

## CHAPTER 4 INTEGRATION

### 4.1 INDEFINITE INTEGRALS

1. (a)  $3x^2$  (b)  $\frac{x^8}{8}$  (c)  $\frac{x^8}{8} - 3x^2 + 8x$
2. (a)  $x^{-3}$  (b)  $-\frac{x^{-3}}{3}$  (c)  $-\frac{x^{-3}}{3} + x^2 + 3x$
3. (a)  $\frac{1}{x^2}$  (b)  $\frac{-1}{4x^2}$  (c)  $\frac{x^4}{4} + \frac{1}{2x^2}$
4. (a)  $\sqrt{x^3}$  (b)  $\sqrt{x}$  (c)  $\frac{2}{3}\sqrt{x^3} + 2\sqrt{x}$
5. (a)  $x^{2/3}$  (b)  $x^{1/3}$  (c)  $x^{-1/3}$
6. (a)  $\cos(\pi x)$  (b)  $-3 \cos x$  (c)  $\frac{-\cos(\pi x)}{\pi} + \cos(3x)$
7. (a)  $\tan x$  (b)  $2 \tan\left(\frac{x}{3}\right)$  (c)  $-\frac{2}{3} \tan\left(\frac{3x}{2}\right)$
8. (a)  $\sec x$  (b)  $\frac{4}{3} \sec(3x)$  (c)  $\frac{2}{\pi} \sec\left(\frac{\pi x}{2}\right)$

9.  $\int (x+1) dx = \frac{x^2}{2} + x + C$       10.  $\int \left(3t^2 + \frac{t}{2}\right) dt = t^3 + \frac{t^2}{4} + C$

11.  $\int \left(\frac{1}{x} - \frac{5}{x^2+1}\right) dx = \ln|x| - 5 \tan^{-1} x + C$

12.  $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx = \int \left(x^{-2} - x^2 - \frac{1}{3}\right) dx = \frac{x^{-1}}{-1} - \frac{x^3}{3} - \frac{1}{3}x + C = -\frac{1}{x} - \frac{x^3}{3} - \frac{x}{3} + C$

13.  $\int (e^{-x} + 4^x) dx = -e^{-x} + \frac{4^x}{\ln 4} + C$

14.  $\int (\sqrt{x} + \sqrt[3]{x}) dx = \int (x^{1/2} + x^{1/3}) dx = \frac{x^{3/2}}{3/2} + \frac{x^{4/3}}{4/3} + C = \frac{2}{3}x^{3/2} + \frac{3}{4}x^{4/3} + C$

15.  $\int \left(\frac{2}{\sqrt{1-y^2}} - \frac{1}{y^{1/4}}\right) dy = 2 \sin^{-1} y - \frac{4}{3}y^{3/4} + C$

$$\int \frac{dx}{2x-1} = \int \frac{du}{2u} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|2x-1| + C = \ln \sqrt{|2x-1|} + C$$

30. Let  $u = x^2 + 4 \Rightarrow du = 2x dx \Rightarrow \frac{du}{2} = x dx$

$$\int \frac{x dx}{x^2+4} = \int \frac{du}{2u} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|x^2+4| + C = \frac{1}{2} \ln(x^2+4) + C = \ln \sqrt{x^2+4} + C$$

31. Let  $u = 2 - \cos t \Rightarrow du = \sin t dt$

$$\int \frac{\sin t}{2 - \cos t} dt = \int \frac{du}{u} = \ln|u| + C = \ln|2 - \cos t| + C$$

32.  $\int \frac{1}{\sin t \cos t} dt = \int \sec t \csc t dt = \sec t + C$

33.  $\int \frac{1}{\sqrt{1-4x^2}} dx = \frac{1}{2} \int \frac{2}{\sqrt{1-(2x)^2}} dx = \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}}$ , where  $u = 2x$  and  $du = 2 dx$

$$= \frac{1}{2} \sin^{-1} u + C = \frac{1}{2} \sin^{-1}(2x) + C$$

34.  $\int \frac{dx}{9+3x^2} = \frac{1}{9} \int \frac{dx}{1+3x^2/9}$ ; Let  $u = \frac{\sqrt{3}x}{3} \Rightarrow du = \frac{\sqrt{3}}{3} dx \Rightarrow \sqrt{3} du = dx$ ;

$$\frac{1}{9} \int \frac{dx}{1+3x^2/9} = \left(\frac{\sqrt{3}}{9}\right) \int \frac{du}{1+u^2} = \frac{\sqrt{3}}{9} \tan^{-1} u + C = \frac{\sqrt{3}}{9} \tan^{-1} \frac{\sqrt{3}x}{3} + C = \frac{1}{3\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + C$$

35.  $\int \frac{dx}{x\sqrt{25x^2-2}} = \int \frac{du}{u\sqrt{u^2-2}}$ , where  $u = 5x$  and  $du = 5 dx$

$$= \frac{1}{\sqrt{2}} \sec^{-1} \left| \frac{u}{\sqrt{2}} \right| + C = \frac{1}{\sqrt{2}} \sec^{-1} \left| \frac{5x}{\sqrt{2}} \right| + C$$

