the function $f(x) = x^3 + x - 1$ in Problem 65, on $0 < x < 1$, $\min f'(x) = \min\left(3x^2 + 1\right) =$ ____________ and $\max f''(x) = \max 6x =$ ____________.

Thus,

$$|r - x_{n+1}| \le \frac{1}{2}\left|\frac{6}{1}\right|(r - x_n)^2.$$

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65. $-1, 1, 3x^2 + 1, 3\left(\frac{1}{2}\right)^2 + 1, 3, 5, x_2^3 + x_2 - 1, \left(\frac{5}{7}\right)^3 + \left(\frac{5}{7}\right) - 1, 27, 593$

66. $\frac{1}{2}\frac{\max|f''|}{\min|f'|}(r - x_n)^2, 1, 6$

CHAPTER 3 SELF-TEST

1. Find the absolute maximum and minimum values (if they exist) of $f(x) = x^3 - x^2 - x + 2$ over the interval $0 \le x < 2$.

In Problems 2–4, sketch the curves. Find the intervals of values of x for which the curve is increasing, decreasing, concave up, and concave down. Locate all asymptotes.

2. $y = \dfrac{x}{\sqrt{1+x^2}}$

3. $y = \dfrac{4x}{x^2+1}$

4. $y = 1 - (x+1)^{1/3}$

5. Apply Rolle's Theorem to show that the equation $\cos x = \sqrt{x}$, $x \ge 0$, has exactly one real solution.

6. Find all the numbers c which satisfy the conclusion of the Mean Value Theorem for $f(x) = 1 + 2x^2$ over $-1 \le x \le 1$.

7. Let $f(x) = \dfrac{1}{x}$. Show that there is no c in the interval $-1 \le x < 2$ such that $f'(c) = \dfrac{f(2) - f(-1)}{2-(-1)}$. Explain why this does *not* contradict the Mean Value Theorem.

In Problems 8 and 9, find all stable and unstable equilibria of the autonomous differential equation.

8. $\dfrac{dy}{dt} = y + 1$

9. $\dfrac{dy}{dt} = -y^2(y+2)(y-1)$

10. Suppose a company can sell x items per week at a price $P = 200 - 0.01x$ cents, and that it costs $C = 50x + 20{,}000$ cents to produce the x items. How much should the company charge per item in order to maximize its profits?

11. The weight W (lbs/sec) of flue gas passing up a chimney at different temperatures T is represented by $W = A(T - T_0)(1 + \alpha T)^{-2}$, where A is a positive constant, T the absolute temperature of the hot gases passing up the chimney, T_0 the temperature of the outside air (all in °C), and $\alpha = \dfrac{1}{273}$ is the coefficient of expansion of the gas. For a given $T_0 = 15°\text{C}$, find the temperature T at which the greatest amount of gas will pass up the chimney.

12. Find the linearization of $f(x) = \sqrt{x + \frac{1}{x}}$ at $x = 4$.

13. Use the linearization $L(x)$ to estimate the value of $\sin 29°$.

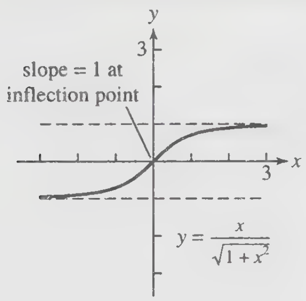
14. Beginning with the estimate $x_1 = \dfrac{\pi}{2}$, apply Newton's method once to calculate a positive solution to the equation $\sin x = \dfrac{2}{3}x$.

SOLUTIONS TO CHAPTER 3 SELF-TEST

1. $f(x) = x^3 - x^2 - x + 2$ for $0 \le x < 2$, and $f'(x) = 3x^2 - 2x - 1 = (3x+1)(x-1)$. Thus $f'(x) = 0$ implies $x = -\frac{1}{3}$ or $x = 1$. Then $x = 1$ is the only critical point in the interval [0, 2). Next note that $f'(x) < 0$ in $[0,1)$, so f is decreasing to the left of $x = 1$, and $f'(x) > 0$ in (1, 2) so f is increasing to the right of $x = 1$. Also, $f(0) = 2, f(1) = 1,$ and $f(2) = 4$. Since $x = 2$ is not in the interval, there is no absolute maximum. The absolute minimum value is $f(1) = 1$ (which is also a relative minimum).

