Chapter 2: Derivatives

2.1 THE DERIVATIVE AS A FUNCTION

OBJECTIVE A: Use the three algebraic steps to find the derivative $\frac{dy}{dx}$ of a function $y = f(x)$.

1. The derivative of a function f at the point x is given by the limit $f'(x) =$ ____________ whenever this limit exists.

2. The fraction $\frac{f(x+h)-f(x)}{h}$ is called the ____________ for f at x.

3. When the number $f'(x)$ exists it is called the ____________ of the curve $y = f(x)$ at x. The line through the point $(x, f(x))$ with slope $f'(x)$ is the ____________ to the curve at x.

4. For the function $f(x) = \sqrt{3-x}$,

STEP 1: Form $f(x+h) =$ ____________ and $f(x) = \sqrt{3-x}$.

STEP 2: Expand and simplify the difference quotient:

$$\frac{f(x-h)-f(x)}{h} = \underline{\qquad\qquad} = \frac{\sqrt{3-x-h}-\sqrt{3-x}}{h} \cdot \frac{\sqrt{3-x-h}+\sqrt{3-x}}{\sqrt{3-x-h}+\sqrt{3-x}}$$

$$= \frac{\underline{\qquad\qquad}}{h\left(\sqrt{3-x-h}+\sqrt{3-x}\right)} = \frac{-1}{\underline{\qquad\qquad}}$$

STEP 3: Take the limit as $h \to 0$: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} =$ ____________.

5. For the function $f(x) = \frac{x-1}{x+1}$,

STEP 1: Form $f(x+h) =$ ____________ and $f(x) = \frac{x-1}{x+1}$.

STEP 2: Expand and simplify the difference quotient:

$$\frac{f(x-h)-f(x)}{h} = \frac{\frac{(x+h)-1}{(x+h)+1} - \frac{(x-1)}{(x+1)}}{h} = \frac{(x+1)(x+h-1)-(x-1)(x+h+1)}{h(\underline{\qquad\qquad})}$$

$$(\underline{\qquad\qquad}) = \frac{\left(x^2+xh-x+x+h-1\right)}{h\left(x^2+xh+x+x+h+1\right)} = \frac{2h}{\underline{\qquad\qquad}} = \frac{2}{x^2+xh+2x+h+1}$$

STEP 3: Take the limit a $h \to 0$: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} =$ ____________ $= \frac{2}{(x+1)^2}$.

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1. $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ 2. difference quotient 3. slope, tangent

4. $\sqrt{3-(x+h)}$, $\frac{\sqrt{3-(x+h)}-\sqrt{3-x}}{h}$, $(3-x-h)-(3-x)$, $\sqrt{3-x-h}+\sqrt{3-x}$, $\frac{-1}{2\sqrt{3-x}}$

OBJECTIVE B: Write an equation of the tangent line to the curve $y = f(x)$ at a specified value $x = a$.

6. To find an equation of the tangent line to the curve $f(x) = \frac{x-1}{x+1}$ when $x = 2$, we calculate the slope m. By definition the slope is the limit of a secant through $P\left(2, \frac{1}{3}\right)$ and a point Q nearby on the curve. The symbolic notation for this slope is _______________. From our calculation in the previous Problem 5, that slope has the value _______________. The point on the curve corresponding to $x = 2$ has coordinates _______________. Therefore, the point-slope form gives an equation of the tangent line as _______________.

7. If $y = mx + b$ is a straight line, then the derivative $\frac{dy}{dx}$ always has the value _______________. That is, the derivative equals the _______________ of the straight line.

OBJECTIVE C: Know the basic elementary facts about the derivative.

8. Differentiable functions are continuous. That is, if f has a _______________ at $x = c$, the f is _______________ at $x = c$.

9. Can a continuous function fail to have a derivative at a point? _______________

10. What are four conditions under which a function whose graph is otherwise smooth fails to have a derivative at a point?

1. _______________,
2. _______________,
3. _______________,
4. _______________.

11. Is every function the derivative of some function? _______________

____

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OBJECTIVE D: Know from memory the following five derivative rules: derivative of a constant, positive integer power, constant multiple, sum , and difference.

12. If c is a constant and $y = c$, then $\frac{dy}{dx} =$ _________.

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5. $\frac{(x+h)-1}{(x+h)+1}$, $(x+h+1)(x+1)$, $\left(x^2 + xh + x - x - h - 1\right)$, $h\left(x^2 + hx + 2x + h + 1\right)$, $\frac{2}{\left(x^2 + 2x + 1\right)}$

6. $f'(2)$, $\frac{2}{9}$, $\left(2, \frac{1}{3}\right)$, $y = \frac{1}{3} + \frac{2}{9}(x-2)$ or $y = \frac{2}{9}x - \frac{1}{9}$

7. m, slope

8. derivative, continuous

9. Yes, the absolute value function fails to have a derivative at $x = 0$.

10. the graph has a corner, a cusp, a vertical tangent, a discontinuity

11. No, because a function cannot be a derivative on an interval unless it has the intermediate value property there.

12. 0

13. If n is any positive integer and $y = x^n$, then $\frac{dy}{dx} =$ ________.

14. If u is a differentiable function of x, and if $y = cu$ where c is a constant, then $\frac{dy}{dx} =$ ________.

15. If u and v are differentiable functions of x, then $y = u + v$ is a ______________ function of x, and $\frac{dy}{dx} =$ ________. Likewise, $\frac{d}{dx}(u - v) =$ ______________.

OBJECTIVE E: Calculate the derivatives of any polynomial function.

16. $\frac{d}{dx}\left(3x^2 - 12x + 1\right) =$ ______________.

17. $\frac{d}{dx}\left(\sqrt{3}x^4 - \frac{2}{5}x^3 + \frac{1}{3}x^2 - 15x + 109\right) =$ ______________.

OBJECTIVE F: Calculate second and higher-order derivatives.

18. If $y = f(x)$, the second derivative of y with respect to x is the derivative of ______________. The second derivative is denoted by ______________ or ______________ or ______________.

19. In general, the n^{th} derivative of $y = f(x)$ with respect to x is the derivative of ______________, and is denoted by ______________ or ______________ or ______________.

20. $\frac{d^2}{dx^2}\left(4x^5 - 3x^2 + 2x - 20\right) = \frac{d}{dx}($______________$) =$ ______________.

21. $\frac{d^2}{dx^2}\left(2x^2 - 1\right)(x - 3) = \frac{d^2}{dx^2}($______________$) = \frac{d}{dx}($______________$) =$ ______________.

2.2 THE DERIVATIVE AS A RATE OF CHANGE

OBJECTIVE A: Know the definitions for applications of the derivative.

22. If $s = f(t)$ gives the position of a body moving along a line from position $s = f(t)$ to position $s = f(t + \Delta t)$, then the *average velocity* over the time interval Δt is ______________________________. The *instantaneous velocity* at time t is ______________.