36.  $\int \frac{3 dr}{\sqrt{1-4(r-1)^2}} = \frac{3}{2} \int \frac{du}{\sqrt{1-u^2}}$ , where  $u = 2(r-1)$  and  $du = 2 dr$

$$= \frac{3}{2} \sin^{-1} u + C = \frac{3}{2} \sin^{-1} 2(r-1) + C$$

37.  $\int \frac{dx}{1+(3x+1)^2} = \frac{1}{3} \int \frac{du}{1+u^2}$ , where  $u = 3x+1$  and  $du = 3 dx$

$$= \frac{1}{3} \tan^{-1} u + C = \frac{1}{3} \tan^{-1}(3x+1) + C$$

38.  $\int \frac{y dy}{\sqrt{1-y^4}} = \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}}$ , where  $u = y^2$  and  $du = 2y dy$

$$= \frac{1}{2} \sin^{-1} u + C = \frac{1}{2} \sin^{-1} y^2 + C$$

39. Let  $u = e^x \Rightarrow du = e^x dx$ ;  $\int \frac{e^x}{1+e^{2x}} dx = \int \frac{e^x}{1+(e^x)^2} dx = \int \frac{du}{1+u^2} = \tan^{-1} u + C = \tan^{-1}(e^x) + C$

40. Let  $u = \ln t \Rightarrow du = \frac{1}{t} dt$ ;  $\int \frac{4 dt}{t(1+\ln^2 t)} = 4 \int \frac{du}{1+u^2} = 4 \tan^{-1} u + C = 4 \tan^{-1}(\ln t) + C$

41. (a) Let  $u = \tan x \Rightarrow du = \sec^2 x dx$ ;  $v = u^3 \Rightarrow dv = 3u^2 du \Rightarrow 6 dv = 18u^2 du$ ;  $w = 2 + v \Rightarrow dw = dv$

$$\begin{aligned} \int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)^2} dx &= \int \frac{18u^2}{(2+u^3)^2} du = \int \frac{6 dv}{(2+v)^2} = \int \frac{6 dw}{w^2} = 6 \int w^{-2} dw = -6w^{-1} + C = -\frac{6}{2+v} + C \\ &= -\frac{6}{2+u^3} + C = -\frac{6}{2+\tan^3 x} + C \end{aligned}$$

(b) Let  $u = \tan^3 x \Rightarrow du = 3 \tan^2 x \sec^2 x dx \Rightarrow 6 du = 18 \tan^2 x \sec^2 x dx$ ;  $v = 2 + u \Rightarrow dv = du$

$$\int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)^2} dx = \int \frac{6 du}{(2+u)^2} = \int \frac{6 dv}{v^2} = -\frac{6}{v} + C = -\frac{6}{2+u} + C = -\frac{6}{2+\tan^3 x} + C$$

(c) Let  $u = 2 + \tan^3 x \Rightarrow du = 3 \tan^2 x \sec^2 x dx \Rightarrow 6 du = 18 \tan^2 x \sec^2 x dx$

$$\int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)^2} dx = \int \frac{6 du}{u^2} = -\frac{6}{u} + C = -\frac{6}{2 + \tan^3 x} + C$$

42. (a) Let  $u = x - 1 \Rightarrow du = dx$ ;  $v = \sin u \Rightarrow dv = \cos u du$ ;  $w = 1 + v^2 \Rightarrow dw = 2v dv \Rightarrow \frac{1}{2} dw = v dv$

$$\begin{aligned} \int \sqrt{1 + \sin^2(x-1)} \sin(x-1) \cos(x-1) dx &= \int \sqrt{1 + \sin^2 u} \sin u \cos u du = \int v \sqrt{1+v^2} dv \\ &= \int \frac{1}{2} \sqrt{w} dw = \frac{1}{3} w^{3/2} + C = \frac{1}{3} (1+v^2)^{3/2} + C = \frac{1}{3} (1 + \sin^2 u)^{3/2} + C = \frac{1}{3} (2 + \sin^2(x-1))^{3/2} + C \end{aligned}$$

(b) Let  $u = \sin(x-1) \Rightarrow du = \cos(x-1) dx$ ;  $v = 1 + u^2 \Rightarrow dv = 2u du \Rightarrow \frac{1}{2} dv = u du$

$$\begin{aligned} \int \sqrt{1 + \sin^2(x-1)} \sin(x-1) \cos(x-1) dx &= \int u \sqrt{1+u^2} du = \int \frac{1}{2} \sqrt{v} dv = \int \frac{1}{2} v^{1/2} dv \\ &= \left( \frac{1}{2} \left( \frac{2}{3} \right) v^{3/2} \right) + C = \frac{1}{3} v^{3/2} + C = \frac{1}{3} (1+u^2)^{3/2} + C = \frac{1}{3} (1 + \sin^2(x-1))^{3/2} + C \end{aligned}$$

(c) Let  $u = 1 + \sin^2(x-1) \Rightarrow du = 2 \sin(x-1) \cos(x-1) dx \Rightarrow \frac{1}{2} du = \sin(x-1) \cos(x-1) dx$

$$\begin{aligned} \int \sqrt{1 + \sin^2(x-1)} \sin(x-1) \cos(x-1) dx &= \int \frac{1}{2} \sqrt{u} du = \int \frac{1}{2} u^{1/2} du = \frac{1}{2} \left( \frac{2}{3} u^{3/2} \right) + C \\ &= \frac{1}{3} (1 + \sin^2(x-1))^{3/2} + C \end{aligned}$$

