Chapter 1: Limits and Continuity

1.1 RATES OF CHANGE AND LIMITS

OBJECTIVE A: Find the average rate of change of a function $y = f(x)$ over an interval $[x_1, x_2]$.

1. The average rate of change of $y = f(x)$ over $[x_1, x_2]$ is the change in y, $\Delta y =$ ______________ divided by $\Delta x =$ ______________, the length of the interval over which the change occurred.

2. Geometrically, an average rate of change is a ____________________.

3. The average rate of change of $f(x) = x^2 - x + 1$ over [0, 2] is
$$\frac{\Delta y}{\Delta x} = \frac{f(2) - \_\_\_\_\_}{2-0} = \frac{(2^2 - 2 + 1) - \_\_\_\_\_\_\_\_\_}{2} = \_\_\_\_\_\_\_\_.$$

OBJECTIVE B: Know the informal definition of the limit $\lim_{x \to x_0} f(x) = L$.

4. According to the informal definition, we write $\lim_{x \to x_0} f(x) = L$ if the values of __________ approach the value L as x approaches __________.

5. If f is the identity function $f(x) = x$, then for any value of x_0, $\lim_{x \to x_0} f(x) =$ _______.

6. If f is the constant function $f(x) = k$, then for any value of x_0, $\lim_{x \to x_0} f(x) =$ _______.

OBJECTIVE C: Find limits of elementary functions by substitution, if possible.

7. $\lim_{x \to \frac{1}{4}} (8x - 3) =$ _______ $- 3 =$ _______.

8. $$\lim_{x \to \frac{1}{2}} \frac{6x^2 + \frac{1}{2}}{4x - 1} = \frac{6\left(\frac{1}{4}\right) + \frac{1}{2}}{(\_\_\_\_\_)} = \_\_\_\_\_\_\_.$$

9. $$\lim_{x \to \frac{\pi}{4}} (\cos x)(2 \sin x) = \_\_\_\_\_\_\_\left(\frac{2}{\sqrt{2}}\right) = \_\_\_\_\_\_\_.$$

10. A function may fail to have a limit as x approaches x_0 because it .

    a. ____,

    b. ____,

    c. ____.

---

1. $f(x_2) - f(x_1)$, $x_2 - x_1$
2. secant slope
3. $f(0)$, $(0^2 - 0 + 1)$, $\frac{2}{2} = 1$
4. $f(x)$, x_0
5. x_0
6. k
7. 2, −1
8. 2 − 1, 2
9. $\frac{1}{\sqrt{2}}$, 1
10. a. jumps b. grows too large c. oscillates too much

OBJECTIVE D: Given a function $y = f(x)$, a positive number ε, a point x_0, and a target value L, determine an interval about x_0 in which we must hold x to be sure that $y = f(x)$ lies within ε units of L.

11. Suppose $y = -3x+1$, $\varepsilon = 1$, $x_0 = 2$, $L = -5$. We must know in what interval of values to hold x to make to make y satisfy the inequality $|y-(-5)| < 1$.

Substituting for y, we find $|(-3x+1)-(-5)| < 1$ or __________ < 1. Thus, $|-3(x-2)| < 1$ or $|x-2| <$ __________. Thus, $-\frac{1}{3} < x-2 < \frac{1}{3}$, or x satisfies the inequality __________ $< x <$ __________.

In summary, the interval $|x-2| < \frac{1}{3}$ contains the values near $x_0 = 2$ to which x must be held to ensure $y = -3x+1$ is within $\varepsilon = 1$ of $L = -5$.

OBJECTIVE E: Write the formal definition of the limit of a function $f(x)$ as x approaches a number x_0.

12. Let f be a function defined on an open interval containing the point x_0, except possibly at x_0 itself. Then the limit of f as x approaches x_0 is L, written ____________________, if, given any number ε, there is a corresponding positive number δ such that ____________________ holds whenever ____________________.

13. As an application of the definition, consider the limit of the function $f(x) = 3-2x$ as x approaches 5. The limit is $L = -7$. To show this it is required to establish that: For any positive number ε, there is a positive number δ such that ____________________ when $0 <$ ____________________ $< \delta$.

Now, $|(3-2x)-(-7)| = 2 \cdot$ ____________________. Thus, $2|x-5| < \varepsilon$ provided $|x-5| <$ ____________________.
Therefore, if $\delta =$ ____________________, then $|(3-2x)-(-7)| < \varepsilon$ whenever ____________________.
That is, $\lim_{x\to 5}(3-2x) = -7$.