23. The *speed* is the ______________________________________.

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13. nx^{n-1}

14. $c\frac{du}{dx}$

15. differentiable, $\frac{du}{dx} + \frac{dv}{dx}, \frac{du}{dx} - \frac{dv}{dx}$

16. $6x - 12$

17. $4\sqrt{3}x^3 - \frac{6}{5}x^2 + \frac{2}{3}x - 15$

18. $\frac{dy}{dx} = f'(x)$, $\frac{d^2y}{dx^2}$, y'', or $f''(x)$

19. $\frac{d^{n-1}y}{dx^{n-1}}$, $\frac{d^ny}{dx^n}$, $y^{(n)}$, or $f^{(n)}(x)$

20. $20x^4 - 6x + 2$, $80x^3 - 6$

21. $2x^3 - 6x^2 - x + 3$, $6x^2 - 12x - 1$, $12x - 12$

22. $\frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$, $\frac{ds}{dt}$

23. absolute value of velocity

24. *Acceleration* is the derivative of ______________________________.

OBJECTIVE B: If $s = f(t)$ gives the position of a body moving along a line as a function of time t, find and interpret the velocity and acceleration at a specified instant.

25. Suppose a particle is moving along a straight line, negative to the left and positive to the right, according to the law $s = t^3 - 3t^2 - 9t + 5$. Then the velocity is given by $\frac{ds}{dt} =$ ______________ = ______________. Thus, the velocity is positive when ______________; the velocity is ______________ when $-1 < t < 3$ so the particle is moving to the ______________.

26. The acceleration of the particle is $\frac{d^2s}{dt^2} =$ ______________. When the velocity is zero, $t =$ ______________ or ______________ and the acceleration has the value ______________ or ______________, respectively.

27. Suppose the law of motion of a particle is given by $s = t^3 - 6t^2 + 2$. Then the instantaneous velocity is given by $v = \frac{ds}{dt} =$ ______________. When t = 2.3 sec, the velocity of the particle is $v(2.3) =$ ______________. If our coordinate axis of motion is such that the positive direction is to the right (which is conventional), the interpretation of this negative velocity means that the particle is moving to the ______________. When $t = 4$ sec, the velocity of the particle is ______________ and the particle is at rest. When $t = 4.5$ sec, the velocity of the particle is ______________ and the particle is moving to the ______________.

28. Suppose a ball is thrown directly upward with a speed of 96 ft/sec and moves according to the law $y = 96t - 16t^2$, where y is the height in feet above the starting point, and t is the time in seconds after it is thrown. The velocity of the ball at any time t is $v(t) = \frac{dy}{dt} =$ ______________. Hence when $t = 2$ sec, the velocity of the ball is ______________. Since $v(2)$ is positive, the ball is still rising. At its highest point the velocity of the ball is ______________, and this occurs when $t =$ ______________ seconds. The height corresponding to this time is $y =$ ______________ feet, and this is the highest point reached. Notice the acceleration is a constant ______________ ft / sec^2.

OBJECTIVE C: Given a functional relationship $y = f(x)$ between two variables x and y, calculate the average rate of change and the instantaneous rate of change of y with respect to x.

29. Every derivative may be interpreted as the instantaneous rate of change of one variable per unit change in the other. If $y = f(x)$, then $\frac{\Delta y}{\Delta x} =$ ______________ is interpreted as the ______________ rate of change of y by a change of one unit in ______________. Passage to the limit as $\Delta x \to 0$ gives $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} =$ ______________ as the ______________ rate of change of ______________ with respect to ______________.

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24. velocity with respect to time

25. $3t^2 - 6t - 9$, $3(t-3)(t+1)$, $t < -1$ or $t > 3$, right, negative, left

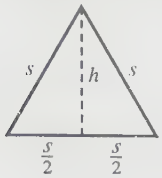
26. $6t - 6$, -1 or 3, -12 or 12

27. $3t^2 - 12t$, -11.73 units/sec., left, 0, 6.75 units/sec., right

28. $96 - 32t$, 32 ft/sec, zero, 3, 144 ft, –32

29. $\frac{f(x+\Delta x) - f(x)}{\Delta x}$, average, x, $f'(x)$, instantaneous, y, x

30. Consider the equilateral triangle pictured below.



By the Pythagorean theorem $s^2 =$ ______________ or, solving for h, $h =$ ______________. Then, the area of the triangle is given by $A = \frac{1}{2}$ base $\cdot$ height $=$ ______________. The average rate of change of area with respect to side length is $\frac{\Delta A}{\Delta s} = \frac{\text{\_\_\_\_\_\_\_\_\_\_\_\_}}{\Delta s} =$ ______________. Taking the limit as Δs tends to zero gives, $\frac{dA}{ds} \lim_{\Delta s \to 0} \frac{\Delta A}{\Delta s} =$ ______________.

31. Suppose it costs $C(x)$ thousand dollars per year to produce x thousand gallons of antifreeze, where $C(x)$ is given by the table

x	0.25	0.5	0.75	1.0	1.25	1.50	1.75	2.0	2.25	2.5
$C(x)$	5.875	8.5	10.875	13.0	14.875	16.5	17.875	19.0	19.875	20.0

The ______________ cost at any x is the value of the derivative $C'(x)$. Using the table, we estimate $C'(1.75)$ as follows: $C'(1.75) \approx \frac{\Delta C}{\Delta x} = \frac{\text{\_\_\_\_\_\_\_\_}}{2.0 - 1.75} =$ ______________. Here we have estimated the marginal cost by the __________ cost.

2.3 PRODUCTS, QUOTIENTS, AND NEGATIVE POWERS

OBJECTIVE A: Know from memory the derivative rules for product, quotient, and negative integer power.

32. If u and v are differentiable functions of x, then the derivative $\frac{d}{dx}(uv) =$ ______________.

33. If u and v are differentiable functions of x, then the derivative $\frac{d}{dx}\left(\frac{u}{v}\right) =$ ______________ when $v \neq 0$.

34. If n is any negative integer and $y = x^n$, then $\frac{dy}{dx} =$ ______________.

OBJECTIVE B: Find the derivative of a product of polynomial or power functions.

35. If $y = (x^2 - 2)(2x^3 - 5)$, then

$y' = (x^2 - 2)\frac{d}{dx}(2x^3 - 5) + (2x^3 - 5)$______________ $= (x^2 - 2)($__________$) + (2x^3 - 5)(2x)$

$=$ ______________ $+ 4x^4 - 10x =$ ______________.

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30. $h^2 + \frac{s^2}{4}$, $\frac{\sqrt{3}}{2}s$, $\frac{\sqrt{3}}{8}s^2$, $\frac{\sqrt{3}}{8}(s + \Delta s)^2 - \frac{\sqrt{3}}{8}s^2$, $\frac{\sqrt{3}}{8}(2s + \Delta s)$, $\frac{\sqrt{3}}{4}s$

31. marginal, 19.0 − 17.875, 4.5, average

32. $u\frac{dv}{dx} + v\frac{du}{dx}$

33. $\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

34. nx^{n-1}

36. $\frac{d}{dx}\left[(3x^2-2x+1)(5x-4)\right]=(3x^2-2x+1)(\_\_\_\_\_\_\_\_)+(5x-4)(\_\_\_\_\_\_\_\_)$
$=(15x^2-10x+5)+(\_\_\_\_\_\_\_\_)=\_\_\_\_\_\_\_\_.$

OBJECTIVE C: Find the derivative of a quotient of polynomial or power functions.