43. Let  $u = 3(2r-1)^2 + 6 \Rightarrow du = 6(2r-1)(2) dr \Rightarrow \frac{1}{12} du = (2r-1) dr$ ;  $v = \sqrt{u} \Rightarrow dv = \frac{1}{2\sqrt{u}} du \Rightarrow \frac{1}{6} dv = \frac{1}{12\sqrt{u}} du$

$$\int \frac{(2r-1) \cos \sqrt{3(2r-1)^2 + 6}}{\sqrt{3(2r-1)^2 + 6}} dr = \int \left( \frac{\cos \sqrt{u}}{\sqrt{u}} \right) \left( \frac{1}{12} du \right) = \int (\cos v) \left( \frac{1}{6} dv \right) = \frac{1}{6} \sin v + C = \frac{1}{6} \sin \sqrt{u} + C$$

$$= \frac{1}{6} \sin \sqrt{3(2r-1)^2 + 6} + C$$

44. Let  $u = \cos \sqrt{\theta} \Rightarrow du = (-\sin \sqrt{\theta}) \left( \frac{1}{2\sqrt{\theta}} \right) d\theta \Rightarrow -2 du = \frac{\sin \sqrt{\theta}}{\sqrt{\theta}} d\theta$

$$\int \frac{\sin \sqrt{\theta}}{\sqrt{\theta} \cos^3 \sqrt{\theta}} d\theta = \int \frac{\sin \sqrt{\theta}}{\sqrt{\theta} \cos^3 \sqrt{\theta}} d\theta = \int \frac{-2 du}{u^{3/2}} = -2 \int u^{-3/2} du = -2(-2u^{-1/2}) + C = \frac{4}{\sqrt{u}} + C$$

$$= \frac{4}{\sqrt{\cos \sqrt{\theta}}} + C$$

45. Let  $u = 3t^2 - 1 \Rightarrow du = 6t dt \Rightarrow 2 du = 12t dt$

$$s = \int 12t(3t^2 - 1)^3 dt = \int u^3 (2 du) = 2 \left( \frac{1}{4} u^4 \right) + C = \frac{1}{2} u^4 + C = \frac{1}{2} (3t^2 - 1)^4 + C;$$

$$s = 3 \text{ when } t = 1 \Rightarrow 3 = \frac{1}{2} (3 - 1)^4 + C \Rightarrow 3 = 8 + C \Rightarrow C = -5 \Rightarrow s = \frac{1}{2} (3t^2 - 1)^4 - 5$$

46. Let  $u = x^2 + 8 \Rightarrow du = 2x dx \Rightarrow 2 du = 4x dx$

$$y = \int 4x(x^2 + 8)^{-1/3} dx = \int u^{-1/3} (2 du) = 2 \left( \frac{3}{2} u^{2/3} \right) + C = 3u^{2/3} + C = 3(x^2 + 8)^{2/3} + C;$$

$$y = 0 \text{ when } x = 0 \Rightarrow 0 = 3(8)^{2/3} + C \Rightarrow C = -12 \Rightarrow y = 3(x^2 + 8)^{2/3} - 12$$

47. Let  $u = t + \frac{\pi}{12} \Rightarrow du = dt$

$$s = \int 8 \sin^2 \left( t + \frac{\pi}{12} \right) dt = \int 8 \sin^2 u du = 8 \left( \frac{u}{2} - \frac{1}{4} \sin 2u \right) + C = 4 \left( t + \frac{\pi}{12} \right) - 2 \sin \left( 2t + \frac{\pi}{6} \right) + C;$$

$$s = 8 \text{ when } t = 0 \Rightarrow 8 = 4 \left( \frac{\pi}{12} \right) - 2 \sin \left( \frac{\pi}{6} \right) + C \Rightarrow C = 8 - \frac{\pi}{3} + 1 = 9 - \frac{\pi}{3} \Rightarrow s = 4t - 2 \sin \left( 2t + \frac{\pi}{6} \right) + 9$$

48.  $\frac{dy}{dx} = 1 + \frac{1}{x}$  at  $(1, 3) \Rightarrow y = x + \ln |x| + C$ ;  $y = 3$  at  $x = 1 \Rightarrow C = 2 \Rightarrow y = x + \ln |x| + 2$

49.  $\frac{dy}{dt} = e^t \sin(e^t - 2) \Rightarrow y = \int e^t \sin(e^t - 2) dt$ ;

$$\text{let } u = e^t - 2 \Rightarrow du = e^t dt \Rightarrow y = \int \sin u du = -\cos u + C = -\cos(e^t - 2) + C; y(\ln 2) = 0$$

$$\Rightarrow -\cos(e^{\ln 2} - 2) + C = 0 \Rightarrow -\cos(2 - 2) + C = 0 \Rightarrow C = \cos 0 = 1; \text{ thus, } y = 1 - \cos(e^t - 2)$$

50.  $\frac{dy}{dt} = e^{-t} \sec^2(\pi e^{-t}) \Rightarrow y = \int e^{-t} \sec^2(\pi e^{-t}) dt$ ;

$$\begin{aligned} \text{let } u = \pi e^{-t} &\Rightarrow du = -\pi e^{-t} dt \Rightarrow -\frac{1}{\pi} du = e^{-t} dt \Rightarrow y = -\frac{1}{\pi} \int \sec^2 u \, du = -\frac{1}{\pi} \tan u + C \\ &= -\frac{1}{\pi} \tan(\pi e^{-t}) + C; y(\ln 4) = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi} \tan(\pi e^{-\ln 4}) + C = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi} \tan\left(\pi \cdot \frac{1}{4}\right) + C = \frac{2}{\pi} \\ &\Rightarrow -\frac{1}{\pi}(1) + C = \frac{2}{\pi} \Rightarrow C = \frac{3}{\pi}; \text{ thus, } y = \frac{3}{\pi} - \frac{1}{\pi} \tan(\pi e^{-t}) \end{aligned}$$

$$51. \text{ Let } u = 2t - \frac{\pi}{2} \Rightarrow du = 2 \, dt \Rightarrow -2 \, du = -4 \, dt$$