OBJECTIVE F: Given a function $f(x)$, a point x_0, and a positive number ε, find a number $\delta > 0$ such that for all x $0 < |x-x_0| < \delta$ implies $|f(x)-L| < \varepsilon$, where $L = \lim_{x\to x_0} f(x)$.

14. For the limit $\lim_{x\to 4}\sqrt{2x+1} = 3$, find a $\delta > 0$ that works for $\varepsilon = 1$.

Solution:

STEP 1: We first find an interval about $x_0 = 4$ on which the inequality $\left|\sqrt{2x+1}-3\right| < 1$ holds for $x \neq 4$.

$$\left|\sqrt{2x+1}-3\right| < 1$$

$\Leftrightarrow$	__________	$<$	$\sqrt{2x+1}-3$	$<$	__________
$\Leftrightarrow$	2	$<$	__________	$<$	__________
$\Leftrightarrow$	4	$<$	__________	$<$	16
$\Leftrightarrow$	3	$<$	$2x$	$<$	15
$\Leftrightarrow$	__________	$<$	x	$<$	__________.

---

11. $|-3x+6|, \frac{1}{3}, \frac{5}{3}, \frac{7}{3}$

12. $\lim_{x\to x_0} f(x) = L$, $|f(x)-L| < \varepsilon$, $0 < |x-x_0| < \delta$

13. $|(3-2x)-(-7)| < \varepsilon$, $|x-5|$, $|x-5|$, $\frac{\varepsilon}{2}$, $\frac{\varepsilon}{2}$, $0 < |x-5| < \delta$

The inequality holds for all x in the open interval $\left(\frac{3}{2}, \frac{15}{2}\right)$.

STEP 2: We now find an interval centered at 4. The distance from 4 to the nearest endpoint of $\left(\frac{3}{2}, \frac{15}{2}\right)$ is ____________. If we take $\delta =$ ________ or any smaller positive number, the inequality $0 < |x-4| < \delta$ will automatically place x between $\frac{3}{2}$ and $\frac{15}{2}$ to make the inequality ________________ hold.

1.2 FINDING LIMITS AND ONE-SIDED LIMITS

OBJECTIVE A: Specify the five important limit rules related to the arithmetic operations, as stated in Theorem 1 of the text.

15. Sum Rule: ____

16. Difference Rule: ____

17. Product Rule: ____

18. Constant Multiple Rule: ____

19. Quotient Rule: ____

____

OBJECTIVE B: Evaluate limits $\lim_{x\to c} f(x)$ when $f(x)$ is a sum, difference, product, or quotient of polynomials.

20. To find the limit as x approaches c of any polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_o$ you simply ______________ the number c for ______________ thus evaluating $f(c)$. Hence, $\lim_{x\to -1}(2x^3 + x^2 - 4x - 3) =$ ______________ $=$ ______.

21. $\lim_{x\to 1} \frac{x^3 - 2x}{x^2 + 3} = \frac{\_\_\_\_}{1+3} =$ ________.

---

14. $-1, 1, \sqrt{2x+1}, 4, 2x+1, \frac{3}{2}, \frac{15}{2}, \frac{5}{2}, \frac{5}{2}, \left|\sqrt{2x+1} - 3\right| < 1$

15. $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

16. $\lim_{x\to c}[f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$

17. $\lim_{x\to c} f(x)\cdot g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$

18. $\lim_{x\to c} k\cdot f(x) = k \lim_{x\to c} f(x)$

19. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ provided that $\lim_{x\to c} g(x)$ is not zero

20. substitute, x, $2(-1)^3 + (-1)^2 - 4(-1) - 3$, 0

21. $1-2$, $-\frac{1}{4}$

22. $\lim_{x\to -2}(4+3x)(x^2-x+1) = \lim_{x\to -2}(4+3x)$ __________ = ________ $(4+2+1) = -14$.

23. $\lim_{x\to 4}\dfrac{x^2+x-2}{x^2-1} = \dfrac{\_\_\_\_\_\_}{\lim_{x\to 4}(x^2-1)} = \dfrac{18}{\_\_\_}$.

OBJECTIVE C: Evaluate limits of functions when the denominator is zero at the limit point c by canceling a common factor, or by creating and canceling a common factor.

24. $\lim_{x\to 1}\dfrac{x^3+x^2-3x+1}{x-1} \neq \dfrac{\lim_{x\to 1}(x^3+x^2-3x+1)}{\_\_\_\_\_\_\_\_}$ because the limit of the denominator is __________.