37. If $y=\frac{3x}{5x^2-1}$, then $y'=\frac{(5x^2-1)(\_\_\_\_\_\_)-(3x)(\_\_\_\_\_\_)}{(5x^2-1)^2}=\frac{\_\_\_\_\_\_\_\_}{(5x^2-1)^2}.$

38. $\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{x(\_\_\_\_\_\_\_\_)-1(\_\_\_\_\_\_\_\_)}{x^2}=\_\_\_\_\_\_\_\_.$

39. $\frac{d}{dx}\left(\frac{x^2-2x+5}{x^3+1}\right)=\frac{(x^3+1)(\_\_\_\_\_\_\_\_)-(\_\_\_\_\_\_\_\_)(3x^2)}{(x^3+1)^2}$
$=\frac{(\_\_\_\_\_\_\_\_)-3x^4+6x^3-15x^2}{(x^3+1)^2}=\frac{(\_\_\_\_\_\_\_\_)}{(x^3+1)^2}$

40. If $y=x^n$, where $x\neq 0$ and n is *any* non zero integer, then $\frac{dy}{dx}=\_\_\_\_\_\_\_\_.$

41. If $y=2x^3+3x^{-2}$, then $y'=\_\_\_\_\_\_\_\_$ and $y''=\_\_\_\_\_\_\_\_.$

42. If $y=\left(x-\frac{1}{x}\right)+3x^2$, then by the product rule
$y'=\left(x-\frac{1}{x}\right)6x+3x^2\cdot\frac{d}{dx}(\_\_\_\_\_\_\_\_)$
$=6x^2-6+3x^2(\_\_\_\_\_\_\_\_)$
$=6x^2-6+3x^2+\_\_\_\_\_\_\_\_=9x^2-3.$

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35. $\frac{d}{dx}(x^2-2)$, $6x^2$, $6x^4-12x^2$, $10x^4-12x^2-10x$

36. 5, $6x-2$, $30x^2-34x+8$, $45x^2-44x+13$

37. 3, $10x$, $-15x^2-3$

38. 0, 1, $-\frac{1}{x^2}$

39. $2x-2$, x^2-2x+5, $2x^4-2x^3+2x-2$, $-x^4+4x^3-15x^2+2x-2$

40. nx^{n-1}

41. $6x^2-6x^{-3}$, $12x+18x^{-4}$

42. $x-\frac{1}{x}$, $1+\frac{1}{x^2}$, 3

OBJECTIVE D: Find an equation of the tangent line to a curve $y = f(x)$ meeting some specified requirement (such as a condition on the slope).

43. Consider the curve $y = x^3 - 9x^2 + 15x - 5$. The derivative y' gives the value of the ______________ of the tangent line at any x. For this particular curve, $y' =$ ______________ $= 3($__________$)(x-5)$. Thus, the tangent line is parallel to the x-axis when $x =$ ______________ or $x =$ ______________. When the slope of the tangent line to the above curve equals 15 the value of x is ______________ or ______________. The corresponding y values of the function are ______________ and ______________, respectively. Equations of the two tangent lines are then given by ______________ and ______________.

2.4 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

OBJECTIVE: Calculate the derivatives of functions involving the trigonometric functions, making use of appropriate rules of differentiation and the derivatives of the sine and cosine functions.

44. $\frac{d}{dx}(\sin x) =$ ______________.

45. $\frac{d}{dx}(\cos x) =$ ______________.

46. $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\_\_\_\_\_\_}\right) = \frac{\cos x \cdot \frac{d}{dx}(\_\_\_\_\_\_) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$

$= \frac{\cos x(\_\_\_\_\_\_) - \sin x(\_\_\_\_\_\_)}{\cos^2 x}$

$= \frac{\cos^2 x + \_\_\_\_\_\_}{\cos^2 x} = \_\_\_\_\_\_$.

47. $\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\_\_\_\_\_\_}\right) = -\frac{1}{\cos^2 x} \cdot \frac{d}{dx}(\_\_\_\_\_\_) = \frac{\_\_\_\_\_\_}{\cos^2 x} = \_\_\_\_\_\_$.

Remark: It will be to your advantage in later work to MEMORIZE the derivative formulas in Problems 46 and 47.

48.

$\frac{d}{dx}\left(3x^2 \sin x\right) = +3x^2 \cdot \frac{d}{dx}(\_\_\_\_\_\_) + \sin x \cdot \frac{d}{dx}(\_\_\_\_\_\_)$

$= \_\_\_\_\_\_ + 6x \sin x.$

49. $\frac{d}{dx}(3\sin x + x \sec x) = 3\frac{d}{dx}(\sin x) + x \cdot \frac{d}{dx}(\sec x) + \_\_\_\_\_\_$

$= \_\_\_\_\_\_ + x \_\_\_\_\_\_ + \sec x$

$= 3\cos x + \sec x(x \tan x + 1).$

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43. slope, $3x^2 - 18x + 15$, $x - 1$, 1, 5, 0, 6, -5, -23, $y + 5 = 15x$ and $y + 23 = 15(x - 6)$

44. $\cos x$

45. $-\sin x$

46. $\cos x$, $\sin x$, $\cos x$, $-\sin x$, $\sin^2 x$, $\sec^2 x$

47. $\cos x$, $\cos x$, $\sin x$, $\sec x \tan x$

48. $\sin x$, $3x^2$, $3x^2 \cos x$

49. $\sec x$, $3\cos x$, $\sec x \tan x$ (from Problem 47)

50. $\frac{d}{dx}\left(\frac{1-\sin x}{x}\right) = \frac{x \cdot \frac{d}{dx}(1-\sin x) - (1-\sin x)\cdot \frac{d}{dx}(\_\_\_\_\_\_)}{x^2}$

$= \frac{x(\_\_\_\_\_\_) - (1-\sin x)}{x^2}$

$= \frac{-x\cos x - 1 + \_\_\_\_\_\_}{x^2}$

$= \frac{\sin x - (1 + x\cos x)}{x^2}$

2.5 THE CHAIN RULE

OBJECTIVE A: If y is a differentiable function of u, and u is a differentiable function of x, use the chain rule to calculate $\frac{dy}{dx}$.

51. If y is a differentiable function of u, and u is a differentiable function of x, then y is a differentiable function of ________, and $\frac{dy}{dx} =$ ________. This rule is known as the ________ rule for the derivative of a composite function.