$$\frac{ds}{dt} = \int -4 \sin\left(2t - \frac{\pi}{2}\right) dt = \int (\sin u)(-2 \, du) = 2 \cos u + C_1 = 2 \cos\left(2t - \frac{\pi}{2}\right) + C_1;$$

$$\text{at } t = 0 \text{ and } \frac{ds}{dt} = 100 \text{ we have } 100 = 2 \cos\left(-\frac{\pi}{2}\right) + C_1 \Rightarrow C_1 = 100 \Rightarrow \frac{ds}{dt} = 2 \cos\left(2t - \frac{\pi}{2}\right) + 100$$

$$\Rightarrow s = \int \left(2 \cos\left(2t - \frac{\pi}{2}\right) + 100\right) dt = \int (\cos u + 50) \, du = \sin u + 50u + C_2 = \sin\left(2t - \frac{\pi}{2}\right) + 50\left(2t - \frac{\pi}{2}\right) + C_2;$$

$$\text{at } t = 0 \text{ and } s = 0 \text{ we have } 0 = \sin\left(-\frac{\pi}{2}\right) + 50\left(-\frac{\pi}{2}\right) + C_2 \Rightarrow C_2 = 1 + 25\pi$$

$$\Rightarrow s = \sin\left(2t - \frac{\pi}{2}\right) + 100t - 25\pi + (1 + 25\pi) \Rightarrow s = \sin\left(2t - \frac{\pi}{2}\right) + 100t + 1$$

$$52. \frac{d^2y}{dx^2} = \sec^2 x \Rightarrow \frac{dy}{dx} = \tan x + C \text{ and } 1 = \tan 0 + C \Rightarrow \frac{dy}{dx} = \tan x + 1 \Rightarrow y = \int (\tan x + 1) \, dx \\ = \ln |\sec x| + x + C_1 \text{ and } 0 = \ln |\sec 0| + 0 + C_1 \Rightarrow C_1 = 0 \Rightarrow y = \ln |\sec x| + x$$

$$53. \frac{d^2y}{dx^2} = 2e^{-x} \Rightarrow \frac{dy}{dx} = -2e^{-x} + C; x = 0 \text{ and } \frac{dy}{dx} = 0 \Rightarrow 0 = -2e^0 + C \Rightarrow C = 2; \text{ thus } \frac{dy}{dx} = -2e^{-x} + 2$$

$$\Rightarrow y = 2e^{-x} + 2x + C_1; x = 0 \text{ and } y = 1 \Rightarrow 1 = 2e^0 + C_1 \Rightarrow C_1 = -1 \Rightarrow y = 2e^{-x} + 2x - 1 = 2(e^{-x} + x) - 1$$

$$54. \frac{d^2y}{dt^2} = 1 - e^{2t} \Rightarrow \frac{dy}{dt} = t - \frac{1}{2}e^{2t} + C; t = 1 \text{ and } \frac{dy}{dt} = 0 \Rightarrow 0 = 1 - \frac{1}{2}e^2 + C \Rightarrow C = \frac{1}{2}e^2 - 1; \text{ thus}$$

$$\frac{dy}{dt} = t - \frac{1}{2}e^{2t} + \frac{1}{2}e^2 - 1 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t + C_1; t = 1 \text{ and } y = -1 \Rightarrow -1 = \frac{1}{2} - \frac{1}{4}e^2 + \frac{1}{2}e^2 - 1 + C_1$$

$$\Rightarrow C_1 = -\frac{1}{2} - \frac{1}{4}e^2 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t - \left(\frac{1}{2} + \frac{1}{4}e^2\right)$$

$$55. \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow dy = \frac{dx}{\sqrt{1-x^2}} \Rightarrow y = \sin^{-1} x + C; x = 0 \text{ and } y = 0 \Rightarrow 0 = \sin^{-1} 0 + C \Rightarrow C = 0 \Rightarrow y = \sin^{-1} x$$

$$56. \frac{dy}{dx} = \frac{1}{x^2+1} - 1 \Rightarrow dy = \left(\frac{1}{1+x^2} - 1\right) dx \Rightarrow y = \tan^{-1}(x) - x + C; x = 0 \text{ and } y = 1 \Rightarrow 1 = \tan^{-1} 0 - 0 + C$$

$$\Rightarrow C = 1 \Rightarrow y = \tan^{-1}(x) - x + 1$$

$$57. \frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}} \Rightarrow dy = \frac{dx}{x\sqrt{x^2-1}} \Rightarrow y = \sec^{-1} |x| + C; x = 2 \text{ and } y = \pi \Rightarrow \pi = \sec^{-1} 2 + C \Rightarrow C = \pi - \sec^{-1} 2$$

$$= \pi - \frac{\pi}{3} = \frac{2\pi}{3} \Rightarrow y = \sec^{-1}(x) + \frac{2\pi}{3}, x > 1$$

$$58. \frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} \Rightarrow dy = \left( \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} \right) dx \Rightarrow y = \tan^{-1} x - 2 \sin^{-1} x + C; x = 0 \text{ and } y = 2$$

$$\Rightarrow 2 = \tan^{-1} 0 - 2 \sin^{-1} 0 + C \Rightarrow C = 2 \Rightarrow y = \tan^{-1} x - 2 \sin^{-1} x + 2$$

$$59. \text{ Let } u = 2t \Rightarrow du = 2 dt \Rightarrow 3 du = 6 dt$$

$$s = \int 6 \sin 2t dt = \int (\sin u)(3 du) = -3 \cos u + C = -3 \cos 2t + C;$$