However, $\dfrac{x^3+x^2-3x+1}{x-1} = \dfrac{(x-1)(\_\_\_\_)}{x-1}$,

so that $\lim_{x\to 1}\dfrac{x^3+x^2-3x+1}{x-1} = \lim_{x\to 1}$ __________ = ________.

25. $\lim_{h\to 0}\dfrac{(1+h)^3-1}{h} = \lim_{h\to 0}\dfrac{(\_\_\_\_\_\_\_\_)-1}{h}$

$= \lim_{h\to 0}\dfrac{\_\_\_\_\_\_\_\_}{h}$

$= \lim_{h\to 0}$ __________ = ________.

26. $\lim_{x\to 9}\dfrac{x^2-81}{3-\sqrt{x}} = \lim_{x\to 9}\dfrac{x^2-81}{3-\sqrt{x}}\cdot\dfrac{\_\_\_\_\_\_}{3+\sqrt{x}}$

$= \lim_{x\to 9}\dfrac{(x^2-81)(3+\sqrt{x})}{(\_\_\_\_\_\_)}$

$= \lim_{x\to 9}\dfrac{(x-9)(\_\_\_\_\_\_)(3+\sqrt{x})}{9-x}$

$= \lim_{x\to 9} -(\_\_\_\_\_\_)(\_\_\_\_\_\_) =$ ________.

OBJECTIVE D: State and use the Sandwich Theorem for limits.

27. If $g(x) \le f(x) \le h(x)$ for $x \ne c$ over some interval containing c, and if $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then

______________.

28. For $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ it is known that $1 \le \dfrac{\tan x}{x} \le \dfrac{1}{\cos x}$. Therefore, since $\lim_{x\to 0}\cos x =$ ________ we have,

$\lim_{x\to 0}\dfrac{\tan x}{x} =$ ________.

---

22. $\lim_{x\to -2}(x^2-x+1)$, $(4-6)$

23. $\lim_{x\to 4}(x^2+x-2)$, 15

24. $\lim_{x\to 1}(x-1)$, 0, x^2+2x-1, x^2+2x-1, 2

25. $1+3h+3h^2+h^3$, $3h+3h^2+h^3$, $3+3h+h^2$, 3

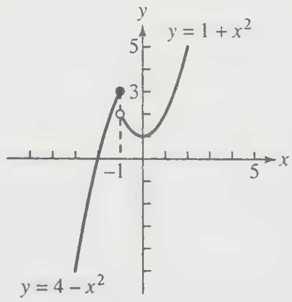
26. $3+\sqrt{x}$, $9-x$, $x+9$, $x+9$, $3+\sqrt{x}$, $-(18)(6) = -108$

27. $\lim_{x\to c} f(x) = L$

28. 1, 1

OBJECTIVE E: For elementary functions $y = f(x)$, find the right-hand and left-hand limits as x approaches a, and from these determine if $\lim_{x\to a} f(x)$ exists.

29. Consider the function defined by $f(x) = \begin{cases} 4-x^2 & \text{if } x \le -1 \\ 1+x^2 & \text{if } x > -1 \end{cases}$



The graph is shown above right.

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}$ __________ = __________,

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}$ __________ = __________,

Since $\lim_{x\to -1^+} f(x) \neq \lim_{x\to -1^-} f(x)$, the limit $\lim_{x\to -1} f(x)$ ________ exist.

30. $\lim_{x\to 1^-} \frac{5x^2-7x+2}{x^2+x-2} = \lim_{x\to 1^-}$ ______________ (factor)

$= \lim_{x\to 1^-} \frac{5x-2}{\_\_\_\_}$

$= \frac{5-2}{\_\_\_\_}$

$=$ ______.

In this problem you *must* factor the numerator and denominator first and cancel the term $(x-1)$ *before* calculating the limit because ____________________ is zero in the original expression: $\lim_{x\to 1^-} (x^2+x-2) = 1^2+1-2 = 0$. Note also that the limit as $x \to 1^+$ equals 1.

OBJECTIVE F: Evaluate limits of trigonometric functions by making use of appropriate trigonometric identities and the limit theorems.

31. One of the most useful facts in calculus is that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} =$ ________. For this limit the angle θ must be measured in ________________.

32. To find $\lim_{x\to 0} \frac{\sin 5x}{3x}$ first substitute $\theta = 5x$, and note that $\theta \to 0$ as $x \to 0$. Then the limit becomes, $\lim_{x\to 0} \frac{\sin 5x}{3x} = \lim_{\theta\to 0}$ __________

$=$ __________ $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \frac{5}{3}($______$) =$ ______.