52. Consider the chain rule in functional form: let $y = f(u)$ and $u = g(x)$ be differentiable functions. Then the composite $y = (f \circ g)(x) = f(g(x))$ is a differentiable function of ________. When $x = x_0$, let $u = g(x_0) = u_0$. According to the chain rule of $(f \circ g)$evaluated at $x = x_0$ is given by $(f \circ g)'(x_0) =$ ________. In this equation, $(f \circ g)'(x_0)$ corresponds to $\left.\frac{dy}{dx}\right|_{x=x_0}$, $P'(u_0)$ corresponds to ________, and ________ corresponds to $\left.\frac{du}{dx}\right|_{x=x_0}$. It is important that you observe that the derivatives in the chain rule equation $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$ are being evaluated at different points: $\frac{dy}{dx}$ and $\frac{du}{dx}$ are evaluated at ________, whereas $\frac{dy}{du}$ is evaluated at $g(x_0) =$ ________. Failure to understand this fact can lead to serious misuse of the chain rule equation.

53. To find $\frac{dy}{dx}$ if $y = u^2 - 2u + 3$ and , calculate $\frac{dy}{du} =$ ________ and then substitute $u = \sqrt{x}$ to obtain $\left.\frac{dy}{du}\right|_{u=\sqrt{x}} =$ ________. According to the chain rule,

$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = (2\sqrt{x} - 2)\cdot$ ________ $=$ ________.

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50. x, $-\cos x$, $\sin x$

51. x, $\frac{dy}{du}\cdot\frac{du}{dx}$, chain

52. x, $f'(u_0)\cdot g'(x_0)$, $\left.\frac{dy}{du}\right|_{u=u_0}$, $g'(x_0)$, $x = x_0$, u_0

53. $2u-2$, $2\sqrt{x}-2$, $\frac{1}{2\sqrt{x}}$, $1-\frac{1}{\sqrt{x}}$

54. Suppose $y = z^{-2} + 3z^{-1}$ and $z = x^2 + 1$. Then $\frac{dy}{dz} =$ ________ so that when $z = x^2 + 1$,

$\left.\frac{dy}{dz}\right|_{z=x^2+1} =$ ____________. Applying the chain rule,

$$\frac{dy}{dx} = \left[-2\left(x^2+1\right)^{-3} - 3\left(x^2+1\right)^{-2}\right] \cdot \_\_\_\_\_\_ = \frac{-2x}{\left(x^2+1\right)^2}\left[\frac{2}{x^2+1} + \_\_\_\_\_\_\right] = \frac{\_\_\_\_}{\left(x^2+1\right)^3}.$$

55. If u is a differentiable function of x and n is a positive integer, then the derivative $\frac{d}{dx}\left(u^n\right) =$ ____________.

56. The above power rule holds when n is a negative integer at all points x where u is ____________.

57. If $y = \left(5x^3 - x^2 + 7\right)^4$, then $y' = 4\left(5x^3 - x^2 + 7\right)^3 \frac{d}{dx}(\_\_\_\_\_\_\_\_) =$ ________.

58. $\frac{d}{dx}[(2x-1)]^{-3} = -3(\_\_\_\_)^{-4} \frac{d}{dx}(\_\_\_\_\_\_) =$ ________, if $x \neq$ ________.

59. Let $y = \frac{2}{(x-1)^3} + \frac{3}{1-x^4}$. Then $y = 2(x-1)^{-3} + 3\left(1-x^4\right)^{-1}$, so that

$$\frac{dy}{dx} = -6(\_\_\_\_)\frac{d}{dx}(x-1) - 3\left(1-x^4\right)^{-2}\frac{d}{dx}(\_\_\_\_) = \_\_\_\_\_\_\_\_.$$

60. $$\frac{d}{dx} x\tan^2 x = x\frac{d}{dx}(\_\_\_\_) + \frac{dx}{dx}\tan^2 x = x \cdot 2\tan x \frac{d}{dx}(\_\_\_\_) + \tan^2 x$$
$$= 2x \_\_\_\_\_\_ + \tan^2 x = \_\_\_\_\_\_\_\_.$$

61. $$\frac{d}{dx}\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{1}{2\,\_\_\_\_\_\_}\frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right) = \frac{1}{2}\left(\frac{1-\cos x}{1+\cos x}\right)^{-1/2}\left[\frac{(1+\cos x)(\_\_\_\_) - (1-\cos x)(\_\_\_\_)}{(1+\cos x)^2}\right]$$
$$= \frac{(1-\cos x)^{-1/2}\sin x}{\_\_\_\_\_\_\_\_}.$$

62. $$\frac{d}{dx}\left(\frac{x-2}{x+1}\right)^5 = 5(\_\_\_\_)^4\frac{d}{dx}(\_\_\_\_) = 5\left(\frac{x-2}{x+1}\right)^4\frac{(x+1)(1) - (\_\_\_\_)(1)}{(x+1)^2} = \frac{\_\_\_\_}{(x+1)^6}.$$

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54. $-2z^{-3} - 3z^{-2}$, $-2\left(x^2+1\right)^{-3} - 3\left(x^2+1\right)^{-2}$, $2x$, 3, $-2x\left(3x^2+5\right)$

55. $nu^{n-1}\frac{du}{dx}$

56. not zero

57. $5x^3 - x^2 + 7$, $4\left(5x^3 - x^2 + 7\right)^3\left(15x^2 - 2x\right)$

58. $2x-1$, $2x-1$, $-6(2x-1)^{-4}$, $\frac{1}{2}$

59. $(x-1)^{-4}$, $1-x^4$, $-6(x-1)^{-4} + 12x^3\left(1-x^4\right)^{-2}$

60. $\tan^2 x$, $\tan x$, $\tan x\sec^2 x$, $\tan x\left(2x\sec^2 + \tan x\right)$

61. $\left(\frac{1-\cos x}{1+\cos x}\right)^{-1/2}$, $\sin x$, $-\sin x$, $(1+\cos x)^{3/2}$

62. $\frac{x-2}{x+1}$, $\frac{x-2}{x+1}$, $x-2$, $15(x-2)^4$

OBJECTIVE B: Given parametric equations $x = f(t)$ and $y = g(t)$, find $\frac{dy}{dx}$ in terms of $\frac{dy}{dt}$ and $\frac{dx}{dt}$. Find $\frac{d^2y}{dx^2}$ in terms of t.

63. The equations $x = f(t)$ and $y = g(t)$, which expresses x and y in terms of t, are called ____________ equations. The variable t is called a ____________. From the chain rule, the derivative $\frac{dy}{dx}$ is given by $\frac{dy}{dx}$ = ____________.

64. Let $x = t^2 - 1$ and $y = \frac{1}{t}$. Then $\frac{dx}{dt}$ = ____________ and $\frac{dy}{dt}$ = ____________. It follows that $y' = \frac{dy}{dx}$ = ____________. To calculate the second derivative $\frac{d^2y}{dx^2}$ we first find $\frac{dy'}{dt}$ = ________________.

Then, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\_\_\_\_}$ = ______________.