2.
$$y = \frac{x}{\left(1+x^2\right)^{1/2}}$$
$$y' = \frac{\left(1+x^2\right)^{1/2} - x\cdot\frac{1}{2}\left(1+x^2\right)^{-1/2}2x}{1+x^2} = \frac{1}{\left(1+x^2\right)^{3/2}}, \text{ and}$$
$$y'' = \frac{-3x}{\left(1+x^2\right)^{5/2}}.$$

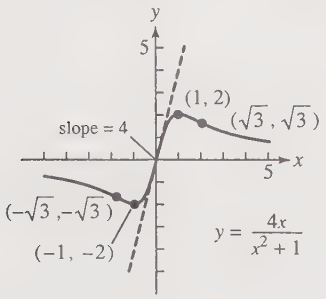
Note that $y = \frac{1}{\left(\frac{1}{x^2}+1\right)^{1/2}}$ for $x \ge 0$ $\left(\text{since } \sqrt{x^2} = |x|\right)$ and that $y = \frac{-1}{\left(\frac{1}{x^2}+1\right)^{1/2}}$ for $x < 0$. Thus $\lim_{x\to\infty} y = 1$ and $\lim_{x\to-\infty} y = -1$. Hence the lines $y = 1$ and $y = -1$ are *horizontal asymptotes*. Since y' exists for all x and is never zero, there are *no critical points*. At $x = 0$, $y'' = 0$ so that $x = 0$ is a *point of inflection* where the graph has slope 1. On $(-\infty, 0)$ $y'' > 0$ and the function is concave up; on $(0, \infty)$ it is concave down. Since $y' > 0$ for all x, the graph is everywhere an increasing function of x. This information yields the graph sketched below.



3. Since $\lim_{x\to\pm\infty} y = \lim_{x\to\pm\infty} \frac{\frac{4}{x}}{1+\frac{1}{x^2}} = 0$, the x-axis is a *horizontal asymptote*. Next, $y' = \frac{4\left(1-x^2\right)}{\left(x^2+1\right)^2}$ and

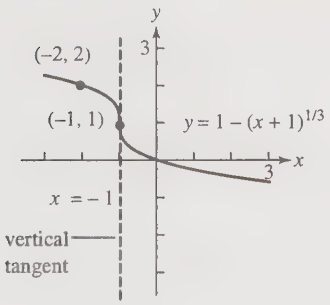
$y'' = \frac{8x\left(x^2-3\right)}{\left(x^2+1\right)^3}$. Hence, $y'=0$ implies $x=\pm1$. Since $y'>0$ at $x=-1$ and $y''<0$ at $x=1$, it follows from the second derivative test that $y(-1)=-2$ is a *relative minimum* and $y(1)=2$ is a *relative maximum.*

Next, $y''=0$ when $x=0,\ -\sqrt{3}$, and $\sqrt{3}$, so that these values for x are *points of inflection.* Moreover, for $x<-\sqrt{3}$, $y''<0$ and the graph of y is concave down; $-\sqrt{3}<x<0$, $y''>0$ and the graph of y is concave up; $0<x<\sqrt{3}$, $y''<0$ and the graph of y is concave down; $x>\sqrt{3}$, $y''>0$ and the graph of y is concave up. Note that at $x=0$, $y'=4$. The graph of y is sketched below. Note the symmetry about the origin.



4. $y = 1-(x+1)^{1/3}$, $y' = -\frac{1}{3}(x+1)^{-2/3}$, and $y'' = \frac{2}{9}(x+1)^{-5/3}$. The derivative y' does not exist when $x=-1$, although the curve y is continuous at $x=-1$. Since $\lim_{x\to-1} \frac{dx}{dy} = \lim_{x\to-1} -3(x+1)^{2/3} = 0$, the graph has a vertical tangent at $x=-1$. Since $y'<0$ for all $x\neq-1$, the curve is everywhere decreasing.

We note that y'' is never zero. However, y'' fails to exist at $x=-1$. When $x<-1$, $y''<0$ and the curve is concave down; when $x>-1$, $y''>0$ and the curve is concave up. Therefore, $x=-1$ is a point of inflection. The graph is sketched in the figure below.



5. Let $f(x)=\cos x-\sqrt{x}$. Since $|\cos x|\le 1$, we see that $f(x)<0$ if $x>1$. Thus, the only possible root must lie within the interval [0,1]. Now, $f(0)=1$ and $f\left(\frac{\pi}{2}\right) = -\sqrt{\frac{\pi}{2}}$, so the Intermediate Value Theorem guarantees a root in the interval $\left[0,\frac{\pi}{2}\right]$: we know in fact that the root must lie in [0, 1]. Calculating the derivative , $f'(x) = -\sin x - \frac{1}{2\sqrt{x}}$, we see that f' is negative in the interval (0, 1). Since f' is different from zero for all values of x between 0 and 1, we conclude that there is exactly one real root to the equation $f(x)=0$ for $x\ge0$.

6. $f(-1)=3$ and $f(1)=3$; $f'(x)=4x$. Hence $\frac{f(1)-f(-1)}{1-(-1)} = f'(c)$ translates into $0=4c$ or $c=0$.