---

29. $1+x^2$, 2, $4-x^2$, 3, does not

30. $\frac{(5x-2)(x-1)}{(x+2)(x-1)}$, $x+2$, $1+2$, 1, the limit of the denominator

31. 1, radians

32. $\frac{\sin\theta}{3\left(\frac{\theta}{5}\right)}$, $\frac{5}{3}$, 1, $\frac{5}{3}$

33. $\lim_{x\to 0} \frac{1-\cos x}{1-\sin x} = \frac{\lim_{x\to 0}(1-\cos x)}{\_\_\_\_\_\_} = \frac{(1-\_\_\_\_\_\_)}{(1-0)} = \_\_\_\_\_\_.$

34. $\lim_{h\to 0} \frac{h}{\sin h} = \lim_{h\to 0} \frac{1}{\_\_\_\_\_\_} = \frac{1}{\lim_{h\to 0} \frac{\sin h}{\_\_\_}} = \frac{1}{\_\_\_\_\_\_} = \_\_\_\_\_\_.$

1.3 LIMITS INVOLVING INFINITY

OBJECTIVE A: Calculate the limit of $f(x)$ as x approaches $+\infty$ or $-\infty$, whenever the limit exists.

35. $\lim_{x\to\infty} \frac{5x^3}{1+3x-2x^3} = \lim_{h\to 0} \frac{\frac{5}{h^3}}{\_\_\_\_\_\_} = \lim_{h\to 0} \frac{5}{\_\_\_\_\_\_} = \_\_\_\_\_\_.$

36. $\lim_{x\to\infty} \frac{1-3x^2}{4x^3+2x-5} = \lim_{x\to\infty} \frac{\_\_\_\_\_\_}{4+\left(\frac{2}{x^2}\right)-\left(\frac{5}{x^3}\right)} = \frac{0-\_\_\_\_}{4+0-0} = \_\_\_\_\_\_.$

37. $\lim_{x\to-\infty} \frac{3-x^2}{6x+1} = \lim_{x\to-\infty} \frac{\left(\frac{3}{x}\right)-x}{\_\_\_\_\_\_} = \_\_\_\_\_\_.$

OBJECTIVE B: Analyze the behavior of the graph of a function $y = f(x)$ as $x \to \pm\infty$.

38. A line $y = b$ is a __________ asymptote of the graph of a function $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or __________.

39. A line $x = a$ is a __________ asymptote of the graph of a function $y = f(x)$ if either $\lim_{x\to a^-} f(x) = \pm\infty$ or __________.

40. Consider the function $y = \frac{2x-5}{x^2-2x-3} + 4$. If we divide the top and bottom of the rational expression by x, we obtain $y =$ __________. As $x \to \pm\infty$, the numerator of the rational expression approaches ________ and the denominator goes to ________. So the rational expression approaches ________. Thus the function y has a ________ asymptote at ________. Furthermore, in the original expression for y the denominator of the rational expression can be factored to obtain $y =$ ________, from which it is evident that y has ________ asymptotes at ________ and ________.

---

33. $\lim_{x\to 0}(1-\sin x)$, 1, 0

34. $\frac{\sin h}{h}$, h, 1, 1

35. $1+\frac{3}{h}-\frac{2}{h^3}$, h^3+3h^2-2, $-\frac{5}{2}$

36. $\frac{1}{x^3}-\frac{3}{x}$, 0, 0

37. $6+\frac{1}{x}$, $+\infty$

38. horizontal, $\lim_{x\to-\infty} f(x) = b$

39. vertical, $\lim_{x\to a^+} f(x) = \pm\infty$

40. $\frac{2-\frac{5}{x}}{x-2-\frac{3}{x}}+4$, 2, $\pm\infty$, 0, horizontal, $y = 4$, $\frac{2x-5}{(x+1)(x-3)}+4$, vertical, $x=-1, x=3$

OBJECTIVE C: Find "infinite limits" such as $\lim_{x\to a} f(x)=\infty$, $\lim_{x\to a^-} f(x)=\infty$, $\lim_{x\to a^+} f(x)=-\infty$, and so forth.

41. $\lim_{x\to 1^+} \frac{|x|+1}{x^2-1} = \lim_{x\to 1^+} \frac{|x|+1}{(x+1)(\_\_\_\_\_\_)} = \lim_{x\to 1^+}$ __________, since $|x|+1=$ ________ for x near 1. Therefore, $\lim_{x\to 1^+} \frac{|x|+1}{x^2-1} =$ __________.