65. If $x = t^2$ and $y = t^2 - 2t$, then $\frac{dx}{dt}$ = ____________ and $\frac{dy}{dt}$ = ____________. Thus $y' = \frac{dy}{dx}$ = ____________. Next, $\frac{d^2y}{dx^2} = \frac{\_\_\_\_}{\frac{dx}{dt}} = \frac{\_\_\_\_}{2t}$ = ____________. When $t = 2$, x = ________, y = __________ and $\frac{dy}{dx}$ = ____________.
Thus an equation of the line tangent to the curve at (4, 0) is ____________________________.

2.6 IMPLICIT DIFFERENTIATION

OBJECTIVE A: Compute first and second derivatives by the technique of implicit differentiation.

66. An equation involving the variables x and y is said to determine y ________________ as a function of x, say $y = f(x)$, provided that f satisfies the equation.

67. For instance, consider the equation $x^2 + y^2 = 2$. If we substitute $y = \sqrt{2 - x^2}$ into the equation we obtain $x^2 + \left(\sqrt{2 - x^2}\right)^2 = x^2 + (\_\_\_\_\_\_) =$ ____________, so the equation is satisfied. Similarly, if we substitute $y = -\sqrt{2 - x^2}$ into the equation $x^2 + \left(-\sqrt{2 - x^2}\right)^2 = x^2 + (\_\_\_\_\_\_) =$ ____________, and the equation is again satisfied. Therefore, each of the two functions $y = \sqrt{2 - x^2}$ and $y = -\sqrt{2 - x^2}$ is defined ____________ by the equation $x^2 + y^2 = 2$.

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63. parametric, parameter, $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

64. $2t$, $-\frac{1}{t^2}$, $-\frac{1}{2t^3}$, $\frac{3}{2}t^{-4}$, $\frac{dx}{dt}$, $\frac{3}{4t^5}$

65. $2t$, $2t - 2$, $1 - \frac{1}{t}$, $\frac{dy'}{dt}$, $\frac{1}{t^2}$, $\frac{1}{2t^3}$, 4, 0, $\frac{1}{2}$, $y = \frac{1}{2}(x - 4)$ or $2y - x + 4 = 0$

66. implicitly

67. $2 - x^2$, 2, $2 - x^2$, 2, implicitly

68. To calculate the derivative $\frac{dy}{dx}$ for $2xy - y^2 = 3$, differentiate both sides of the equation with respect to x and solve for $\frac{dy}{dx}$: $\frac{d}{dx}\left(2xy - y^2\right) = \frac{d}{dx}(3)$ or $\frac{d}{dx}(\_\_\_\_\_\_\_\_) - \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(3)$. Thus, $2\left(y + x\frac{dy}{dx}\right) - \_\_\_\_\_\_\_\_ = 0$ and solving for $\frac{dy}{dx}$, $\frac{dy}{dx} = \_\_\_\_\_\_\_\_\_\_\_\_$.

69. To calculate $\frac{d^2y}{dx^2}$ for $2xy - y^2 = 3$, differentiate both sides of the derivative equation $\frac{dy}{dx} = \frac{-y}{x-y}$ with respect to x: $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-y}{x-y}\right)$ (from Problem 68), or

$$\frac{d^2y}{dx^2} = \frac{-(x-y)\frac{dy}{dx} + y\frac{d}{dx}(\_\_\_\_\_\_)}{(x-y)^2} \quad \text{(quotient rule)}$$

$$= \frac{-(x-y)\frac{dy}{dx} + y(\_\_\_\_\_\_\_\_)}{(x-y)^2} = \frac{-x\frac{dy}{dx} + y\frac{dy}{dx} + y - y\frac{dy}{dx}}{(x-y)^2}.$$

Substitution of $\frac{-y}{x-y}$ for $\frac{dy}{dx}$ in the last equation gives

$$\frac{d^2y}{dx^2} = \frac{-x(\_\_\_\_\_\_\_\_) + y}{(x-y)^2} = \frac{-xy + y(\_\_\_\_\_\_\_\_)}{(x-y)^3} = \frac{\_\_\_\_\_\_\_\_}{(x-y)^3}.$$

OBJECTIVE B: Find the derivative of $g(x) = x^n$ when n is any rational number $n = \frac{p}{q}$.

70. If $g(x) = x^n$ with $n = \frac{1}{m}$ where m is a positive odd integer, then $g'(x) = \_\_\_\_\_\_\_\_\_\_\_\_$ for x satisfying $\_\_\_\_\_\_\_\_\_\_\_\_$.

71. If $g(x) = x^n$ with $n = \frac{1}{m}$, where m is a positive even integer, then $g'(x) = \_\_\_\_\_\_\_\_\_\_\_\_$ for x satisfying $\_\_\_\_\_\_\_\_\_\_\_\_$.

72. $\frac{d}{dx}\left(x^{1/9}\right) = \_\_\_\_\_\_\_\_$ provided x satisfies $\_\_\_\_\_\_\_\_\_\_\_\_$.

73. $\frac{d}{dx}\left(x^{1/6}\right) = \_\_\_\_\_\_\_\_$ provided x satisfies $\_\_\_\_\_\_\_\_\_\_\_\_$.

74. $\frac{d}{dx}\left(x^{3/5}\right) = \_\_\_\_\_\_\_\_$ provided x satisfies $\_\_\_\_\_\_\_\_\_\_\_\_$.

75. $\frac{d}{dx}\left(x^{-3/4}\right) = \_\_\_\_\_\_\_\_$ provided x satisfies $\_\_\_\_\_\_\_\_\_\_\_\_$.

---

68. $2xy$, $2y\frac{dy}{dx}$, $\frac{-y}{x-y}$

69. $x - y$, $1 - \frac{dy}{dx}$, $\frac{y}{x-y}$, $x - y$, $2yx - y^2$

70. nx^{n-1}, $x \neq 0$

71. nx^{n-1}, $x > 0$

72. $\frac{1}{9}x^{-8/9}, x \neq 0$

73. $\frac{1}{6}x^{-5/6}, x > 0$

74. $\frac{3}{5}x^{-2/5}$, $x \neq 0$

75. $-\frac{3}{4}x^{-7/4}$, $x > 0$

76. Let $y=\sqrt{\frac{x+3}{x-3}}$, so $y=u^{1/2}$ where $u=\frac{x+3}{x-3}$. Then $\frac{dy}{dx}=\frac{1}{2}u^{-1/2}\frac{du}{dx}$ whenever $u>0$. Thus

$$\frac{dy}{dx}=\frac{1}{2}(\underline{\qquad\qquad})^{-1/2}\frac{(x-3)(1)-(\underline{\qquad\qquad})}{(x-3)^2}=\frac{1}{2\sqrt{\frac{x+3}{x-3}}}\underline{\qquad\qquad} \text{ whenever } \frac{x+3}{x-3}>0.$$

OBJECTIVE C: Find the lines that are tangent and normal to a specified curve at a given point.