$$\text{at } t = 0 \text{ and } s = 0 \text{ we have } 0 = -3 \cos 0 + C \Rightarrow C = 3 \Rightarrow s = 3 - 3 \cos 2t \Rightarrow s\left(\frac{\pi}{2}\right) = 3 - 3 \cos(\pi) = 6 \text{ m}$$

$$60. \text{ Let } u = \pi t \Rightarrow du = \pi dt \Rightarrow \pi du = \pi^2 dt$$

$$v = \int \pi^2 \cos \pi t dt = \int (\cos u)(\pi du) = \pi \sin u + C_1 = \pi \sin(\pi t) + C_1;$$

$$\text{at } t = 0 \text{ and } v = 8 \text{ we have } 8 = \pi(0) + C_1 \Rightarrow C_1 = 8 \Rightarrow v = \frac{ds}{dt} = \pi \sin(\pi t) + 8 \Rightarrow s = \int (\pi \sin(\pi t) + 8) dt$$

$$= \int \sin u du + 8t + C_2 = -\cos(\pi t) + 8t + C_2; \text{ at } t = 0 \text{ and } s = 0 \text{ we have } 0 = -1 + C_2 \Rightarrow C_2 = 1$$

$$\Rightarrow s = 8t - \cos(\pi t) + 1 \Rightarrow s(1) = 8 - \cos \pi + 1 = 10 \text{ m}$$

$$61. \frac{d}{dx} \left( \sin^{-1} \frac{x}{a} + C \right) = \frac{d}{dx} \sin^{-1} \frac{x}{a} + \frac{dC}{dx} = \frac{1}{\sqrt{1-(x/a)^2}} \cdot \frac{1}{a} = \frac{1}{\sqrt{a^2-x^2}}, \text{ which verifies formula 10.}$$

$$62. \frac{d}{dx} \left( \frac{1}{a} \tan^{-1} \frac{x}{a} + C \right) = \frac{1}{a} \frac{d}{dx} \tan^{-1} \frac{x}{a} + \frac{dC}{dx} = \frac{1}{a(1+(x/a)^2)} \cdot \frac{1}{a} + 0 = \frac{1}{a^2+x^2}, \text{ which verifies formula 11.}$$

$$63. \text{ (a) } \int \frac{1}{\sqrt{3^2-x^2}} dx = \sin^{-1} \frac{x}{3} + C \quad \text{(b) } \int \frac{1}{(\sqrt{3})^2+x^2} dx = \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + C$$

$$64. \text{ (a) } \int \frac{1}{\sqrt{16-25x^2}} dx = \frac{1}{5} \int \frac{1}{\sqrt{(4/5)^2-x^2}} dx = \frac{1}{5} \sin^{-1} \frac{5x}{4} + C$$

$$\text{(b) } \int \frac{8}{1+4x^2} dx = 2 \int \frac{1}{x^2+(1/2)^2} dx = 2[2 \tan^{-1} 2x + \bar{C}] = 4 \tan^{-1} 2x + C, \text{ where } C = 2\bar{C}$$

$$65. \int \sec x dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$= \left[ \begin{array}{l} \text{Let } u = \sec x + \tan x \text{ and} \\ du = (\sec^2 x + \sec x \tan x) dx \end{array} \right] = \int \frac{du}{u} = \ln|\sec x + \tan x| + C$$

66. All three integrations are correct. In each case, the derivative of the function on the right is the integrand on the left, and each formula has an arbitrary constant for generating the remaining antiderivatives. Moreover,  $\sin^2 x + C_1 = 1 - \cos^2 x + C_1 \Rightarrow C_2 = 1 + C_1$ ; also  $-\cos^2 x + C_2 = -\frac{\cos 2x}{2} - \frac{1}{2} + C_2 \Rightarrow C_3 = C_2 - \frac{1}{2} = C_1 + \frac{1}{2}$ .

## 4.3 ESTIMATING WITH FINITE SUMS

1. Using values of the function taken from the graph at the midpoints of the intervals,

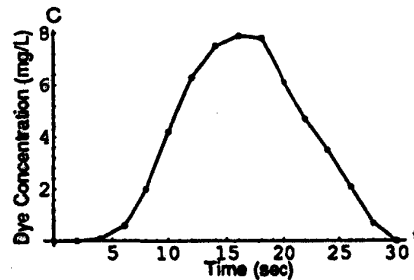
$$\begin{aligned} \text{Area} &\approx (0.25)(2) + (1.0)(2) + (2.0)(2) + (3.25)(2) + (4.0)(2) + (4.0)(2) \\ &+ (3.35)(2) + (2.25)(2) + (1.3)(2) + (0.75)(2) + (0.25)(2) = 44.8 \text{ mg} \cdot \text{sec/L.} \quad \text{Cardiac output} \\ &= \frac{\text{amount of dye}}{\text{area under curve}} \times 60 \approx \frac{5 \text{ mg}}{44.5 \text{ mg} \cdot \text{sec/L}} \times 60 \frac{\text{sec}}{\text{min}} \approx 6.7 \text{ L/min.} \end{aligned}$$