OBJECTIVE D: Find an end behavior model for a function $y=f(x)$.

42. Consider the rational function $y=\frac{x^2-3x-1}{x-3}$. If we divide $x-3$ into x^2-3x-1 we obtain $y=x-$ ________. Thus if x is large ($\pm\infty$) the curve that behaves like $u=$ ________ is both a __________ and __________ end behavior model for y.

43. Consider the function $y=x\sqrt{x^2+1}$. Since for large x, $\sqrt{x^2+1}\approx\sqrt{\_\_\_\_}=x$, one might expect that $u=x\cdot$ ______ $=$ ______ is a right behavior model for y. To verify this, note that

$$\lim_{x\to\infty}\frac{\_\_\_\_}{x^2} = \lim_{x\to\infty}\frac{\_\_\_\_}{1} = \frac{\lim_{x\to\infty}\_\_\_\_}{1} = \frac{\_\_\_\_}{1} = \_\_\_\_\_\_.$$

1.4 CONTINUITY

OBJECTIVE A: Specify the test for a function f to be continuous at an interior point $x=c$ of its domain.

44. The three conditions that must be satisfied if the function f is to be continuous at the point $x=c$ are that __________ exists, __________ exists, and __________.

45. A function is continuous over an interval if it is continuous at ____________ within that interval.

46. If a function f is not continuous at the point $x=c$, it is said to be ____________ at c.

OBJECTIVE B: Given an elementary function $y=f(x)$, determine its points of continuity and discontinuity. Be able to justify your conclusions.

---

41. $x-1$, $\frac{1}{x-1}$, $x+1$, $+\infty$

42. $\frac{1}{x-3}$, x, right, left

43. x^2, x, x^2, $x\sqrt{x^2+1}$, $\sqrt{1+\frac{1}{x^2}}$, $\sqrt{1+\frac{1}{x^2}}$, 1, 1

44. $f(c)$, $\lim_{x\to c} f(x)$, $\lim_{x\to c} f(x)=f(c)$

45. all points

46. discontinuous

47. Consider $f(x) = \{ {}^{x+4 \text{ if } x<-1}_{-x \text{ if } x\geq -1}$. Observe that $c=-1$ belongs to the domain of f. $f(-1)=1$. Does f have a limit as $x \to -1$? To answer that question, we calculate the right- and left-hand limits:

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (\_\_\_\_\_\_\_\_) = \_\_\_\_\_\_\_\_,$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (\_\_\_\_\_\_\_\_) = \_\_\_\_\_\_\_\_.$$

Since $\lim_{x\to -1^-} f(x) = 3 \neq \lim_{x\to -1^+} f(x) = 1$,, then $\lim_{x\to -1} f(x)$ __________, we conclude that f is ____________ at $x=-1$. Sketch a graph of f.

48. Let $f(x) = \dfrac{x}{x-1}$. Since $x =$ __________ does not belong to the domain of f we conclude that f is __________ at 1. Also, $\lim_{x\to 1^-} f(x) =$ __________ and $\lim_{x\to 1^+} f(x) =$ __________ so f does not have a finite limit as $x \to 1$. However, as $x \to +\infty$ or $x \to -\infty$, $f(x) \to$ __________. Sketch a graph of f. Observe that f is continuous at all points except $x = 1$.

OBJECTIVE C: Specify the main facts related to continuous functions.

49. Every constant function is continuous ____________________.

50. Every polynomial function is continuous ______________________________.

51. Every rational function is continuous ______________________________.

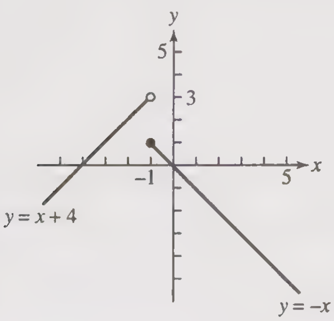
52. If f and g are continuous at c, then $f+g$, $f-g$, and $f \cdot g$ are ______________ at c.

53. If f and g are continuous at c, then $\frac{f}{g}$ is ______________ at c provided that ____________.

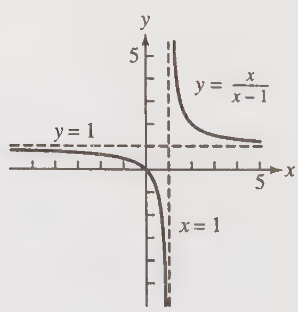
54. If f is continuous at c, and k is any constant, then kf is ______________ at c.