77. By direct substitution the point (1, 1) lies on the curve $xy=1$. Differentiating both sides with respect to x yields $y+\underline{\qquad\qquad}=0$. Solving for $\frac{dy}{dx}$ gives $\frac{dy}{dx}=\underline{\qquad\qquad}$. Thus, at (1, 1) the slope of the curve is $\left.\frac{dy}{dx}\right|_{(1,1)}=\underline{\qquad\qquad}$. Therefore, the tangent to the curve at the point (1, 1) is $y-1=\underline{\qquad\qquad}$.

78. For the curve in Problem 77, the slope of the normal is __________ at the point (1, 1). The normal is therefore given by the equation $y-1=\underline{\qquad\qquad}$.

2.7 RELATED RATES

In this section we consider problems that ask us to find the rate at which some variable quantity changes when we know the rate at which another quantity related to it changes. Examples abound for problems of this sort. For instance, the rate of production of a certain commodity may depend upon its rate of sales; the rate of increase or decrease in the water level of a dam or reservoir is essential information to a public utility serving the demands of a growing population; the rate at which oil may be spreading on the sea surface from a stricken tanker depends on the rate at which it may be leaking and so forth.

OBJECTIVE: Solve relate rates problems by using the following strategy, as presented in the text:

1.	Draw a picture and name the variables and constants. Use t for time. Assume all variables are differentiable functions of t.
2.	Write down the numerical information (in terms of the symbols you have chosen).
3.	Write down what you are asked to find (usually a rate, expressed as a derivative).
4.	Write an equation that relates the variables. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variable whose rate you know.
5.	Differentiate with respect to t to express the rate you want in terms of the rate and variables whose values you know.
6.	Evaluate.

79. A plane flying at 1 mile altitude is 2 miles distant from an observer, measured along the ground, and flying directly away from the observer at 400 mph. How fast is the angle of elevation changing?
Solution. We carry out the steps of the basic strategy.

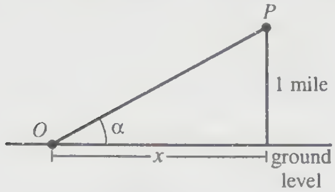
---

76. $\frac{x+3}{x-3}$, $(x+3)(1)$, $\frac{-6}{(x-3)^2}$

77. $x\frac{dy}{dx}$, $-\frac{y}{x}$, -1, $(-1)(x-1)$ or 1 - x

78. 1, $(x-1)$

STEP 1: Pictures and variables:
We picture the plane flying in the coordinate plane using the positive x-axis as the ground pointing in the direction of flight. Let 0 denote the position of the observer at a distance x units (measured along the ground) from the plane P as shown in the figure below. Let α denote the angle of elevation.



STEP 2: Numerical information: At the time in question, $x =$ ________ mi, $y = 1$ mi, $\frac{dx}{dt} = 400$ mi / hr.

STEP 3: To find: the rate ________________.

STEP 4: How the variables are related: The angle α satisfies the equation $\tan\alpha =$ ______________, which holds for all time t.

STEP 5: Differentiate with respect to t: $\sec^2\alpha \cdot ($__________$) = -\frac{1}{x^2} \cdot ($__________$)$, or since

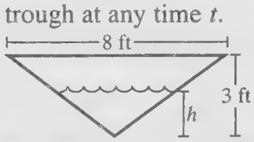
$\sec^2\alpha = 1 + \tan^2\alpha =$ ______________, we solve to find $\frac{d\alpha}{dt} = ($__________$)\frac{dx}{dt}$.

STEP 6: Evaluate: When $x = 2$ and $\frac{dx}{dt} = 400$, $\frac{d\alpha}{dt} =$ ______________ radians per hour, or ____________ rad/sec, or

____________ deg/sec. Notice that the angle of elevation is decreasing because $\frac{d\alpha}{dt}$ is ______________.

80. A trough 10 ft long has a cross section that is an isosceles triangle 3 ft deep and 8 ft across. If water flows in at the rate of 2 ft^3/min, how fast is the surface rising when the water is 2 ft deep?
Solution. W carry out the steps of the basic strategy.

STEP 1: Pictures and variables: We draw a picture of a partially filled trough. A cross section of the trough is shown in the figure below. In the figure h denotes the depth of the water and b its width across the trough at any instant t. Thus, b and h are both functions of __________. We let V denote the volume of water in the trough at any time t.



STEP 2: Numerical information: At the time in question, $h = 2$ ft, $\frac{dV}{dt} =$ ______________ ft^3 / min.

STEP 3: To find: ________________.

---

79. 2, $\frac{d\alpha}{dt}$, $\frac{1}{x}$, $\frac{d\alpha}{dt}$, $\frac{dx}{dt}$, $1 + \frac{1}{x^2}$, $-\frac{1}{x^2}\left(\frac{x^2}{1+x^2}\right)$, -80, $\frac{-80}{3600}$, (approx) -1.3, negative

80. t, 2, $\frac{dh}{dt}$, $\frac{1}{2}bh$, h, $\frac{8}{3}h$, $\frac{40}{3}h^2$, $\frac{80}{3}h\frac{dh}{dt}$, $\frac{3}{80}$

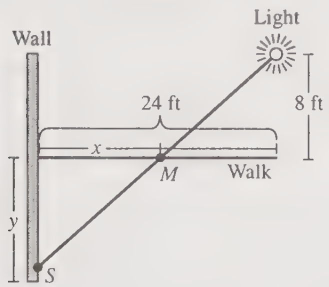
STEP 4: How the variables are related: At any instant of time the volume of water in the trough is given by the formula, $V = 10(\_\_\_\_\_\_\_\_)$. Since the formula for V involves both the variables b and h, we need to write down a formula relating these variables. From the geometry of similar triangles in the figure we have $\frac{b}{\_\_\_\_} = \frac{8}{3}$ or, $b =$ ________. Thus $V =$ ________.

STEP 5: Differentiate with respect to t: $\frac{dV}{dt} =$ ________.

STEP 6: Evaluate. Solving for $\frac{dh}{dt}$ when $\frac{dV}{dt} = 2$ and $h = 2$ gives $\frac{dh}{dt} =$ ________ ft / min.

81. A walk is perpendicular to a long wall, and a woman strolls along it away from the wall at the rate of 3 ft / sec. There is a light 8 ft from the walk and 24 ft from the wall. How fast is her shadow moving along the wall when she is 20 ft from the wall?
Solution. We carry out the steps of the basic strategy.

STEP 1: Picture and variables: The situation is pictured in the figure below. Here M denotes the position of the woman at a distance x units from the wall, S denotes the position of her shadow on the wall at a distance y units from where the wall and the walk intersect, and L denotes the position of the light.



STEP 2: Numerical information: $x = 20$ ft, ________ $= 3$ ft / sec.

STEP 3: To find: ________.

STEP 4: How are the variables related: From similar triangles we can establish a relationship between the variables x and y, $\frac{x}{y} = \frac{\_\_\_\_}{8}$, or $8x =$ ________. This equation is valid at any instant of time t.