7. $\dfrac{f(2)-f(-1)}{2-(-1)}=\dfrac{\frac{1}{2}-(-1)}{3}=\dfrac{1}{2}$ and $f'(c)=-\dfrac{1}{c^2}$. Since $-\dfrac{1}{c^2}=\dfrac{1}{2}$ is impossible to solve for real values of c, there is no number c in the interval $(-1, 2)$ satisfying the conclusion of the Mean Value Theorem. However, this does not contradict the Theorem because $f(x)=\dfrac{1}{x}$ is not continuous over the closed interval $[-1, 2]$: it fails to be continuous at $x=0$. Thus the hypotheses of the theorem are not satisfied.

8. $\dfrac{dy}{dt}=y+1=0$ only when $y=-1$, so that is the only equilibrium. It is unstable because y increases ($y'<0$) for $y>-1$ and decreases ($y'<0$) for $y<-1$. There are no stable equilibria.

9. $\dfrac{dy}{dt}=-y^2(y+2)(y-1)=0$ when $y=0,\ y=-2,$ or $y=1$, so those are the three equilibria. Near $y=0$, y' is positive when $y>0$ and when $y<0$, so $y=0$ is a stable equilibrium. Near $y=-2$, y' is positive when $y>-2$ and negative when $y<-2$, so $y=2$ is an unstable equilibrium. And $y=1$ is an unstable equilibrium for the same reasons.

10. Let Q denote the profit function. Then, $Q(x)=xP-C=150x-0.01x^2-20{,}000$. The maximum occurs when $\dfrac{dQ}{dx}=0$, or $150-0.02x=0$; thus, $x=7500$ items. Since $\dfrac{d^2Q}{dx^2}=-0.02<0$, this provides a *maximum* profit. The price per item is then given by $P(7500)=200-(0.01)(7500)=125$ cents, the price required to obtain the maximum profit $Q(7500)=\$5{,}425.00$.

11. We want to maximize the weight function W. Now, $W=A(T-T_0)(1+\alpha T)^{-2}$,
$\dfrac{dW}{dT}=A(1+\alpha T)^{-2}-2A\alpha(T-T_0)(1+\alpha T)^{-3}$. Setting $\dfrac{dW}{dT}=0$, and simplifying algebraically, gives $(1+\alpha T)-2\alpha(T-T_0)=0$, or $T=\dfrac{1+2\alpha T_0}{\alpha}$. Thus, for $T_0=15°C$ and $\alpha=1/273$ as given,
$T=\dfrac{1}{\alpha}+2T_0=273+30=303°\text{C}$. Setting $\dfrac{dW}{dT}>0$ if $T<303$, and $\dfrac{dW}{dT}<0$ if $T>303$, it is clear that $T=303°$ provides an absolute maximum for W.

12. $f'(x)=\dfrac{1}{2}\left(x+\dfrac{1}{x}\right)^{-1/2}\cdot\dfrac{d}{dx}\left(x+\dfrac{1}{x}\right)=\dfrac{1}{2}\left(x+\dfrac{1}{x}\right)^{-1/2}\left(1-\dfrac{1}{x^2}\right)$. Thus,
$f'(4)=\dfrac{1}{2}\left(4+\dfrac{1}{4}\right)^{-1/2}\left(1-\dfrac{1}{16}\right)=\dfrac{15}{16\sqrt{17}}\approx 0.227$ and $f(4)=\sqrt{\dfrac{17}{4}}\approx 2.062$. Therefore,
$L(x)=\dfrac{\sqrt{17}}{2}+\dfrac{15}{16\sqrt{17}}(x-4)\approx 2.062+0.227(x-4)$.

13. The calculation must be done when $y=\sin x$ for x measured in *radians*. Thus, $\sin 29°\approx\sin\dfrac{\pi}{6}+dy$, where
$dy=\dfrac{dy}{dx}dx$ when $x=\dfrac{\pi}{6}$ and $dx=-\dfrac{\pi}{180}$ radians. Now, $\left.\dfrac{dy}{dx}\right|_{\pi/6}=\cos\dfrac{\pi}{6}=\dfrac{\sqrt{3}}{2}$, so that
$\sin 29°\approx\dfrac{1}{2}+\left(\dfrac{\sqrt{3}}{2}\right)\left(-\dfrac{\pi}{180}\right)\approx 0.48489$.

14. Let $f(x)=\sin x-\dfrac{2}{3}x=0$, $f'(x)=\cos x-\dfrac{2}{3}$. By Newton's method,
$x_2=x_1-\dfrac{\sin x_1-\frac{2}{3}x_1}{\cos x_1-\frac{2}{3}}=\dfrac{\pi}{2}-\dfrac{1-\frac{\pi}{3}}{0-\frac{2}{3}}=\dfrac{\pi}{2}+\dfrac{3}{2}\left(1-\dfrac{\pi}{3}\right)=\dfrac{3}{2}$.

NOTES.