---

47. $x+4$, 3, $-x$, 1, does not exist, discontinuous



48. 1, discontinuous, $-\infty$, $+\infty$, 1



49. at every number

50. at every number

51. at every number at which the denominator is not zero

52. continuous

53. continuous, $g(c) \neq 0$

54. continuous

55. If f is continuous at c and g *is* continuous at $f(c)$, then the composite ______________ is continuous at ______________.

OBJECTIVE D: Understand the Intermediate Value Theorem

56. Suppose that $f(x)$ is continuous for all x in the closed interval $[a,b]$, and that N is any number between $f(a)$ and $f(b)$ What is your conclusion? ____

____.

1.5 TANGENT LINES

OBJECTIVE A: Find the slope of the function $y = f(x)$ at a given point $P(x_0, f(x_0))$ using the definition

$$m = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

57. Find the slope of the curve $y=(x+1)^2$ at $P(0,1)$.

Solution. Here $f(x)=(x+1)^2$ and $x_0 =$ ____________.

STEP 1. Calculate $f(x_0)$ and $f(x_0+h)$: $f(x_0) = f(0) =$ __________ and $f(x_0+h) = f(h) =$ __________.

STEP 2. $$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{\_\_\_\_\_\_}{h}$$
$$= \lim_{h\to 0} \frac{h^2 + \_\_\_\_\_\_}{h}$$
$$= \lim_{h\to 0} (h + \_\_\_\_\_\_) = \_\_\_\_\_\_$$

OBJECTIVE B: Find an equation for the tangent to a curve $y = f(x)$ at a given point $P(x_0, f(x_0))$.

58. Find an equation for the tangent to $y=(x+1)^2$ at $P(0,1)$.

Solution. From Problem 57, the slope of the tangent line is $m = 2$. An equation of the tangent in point-slope form is ______________________.

---

55. $g(f(x)), x = c$

56. There is at least one number c between a and b such that $f(c) = N$.

57. $0, 1, (h+1)^2, (h+1)^2 - 1, 2h, 2, 2$

58. $y = 1 + 2(x-0)$

59. Find an equation for the tangent to $y = \frac{1}{\sqrt{x-1}}$ when $x = 2$.

Solution. Here $f(x) = \frac{1}{\sqrt{x-1}}$.

STEP 1. Calculate $f(2)$ and $f(2+h)$: $f(2) =$ ________________ and $f(2+h) =$ ________________.

STEP 2. Calculate the slope m:

$$m = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{1+h}} - 1}{h}$$

$$= \lim_{h\to 0} \frac{1 - \_\_\_\_\_\_}{h\sqrt{1+h}} = \lim_{h\to 0} \frac{\left(1-\sqrt{1+h}\right)\left(1+\sqrt{1+h}\right)}{h\sqrt{1+h}\left(1+\sqrt{1+h}\right)}$$

$$= \lim_{h\to 0} \frac{\_\_\_\_\_\_}{h\sqrt{1+h}\left(1+\sqrt{1+h}\right)} = \lim_{h\to 0} \frac{\_\_\_\_\_\_}{\sqrt{1+h}\left(1+\sqrt{1+h}\right)} = \_\_\_\_\_\_.$$

STEP 3. Find the tangent line using the point-slope equation: ____________________, or $y = -\frac{1}{2}x + 2$.

OBJECTIVE C: Find the rate of change of a given function $f(x)$ with respect to x at a specified point $x = x_0$.

60. The rate of change of $f(x)$ with respect to x at $x = x_0$ is the same as the ________________ of $y = f(x)$ at $x = x_0$ or the ________________ of f at $x = x_0$.

61. Find the rate of change of the area of a square $\left(A = x^2\right)$ with respect to its side length when the side length is $x = 5$.

Solution. We first calculate $A(5)$ and $A(5+h)$:

$A(5) =$ ________________ and $A(5+h) =$ ________________.

Then the rate of change is

$$\lim_{h\to 0} \frac{A(5+h)-A(5)}{h} = \lim_{h\to 0} \frac{\_\_\_\_\_\_}{h} = \lim_{h\to 0} (10+h) = \_\_\_\_\_\_.$$

That is , the area changes at the rate 10 square units per side length unit when the side length is 5 units.

---

59. 1, $\frac{1}{\sqrt{1+h}}$, $\sqrt{1+h}$, $1-(1+h)$, -1, $-\frac{1}{2}$, $y = 1 + \left(-\frac{1}{2}\right)(x-2)$

60. slope, derivative

61. 25, $(5+h)^2 = 25 + 10h + h^2$, $10h + h^2$, 10

CHAPTER 1 SELF-TEST

Find the limits in Problems 1–6.