STEP 5: Differentiate both sides with respect to t: $8\frac{dx}{dt} = (\_\_\_\_\_\_\_\_)\frac{dy}{dt} -$ ________.

STEP 6: Evaluate. When $x = 20$, $y = \frac{8(20)}{\_\_\_\_} =$ ________, so substitution into the previous derivative equation yields $8 \cdot 3 =$ ________ $- 120$, or $\frac{dy}{dt} =$ ________ ft / sec.

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81. $\frac{dx}{dt}$, $\frac{dy}{dt}$, $24 - x$, $(24 - x)y$, $24 - x$, $y\frac{dx}{dt}$, $24 - 20$, 40, $4\frac{dy}{dt}$, 36

CHAPTER 2 SELF-TEST

1. Use the three algebraic steps to find the slope of the curve $y = x^3 - 2x + 5$ at a point (x, y) on the curve.

2. Write an equation of the tangent line to the curve in Problem 1 at the point when $x = -2$.

3. Find $\frac{dy}{dx}$:

   (a) $y = (2x^3 - x^2 + 7x + 3)^6$ (b) $y = (x^2 - 9)(3x^5 + 7x)$

   (c) $y = \frac{3}{x^4 + 1}$ (d) $y = \frac{x^2 - 1}{3x + 1}$

4. Find $\frac{d^2y}{dx^2}$:

   (a) $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$ (b) $y = (2x^3 - 11)(x^2 - 3)$

   (c) $y = \cot 3x$ (d) $y = 3\sin^2 5x - \sec x$

5. Find $\frac{dy}{dx}$:

   (a) $y = x^{4/3} - 5x^{4/5}$ (b) $y = \sqrt{x + \frac{1}{x}}$

   (c) $y = \frac{1}{2x - \sqrt{x^2 - 1}}$ (d) $y = x\cos(5x - 2)$

6. Find an equation of the tangent line to the graph of $y = \sqrt{1 - x^2}$ when $x = \frac{1}{2}$.

7. A particle moves along a horizontal line (positive to the right) according to the law $s = t^3 - 6t^2 + 2$. During which intervals of time is the particle moving to the right and during which is it moving to the left? What is the acceleration and the velocity when $t = 2.3$?

8. Calculate the instantaneous rate of change of the volume of a sphere with respect to its radius when the radius is 3 cm.

9. Given the parametric equations $x = t^2 - 1$ and $y = t + 1$

   (a) Express dx and dy in terms of t and dt,

   (b) Find $\frac{d^2y}{dx^2}$ in terms of t,

   (c) Find an equation of the tangent line to the curve at the point for which $t = 1$.

10. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x + y^2 = xy$.

11. Find an equation of the tangent line to the curve $x^3 + 3xy^3 + xy^2 = xy$ at the point $(1, -1)$.

12. A photographer is televising a 100-yard dash from a position 10 yards from the track in line with the finish line. When the runners are 10 yards from the finish line, the camera is turning at the rate $\frac{3}{5}$ rad / sec. How fast are the runners moving then?

13. A swimming pool is 40 ft long, 20 ft wide, 8 ft deep at the deep end, and 3 ft deep at the shallow end, the bottom being rectangular. If the pool is filled by pumping water into it at the rate of 40 cu. ft / min, how fast is the water level rising when it is

(a) 3 ft deep at the deep end? (b) 6 ft deep at the deep end?

14. A guy wire is to pass from the top of a pole 36 ft high to an anchorage on the ground 27 ft from the base of the pole. One end of the wire is made fast to the anchorage, and a man climbs the pole with the wire, keeping it taut. If he climbs 2 ft / sec, how fast is he playing out the wire when he reaches the top of the pole?

SOLUTIONS TO CHAPTER 2 SELF-TEST

1. STEP 1: $f(x+h)=(x+h)^3-2(x+h)+5=x^3+3x^2h^2+3xh^2+h^3-2x-2h+5$

STEP 2: Subtracting $f(x+h)-f(x)$:
$f(x+h)-f(x)=3x^2h+3xh^2+h^3-2h$
Dividing by h yields, $\frac{f(x+h)-f(x)}{h}=3x^2+3xh+h^2-2$

STEP 3: As h tends to zero $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=3x^2-2$.

2. The derivative is $f'(x)=3x^2-2$. When $x=-2$, $y=(-2)^3-2(-2)+5=1$, and the slope $f'(-2)$ is $m=3(-2)^2-2=10$. Thus, $y=1+10(x+2)$ or $y=10x+21$ is an equation of the tangent line.

3. (a) $\frac{dy}{dx}=6\left(2x^3-x^2+7x+3\right)^5\left(6x^2-2x+7\right)$

(b) $\frac{dy}{dx}=2x\left(3x^5+7x\right)+\left(x^2-9\right)\left(15x^4+7\right)=21x^6-135x^4+21x^2-63$

(c) $\frac{dy}{dx}=\frac{-3\left(4x^3\right)}{\left(x^4+1\right)^2}$

(d) $\frac{dy}{dx}=\frac{(3x+1)(2x)-\left(x^2-1\right)(3)}{(3x+1)^2}=\frac{3x^2+2x+3}{(3x+1)^2}$

4. (a) $\frac{dy}{dx}=x^2+x-6,\ \frac{d^2y}{dx^2}=2x+1$

(b) $\frac{dy}{dx}=6x^2\left(x^2-3\right)+\left(2x^3-11\right)(2x)=10x^4-18x^2-22x,\ \frac{d^2y}{dx^2}=40x^3-36x-22$

(c) $\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\cos 3x}{\sin 3x}\right)=\frac{(\sin 3x)(-3\sin 3x)-(\cos 3x)(3\cos 3x)}{\sin^2 3x}=\frac{-3\sin^2 3x-3\cos^2 3x}{\sin^2 3x}=-3\csc^2 3x$
$\frac{d^2y}{dx^2}=(-3)(2)(\csc 3x)\cdot\frac{d}{dx}(\csc 3x)=(-6\csc 3x)(-3\csc 3x\cot 3x)=18\csc^2 3x\cot 3x$

(d) $\frac{dy}{dx} = 30\sin 5x\cos 5x - \sec x\tan x = 15\sin 10x - \sec x\tan x$

$\frac{d^2y}{dx^2} = 150\cos 10x - (\sec x\tan x)(\tan x) - (\sec x)(\sec^2 x) = 150\cos 10x - (\sec x)(\tan^2 x + \sec^2 x)$

5. (a) $\frac{dy}{dx} = \frac{4}{3}x^{1/3} - 4x^{-1/5}$

(b) $\frac{dy}{dx} = \frac{1}{2}\left(x + \frac{1}{x}\right)^{-1/2} \cdot \frac{d}{dx}\left(x + \frac{1}{x}\right) = \frac{1}{2}\left(x + \frac{1}{x}\right)^{-1/2}\left(1 - \frac{1}{x^2}\right)$

(c) $\frac{dy}{dx} = -\left(2x - \sqrt{x^2 - 1}\right)^{-2}\left[2 - \frac{1}{2}\left(x^2 - 1\right)^{-1/2} \cdot 2x\right] = -\left(2x - \sqrt{x^2 - 1}\right)^{-2}\left(2 - \frac{x}{\sqrt{x^2 - 1}}\right)$

(d) $\frac{dy}{dx} = \cos(5x - 2) + x[-\sin(5x - 2) \cdot 5] = \cos(5x - 2) - 5x\sin(5x - 2)$

6. $y' = \frac{1}{2}\left(1 - x^2\right)^{-1/2}(-2x) = -x\left(1 - x^2\right)^{-1/2}$, when $x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$ and $y' = -\frac{1}{\sqrt{3}}$ so that

$y = \frac{\sqrt{3}}{2} + \left(-\frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{2}\right)$ or $\sqrt{3}y + x - 2 = 0$ is an equation of the tangent line.