2. Using values of the function taken from the graph

at the midpoints of the intervals,

$$\begin{aligned} \text{Area} &\approx 0(2) + (0.1)(2) + (0.4)(2) + (1.2)(2) + (3.2)(2) \\ &+ (5.3)(2) + (6.8)(2) + (7.6)(2) + (7.7)(2) + (6.9)(2) \\ &+ (5.6)(2) + (4.0)(2) + (2.8)(2) + (1.6)(2) + (0.2)(2) \\ &= 107.0 \text{ mg} \cdot \text{sec/L.} \end{aligned}$$

$$\begin{aligned} \text{Cardiac output} &= \frac{\text{dye concentration}}{\text{area estimate}} \times 60 \\ &= \frac{10 \text{ mg}}{107.0 \text{ mg} \cdot \text{sec/L}} \times 60 \frac{\text{sec}}{\text{min}} = 5.61 \text{ L/min.} \end{aligned}$$



3. (a)  $D \approx (0)(1) + (12)(1) + (22)(1) + (10)(1) + (5)(1) + (13)(1) + (11)(1) + (6)(1) + (2)(1) + (6)(1) = 87$  inches  
 (b)  $D \approx (12)(1) + (22)(1) + (10)(1) + (5)(1) + (13)(1) + (11)(1) + (6)(1) + (2)(1) + (6)(1) + (0)(1) = 87$  inches
4. (a)  $D \approx (1)(300) + (1.2)(300) + (1.7)(300) + (2.0)(300) + (1.8)(300) + (1.6)(300) + (1.4)(300) + (1.2)(300) + (1.0)(300) + (1.8)(300) + (1.5)(300) + (1.2)(300) = 5200$  meters (NOTE: 5 minutes = 300 seconds)  
 (b)  $D \approx (1.2)(300) + (1.7)(300) + (2.0)(300) + (1.8)(300) + (1.6)(300) + (1.4)(300) + (1.2)(300) + (1.0)(300) + (1.8)(300) + (1.5)(300) + (1.2)(300) + (0)(300) = 4920$  meters (NOTE: 5 minutes = 300 seconds)
5. (a)  $D \approx (0)(10) + (44)(10) + (15)(10) + (35)(10) + (30)(10) + (44)(10) + (35)(10) + (15)(10) + (22)(10) + (35)(10) + (44)(10) + (30)(10) = 3490$  feet  $\approx 0.66$  miles  
 (b)  $D \approx (44)(10) + (15)(10) + (35)(10) + (30)(10) + (44)(10) + (35)(10) + (15)(10) + (22)(10) + (35)(10) + (44)(10) + (30)(10) + (35)(10) = 3840$  feet  $\approx 0.73$  miles
6. (a) The distance traveled will be the area under the curve. We will use the approximate velocities at the midpoints of each time interval to approximate this area using rectangles. Thus,  
 $D \approx (20)(0.001) + (51)(0.001) + (72)(0.001) + (89)(0.001) + (102)(0.001) + (112)(0.001) + (120.5)(0.001) + (128.5)(0.001) + (134.5)(0.001) + (139.5)(0.001) \approx 0.969$  miles  
 (b) Roughly, after 0.0063 hours, the car would have gone 0.485 miles, where 0.0060 hours = 22.7 sec. At 22.7 sec, the velocity was approximately 120 mi/hr.
7. (a)  $S_4 = \pi \left[ \sqrt{16 - (-2)^2} \right]^2 (2) + \pi \left[ \sqrt{16 - 0^2} \right]^2 (2) + \pi \left[ \sqrt{16 - (2)^2} \right]^2 (2) = \pi[(16 - 4) + (16 - 0) + (16 - 4)](2) = 80\pi$   
 (b)  $\frac{|V - S_4|}{V} = \frac{\left| \left( \frac{256}{3} \right) \pi - 80\pi \right|}{\left( \frac{256}{3} \right) \pi} = \frac{16}{256} \approx 6\%$

8. (a)  $S_5 = \pi[(25 - (-3)^2) + (25 - (-1)^2) + (25 - (1)^2) + (25 - (3)^2)](2) = \pi(16 + 24 + 24 + 16)(2) = 160\pi$

(b)  $V = \frac{4}{3}\pi r^3 = \frac{500\pi}{3} \Rightarrow \frac{|V - S_5|}{V} = \frac{\left| \left(\frac{500}{3}\right)\pi - 160\pi \right|}{\left(\frac{500}{3}\right)\pi} = \frac{20}{500} = 4\%$

9. (a)  $S_8 = \pi \left[ (16 - 0^2) + \left(16 - \left(\frac{1}{2}\right)^2\right) + (16 - (1)^2) + \left(16 - \left(\frac{3}{2}\right)^2\right) + (16 - (2)^2) + \left(16 - \left(\frac{5}{2}\right)^2\right) \right. \\ \left. + (16 - (3)^2) + \left(16 - \left(\frac{7}{2}\right)^2\right) \right] \left(\frac{1}{2}\right) = \frac{\pi}{2} \left[ 128 - \frac{1}{4} - 1 - \frac{9}{4} - 4 - \frac{25}{4} - 9 - \frac{49}{4} \right] = \frac{372\pi}{8} = \frac{93\pi}{2}$ , overestimates

(b)  $V = \frac{2}{3}\pi r^3 = \frac{128\pi}{3} \Rightarrow \frac{|V - S_8|}{V} = \frac{\left| \left(\frac{128}{3}\right)\pi - \left(\frac{93}{2}\right)\pi \right|}{\left(\frac{128}{3}\right)\pi} = \frac{23}{256} \approx 9\%$

10. (a)  $S_8 = \pi \left[ \left(16 - \left(\frac{1}{2}\right)^2\right) + (16 - (1)^2) + \left(16 - \left(\frac{3}{2}\right)^2\right) + (16 - (2)^2) + \left(16 - \left(\frac{5}{2}\right)^2\right) + (16 - (3)^2) \right. \\ \left. + \left(16 - \left(\frac{7}{2}\right)^2\right) \right] \left(\frac{1}{2}\right) = \frac{\pi}{2} \left[ 112 - \frac{1}{4} - 1 - \frac{9}{4} - 4 - \frac{25}{4} - 9 - \frac{49}{4} \right] = \frac{308\pi}{8} = \frac{77\pi}{2}$ , underestimates