1. $\lim_{t\to 3}\frac{t^2-1}{t-1}$

2. $\lim_{x\to 2}\frac{2x^2-3x-2}{x-2}$

3. $\lim_{x\to 1^+}\frac{3x-1}{5x^3-2x+1}$

4. $\lim_{t\to 0^-}\frac{2t^2+3t-1}{t^3-2t}$

5. $\lim_{t\to\infty}\frac{t^2}{4-t^2}$

6. $\lim_{x\to\infty}\left(\frac{1}{x}\cos x^2\right)$

7. Let f be defined by $f(x)=\begin{cases}2x-3 & , \text{ if } x\ge 0\\ -1 & , \text{ if } x<0\end{cases}$

   (a) Find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$.

   (b) Is f continuous at $x = 0$? Justify your answer.

8. Consider the function $f(x)=\frac{x-1}{x^2-x}$.

   (a) For what values of x is f continuous? Justify your conclusion.

   (b) Is f continuous at $x = 1$? If not, what value can be assigned to $f(1)$ so that the resultant function is continuous there.?

9. It can be shown that for all values of x
   $$\frac{x^2}{2}-\frac{x^4}{24}\le 1-\cos x\le\frac{x^2}{2}.$$
   Use this result to find $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

10. In what interval about $x_0 = -1$ must we hold x to be sure that $y=-\frac{x}{3}+\frac{2}{3}$ lies within $\varepsilon = 0.5$ units of $y_0 = 1$?

11. Justify that the function $f(x) = x^5 + 1$ has at least one real zero; that is, where there is some real number c so that $f(c) = 0$.

12. Given $f(x) = 2x + 7, x_0 = -2$, and $\varepsilon = 0.01$, find $L=\lim_{x\to x_0} f(x)$. Then find $\delta > 0$ such that
    $$0<|x-x_0|<\delta\Rightarrow|f(x)-L|<\varepsilon.$$

13. Find the slope of $f(x)=\frac{2x}{x+1}$ at (1,1). Then find an equation for the line tangent to the graph there.

14. At t seconds after lift off, the height of a rocket is $4t^2$ ft. How fast is the rocket climbing after one minute?

SOLUTIONS TO CHAPTER 1 SELF-TEST

1. $\lim_{t\to 3}\frac{t^2-1}{t-1}=\frac{9-1}{3-1}=4$

2. $\lim_{x\to 2}\frac{2x^2-3x-2}{x-2}=\lim_{x\to 2}\frac{(2x+1)(x-2)}{x-2}=\lim_{x\to 2}(2x+1)=2(2)+1=5$

3. $\lim_{x\to 1^+}\frac{3x-1}{5x^3-2x+1}=\frac{3-1}{5-2+1}=\frac{1}{2}$

4. $\lim\limits_{t\to 0^-}\dfrac{2t^2+3t-1}{t^3-2t}=\lim\limits_{t\to 0^-}\dfrac{2t^2+3t-1}{t\left(t^2-2\right)}=\lim\limits_{t\to 0^-}\dfrac{1}{t}\lim\limits_{t\to 0^-}\dfrac{2t^2+3t-1}{t^2-2}=-\infty$

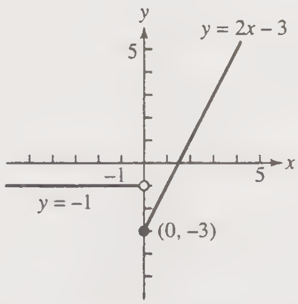
5. $\lim\limits_{t\to\infty}\dfrac{t^2}{4-t^2}=\lim\limits_{h\to 0}\dfrac{\frac{1}{h^2}}{4-\frac{1}{h^2}}=\lim\limits_{h\to 0}=\dfrac{1}{4h^2-1}=\dfrac{1}{0-1}=-1$

6. $0\le\left|\dfrac{1}{x}\cos x^2\right|\le\dfrac{1}{|x|}$ because $\left|\cos x^2\right|\le 1$ for all values of x. Since $\lim\limits_{x\to\infty}\dfrac{1}{|x|}=0$, we have $\lim\limits_{x\to\infty}\left(\dfrac{1}{x}\cos x^2\right)=0$ by the Sandwich Theorem.