7. $\frac{ds}{dt} = 3t^2 - 12t = 3t(t - 4)$; $\frac{d^2s}{dt^2} = 6t - 12$

The particle is moving to the right when $\frac{ds}{dt} > 0$ so $t > 4$ or $t < 0$; it is moving to the left when

$0 < t < 4$ and $\frac{ds}{dt} < 0$. At $t = 2.3$,

$\left.\frac{ds}{dt}\right|_{2.3} = 6(2.3)^2 - 12(2.3) = -11.73$, velocity

$\left.\frac{d^2s}{dt^2}\right|_{2.3} = 6(2.3) - 12 = 1.8$, acceleration

8. The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$, where r is the radius. We seek the value of $\frac{dV}{dx}$ when $r = 3$.

Thus, $\frac{dV}{dx} = V'(r) = 4\pi r^2$, so that $V'(3) = 36\pi$.

9. (a) $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 1$, so $dx = 2t\,dt$ and $dy = dt$

(b) $\frac{dy}{dx} = \frac{1}{2t}$ so that $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{-1}{2t^2}}{2t} = -\frac{1}{4t^3}$

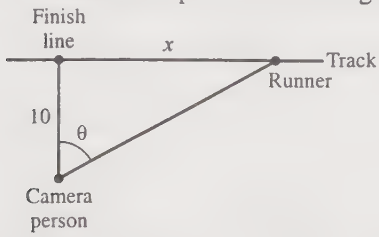
(c) When $t = 1$, $x = 0$, $y = 2$, and $\frac{dy}{dx} = \frac{1}{2}$; thus $y - 2 = \frac{1}{2}(x - 0)$, $y = \frac{1}{2}x + 2$ is an equation of the tangent line.

10. Differentiating implicitly, $1+2yy'=y+xy'$ or $y'=\dfrac{y-1}{2y-x}$

$$\frac{d^2y}{dx^2}=\frac{(2y-x)(y')-(y-1)(2y'-1)}{(2y-x)^2}=\frac{(2-x)y'+(y-1)}{(2y-x)^2}=\frac{(2-x)\frac{y-1}{2y-x}+(y-1)}{(2y-x)^2}=\frac{2(y-1)(y-x+1)}{(2y-x)^3}.$$

11. Since $1^3+3(1)(-1)^3+1(-1)^2=1(-1)$ is true, the point $(1,-1)$ is on the curve. Differentiating implicitly, $3x^2+3y^3+9xy^2y'+y^2+2xyy'=y+xy'$, so evaluation at $(1,-1)$ yields $3-3+9y'+1-2y'=-1+y'$, or $y'=-\dfrac{1}{3}$. Thus an equation of the tangent line is given by $y=-1+\left(-\dfrac{1}{3}\right)(x-1)$ or $x+3y=-2$.

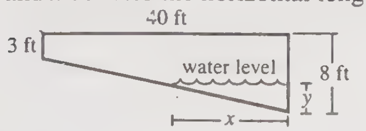
12. The situation is pictured in the figure below.



Thus, $\tan\theta=\dfrac{x}{10}$, or $x=10\tan\theta$. $\dfrac{dx}{dt}=10\sec^2\theta\dfrac{d\theta}{dt}$. Now, when $x=10$ yds, $\theta=\dfrac{\pi}{4}$, and $\dfrac{d\theta}{dt}=\dfrac{3}{5}$ rad/ sec.

Hence, $\left.\dfrac{dx}{dt}\right|_{x=10}=10\left(\sec^2\dfrac{\pi}{4}\right)\left(\dfrac{3}{5}\right)=10\left(\sqrt{2}\right)^2\left(\dfrac{3}{5}\right)=12$ yd / sec.

13. A vertical cross-section of the pool is pictured in the figure below: y denotes the depth of the water at any time t, and x denotes the horizontal length of the water in the bottom of the pool.

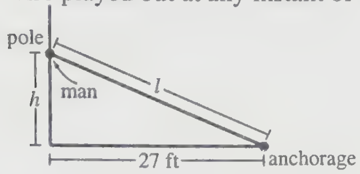


(a) When $y<5$, we have from the geometry of similar triangles in the figure that, $\dfrac{x}{40}=\dfrac{y}{5}$ or $x=8y$. The volume of water in the pool is given by $V=\dfrac{1}{2}x\cdot y\cdot 20=80y^2$. Hence $\dfrac{dV}{dt}=160y\dfrac{dy}{dt}$, and since $\dfrac{dV}{dt}=40$ is given, solving for $\dfrac{dy}{dt}$ yields $\left.\dfrac{dy}{dt}\right|_{y=3}=\dfrac{40}{160(3)}=\dfrac{1}{12}$ ft/ min.

(b) When $y>5$, the total volume of water is given by $V=\dfrac{1}{2}(40)(5)(20)+(40)(20)(y-5)=800y-2000$.

Hence, $\dfrac{dV}{dt}=800\dfrac{dy}{dt}$, and since $\dfrac{dV}{dt}=40$ is given, solving for $\dfrac{dy}{dt}$ yields $\left.\dfrac{dy}{dt}\right|_{y=6}=\dfrac{40}{800}=\dfrac{1}{20}$ ft / min.

14. The situation is pictured below, where h is the height of the man above the ground and ℓ is the length of guy wire played out at any instant of time t.



We want to find $\frac{d\ell}{dt}$ when $h = 36$. Now, $\ell^2 = h^2 + (27)^2$, and differentiation with respect to t gives $2\ell\frac{d\ell}{dt} = 2h\frac{dh}{dt}$. When $h - 36$, we have $\ell = \sqrt{(36)^2 + (27)^2} = 9\sqrt{4^2 + 3^2} = 45$. thus, for $\frac{dh}{dt} = 2$, $\left.\frac{d\ell}{dt}\right|_{h=36} = \frac{36}{45} \cdot 2 = \frac{72}{45} = \frac{8}{5}$ ft / sec.

NOTES.