(b)  $\frac{|V - S_8|}{V} = \frac{\left| \left(\frac{128}{3}\right)\pi - \left(\frac{77}{2}\right)\pi \right|}{\left(\frac{128}{3}\right)\pi} = \frac{25}{256} \approx 10\%$

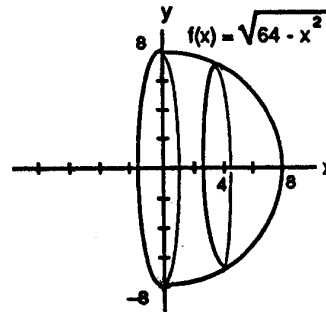
11. (a) To have the same orientation as the hemisphere in Exercise 10, tip the bowl sideways (assume the water is ice). The water covers the interval  $[4, 8]$ . The function which will give us the values of the radii of the approximating cylinders is the equation of the upper semicircle formed by intersecting the

hemisphere with the  $xy$ -plane,  $f(x) = \sqrt{64 - x^2}$ .

Using  $\Delta x = \frac{1}{2}$  and left-endpoints for each interval

$$\begin{aligned} \Rightarrow S_8 &= \pi \left[ (64 - (4)^2) + \left(64 - \left(\frac{9}{2}\right)^2\right) \right. \\ &+ (64 - (5)^2) + \left(64 - \left(\frac{11}{2}\right)^2\right) + (64 - (6)^2) \\ &+ \left(64 - \left(\frac{13}{2}\right)^2\right) + (64 - (7)^2) + \left(64 - \left(\frac{15}{2}\right)^2\right) \left. \right] \left(\frac{1}{2}\right) = \frac{\pi}{2} \left( 512 - 16 - \frac{81}{4} - 25 - \frac{121}{4} - 36 - \frac{169}{4} - 49 - \frac{225}{4} \right) \\ &= \frac{\pi}{2} \left( 386 - \frac{596}{4} \right) = \frac{\pi}{8} (1544 - 596) = \frac{948}{8} \pi = 118.5\pi; \end{aligned}$$

(b)  $\frac{|V - S_8|}{V} = \frac{\left| \left(\frac{320}{3}\right)\pi - \left(\frac{948}{8}\right)\pi \right|}{\left(\frac{320}{3}\right)\pi} = \frac{2844 - 2560}{2560} \approx 11\%$





12. We are using boxes (rectangular parallelepipeds) that are 30 feet wide, 5 feet long, and  $h(x)$  feet deep to approximate the volume of water in the pool.

(a) Using left-hand endpoints in the table:  $S = (30)(5)(6.0) + (30)(5)(8.2) + (30)(5)(9.1) + (30)(5)(9.9) + (30)(5)(10.5) + (30)(5)(11.0) + (30)(5)(11.5) + (30)(5)(11.9) + (30)(5)(12.3) + (30)(5)(12.7) = 15,465 \text{ ft}^3$ .

(b) Using right-hand endpoints in the table:  $S = (30)(5)(8.2) + (30)(5)(9.1) + (30)(5)(9.9) + (30)(5)(10.5) + (30)(5)(11.0) + (30)(5)(11.5) + (30)(5)(11.9) + (30)(5)(12.3) + (30)(5)(12.7) + (30)(5)(13) = 16,515 \text{ ft}^3$ .

13. (a)  $S_5 = \pi [(\sqrt{0})^2 + (\sqrt{1})^2 + (\sqrt{2})^2 + (\sqrt{3})^2 + (\sqrt{4})^2](1) = 10\pi$ , underestimates the volume.

(b)  $\frac{|V - S_5|}{V} = \frac{\left(\frac{25}{2}\right)\pi - 10\pi}{\left(\frac{25}{2}\right)\pi} = \frac{5}{25} = 20\%$

14. (a)  $S_5 = \pi [(\sqrt{1})^2 + (\sqrt{2})^2 + (\sqrt{3})^2 + (\sqrt{4})^2 + (\sqrt{5})^2](1) = 15\pi$ , overestimates the volume.

(b)  $\frac{|V - S_5|}{V} = \frac{15\pi - \left(\frac{25}{2}\right)\pi}{\left(\frac{25}{2}\right)\pi} = \frac{5}{25} = 20\%$

15. (a) Because the acceleration is decreasing, an upper estimate is obtained using left end-points in summing acceleration  $\cdot \Delta t$ . Thus,  $\Delta t = 1$  and speed  $\approx [32.00 + 19.41 + 11.77 + 7.14 + 4.33](1) = 74.65 \text{ ft/sec}$

(b) Using right end-points we obtain a lower estimate: speed  $\approx [19.41 + 11.77 + 7.14 + 4.33 + 2.63](1) = 45.28 \text{ ft/sec}$

- (c) Upper estimates for the speed at each second are:

t	0	1	2	3	4	5
v	0	32.00	51.41	63.18	70.32	74.65

Thus, the distance fallen when  $t = 3$  seconds is  $s \approx [32.00 + 51.41 + 63.18](1) = 146.59 \text{ ft}$ .