7. **(a)** From the graph of f shown below right,

$\lim\limits_{x\to 0^+}f(x)=\lim\limits_{x\to 0^+}(2x-3)=-3$

$\lim\limits_{x\to 0^-}f(x)=\lim\limits_{x\to 0^-}(-1)=-1$



(b) No, $\lim\limits_{x\to 0}f(x)$ does not exist because the left-hand and right-hand limits differ as x tends to zero, so f is not continuous at $x=0$.

8. **(a)** Since division by zero is never permitted, the points $x=0$ and $x=1$ do not belong to the domain of f, and therefore f is discontinuous at those two values; it is continuous for all other values of x.

(b) $f(x)=\dfrac{x-1}{x^2-x}=\dfrac{x-1}{x(x-1)}=\dfrac{1}{x}$ if $x\ne 1$ and $x\ne 0$. Since $\lim\limits_{x\to 1}f(x)=\lim\limits_{x\to 1}\dfrac{1}{x}=1$, if we specify $f(1)=1$ the new function so defined is continuous at $x=1$.

9. From the assumed inequality, we divide through by the positive number x^2 obtaining, $\dfrac{1}{2}-\dfrac{x^2}{24}\le\dfrac{1-\cos x}{x^2}\le\dfrac{1}{2}$.

Applying the Sandwich Theorem , $\lim\limits_{x\to 0}\left(\dfrac{1}{2}-\dfrac{x^2}{24}\right)=\dfrac{1}{2}$ and $\lim\limits_{x\to 0}\dfrac{1}{2}=\dfrac{1}{2}$ so that $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}=\dfrac{1}{2}$.

10. $y_0=1,\ \varepsilon=0.5$

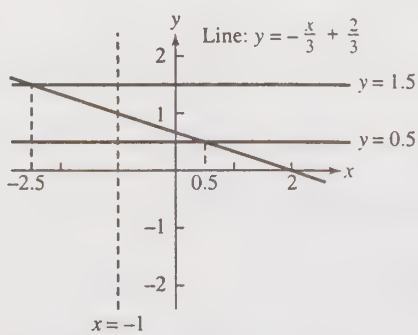
$|y-y_0|=|y-1|=\left|\left(-\dfrac{x}{3}+\dfrac{2}{3}\right)-1\right|<\varepsilon$

$\Leftrightarrow\ \left|-\dfrac{x}{3}-\dfrac{1}{3}\right|=\dfrac{1}{3}|x+1|<0.5$

$\Leftrightarrow\ |x+1|<1.5$

$\Leftrightarrow\ -1.5<x+1<1.5$

$\Leftrightarrow\ -2.5<x<0.5$



11. Notice that $f(-2)=(-2)^5+1=-31$ is a negative and $f(1)=1^5+1=2$ is positive. Since f is a continuous function, the Intermediate Value Theorem guarantees the existence of a real number c satisfying $-2<c<1$ and $f(c)=0$.

12. $\lim_{x\to-2}(2x+7) = 2\lim_{x\to-2} x + \lim_{x\to-2} 7 = 2(-2)+7 = 3.$

$|(2x+7)-3| = |2x+4| = 2|x+2| = 2|x-(-2)|.$

Thus

$|(2x+7)-3| < 0.01$ whenever $|x-(-2)| < 0.005.$

That is, $\delta = 0.005$ for $\varepsilon = 0.01$ in the formal definition of the limit of $f(x) = 2x+7$ as x approaches $x_0 = -2$.

13. The slope is

$$\begin{aligned} m &= \lim_{h\to0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0}\frac{\frac{2(1+h)}{2+h}-\frac{2}{2}}{h} \\ &= \lim_{h\to0}\frac{(2+2h)-(2+h)}{h(2+h)} = \lim_{h\to0}\frac{h}{h(2+h)} \\ &= \lim_{h\to0}\frac{1}{2+h} = \frac{1}{2.} \end{aligned}$$

An equation of the tangent line is $y = 1+\frac{1}{2}(x-1)$ or $y = \frac{1}{2}(x+1)$.

14. The rate of change of $H(t) = 4t^2$ at $t = 60$ seconds is

$$\begin{aligned} &\lim_{h\to0}\frac{H(60+h)-H(60)}{h} = \lim_{h\to0}\frac{4(60+h)^2-4(3600)}{h} \\ &= \lim_{h\to0}\frac{4\left(3600+120h+h^2\right)-4(3600)}{h} \\ &= \lim_{h\to0}\frac{4\left(120h+h^2\right)}{h} = \lim_{h\to0}(480+4h) = 480 \end{aligned}$$

The rocket is climbing at the rate of 480 ft/sec after 1 minute.

NOTES.