16. (a) The speed is a decreasing function of time  $\Rightarrow$  left end-points give an upper estimate for the height (distance) attained. Also

t	0	1	2	3	4	5
v	400	368	336	304	272	240

gives the time-velocity table by subtracting the constant  $g = 32$  from the speed at each time increment  $\Delta t = 1 \text{ sec}$ . Thus, the speed  $\approx 240 \text{ ft/sec}$  after 5 seconds.

- (b) A lower estimate for height attained is  $h \approx [368 + 336 + 304 + 272 + 240](1) = 1520 \text{ ft}$ .

17. Partition  $[0, 2]$  into the four subintervals  $\left[0, \frac{1}{2}\right]$ ,  $\left[\frac{1}{2}, 1\right]$ ,  $\left[1, \frac{3}{2}\right]$ , and  $\left[\frac{3}{2}, 2\right]$ . The midpoints of these subintervals are  $m_1 = \frac{1}{4}$ ,  $m_2 = \frac{3}{4}$ ,  $m_3 = \frac{5}{4}$ , and  $m_4 = \frac{7}{4}$ . The heights of the four approximating

rectangles are  $f(m_1) = \left(\frac{1}{4}\right)^3 = \frac{1}{64}$ ,  $f(m_2) = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$ ,  $f(m_3) = \left(\frac{5}{4}\right)^3 = \frac{125}{64}$ , and  $f(m_4) = \left(\frac{7}{4}\right)^3 = \frac{343}{64}$

$\Rightarrow$  Average value  $\approx \frac{\frac{1}{64} + \frac{27}{64} + \frac{125}{64} + \frac{343}{64}}{4} = \frac{1 + 27 + 125 + 343}{4 \cdot 64} = \frac{496}{256} = \frac{31}{16}$ . Notice that the average value is

approximated by  $\frac{\left(\frac{1}{4}\right)^3 + \left(\frac{3}{4}\right)^3 + \left(\frac{5}{4}\right)^3 + \left(\frac{7}{4}\right)^3}{4} = \frac{1}{2} \left[ \left(\frac{1}{4}\right)^3 \left(\frac{1}{2}\right) + \left(\frac{3}{4}\right)^3 \left(\frac{1}{2}\right) + \left(\frac{5}{4}\right)^3 \left(\frac{1}{2}\right) + \left(\frac{7}{4}\right)^3 \left(\frac{1}{2}\right) \right]$

$= \frac{1}{\text{length of } [0, 2]} \cdot \left[ \begin{array}{l} \text{approximate area under} \\ \text{curve } f(x) = x^3 \end{array} \right]$ . We use this observation in solving the next several exercises.

18. Partition  $[1, 9]$  into the four subintervals  $[1, 3]$ ,  $[3, 5]$ ,  $[5, 7]$ , and  $[7, 9]$ . The midpoints of these subintervals are  $m_1 = 2$ ,  $m_2 = 4$ ,  $m_3 = 6$ , and  $m_4 = 8$ . The heights of the four approximating rectangles are  $f(m_1) = \frac{1}{2}$ ,

$f(m_2) = \frac{1}{4}$ ,  $f(m_3) = \frac{1}{8}$ , and  $f(m_4) = \frac{1}{8}$ . The width of each rectangle is  $\Delta x = 2$ . Thus,

Area  $\approx 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 2\left(\frac{1}{8}\right) + 2\left(\frac{1}{8}\right) = \frac{25}{12} \Rightarrow$  average value  $\approx \frac{\text{area}}{\text{length of } [1, 9]} = \frac{\left(\frac{25}{12}\right)}{8} = \frac{25}{96}$ .

19. Partition  $[0, 2]$  into the four subintervals  $[0, 0.5]$ ,  $[0.5, 1]$ ,  $[1, 1.5]$ , and  $[1.5, 2]$ . The midpoints of the subintervals are  $m_1 = 0.25$ ,  $m_2 = 0.75$ ,  $m_3 = 1.25$ , and  $m_4 = 1.75$ . The heights of the four approximating rectangles are

$f(m_1) = \frac{1}{2} + \sin^2 \frac{\pi}{4} = \frac{1}{2} + \frac{1}{2} = 1$ ,  $f(m_2) = \frac{1}{2} + \sin^2 \frac{3\pi}{4} = \frac{1}{2} + \frac{1}{2} = 1$ ,  $f(m_3) = \frac{1}{2} + \sin^2 \frac{5\pi}{4} = \frac{1}{2} + \left(-\frac{1}{\sqrt{2}}\right)^2$   
 $= \frac{1}{2} + \frac{1}{2} = 1$ , and  $f(m_4) = \frac{1}{2} + \sin^2 \frac{7\pi}{4} = \frac{1}{2} + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1$ . The width of each rectangle is  $\Delta x = \frac{1}{2}$ . Thus,

Area  $\approx (1 + 1 + 1 + 1)\left(\frac{1}{2}\right) = 2 \Rightarrow$  average value  $\approx \frac{\text{area}}{\text{length of } [0, 2]} = \frac{2}{2} = 1$ .

20. Partition  $[0, 4]$  into the four subintervals  $[0, 1]$ ,  $[1, 2]$ ,  $[2, 3]$ , and  $[3, 4]$ . The midpoints of the subintervals are  $m_1 = \frac{1}{2}$ ,  $m_2 = \frac{3}{2}$ ,  $m_3 = \frac{5}{2}$ , and  $m_4 = \frac{7}{2}$ . The heights of the four approximating rectangles are

$f(m_1) = 1 - \left(\cos\left(\frac{\pi\left(\frac{1}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{\pi}{8}\right)\right)^4 = 0.27145$  (to 5 decimal places),

$f(m_2) = 1 - \left(\cos\left(\frac{\pi\left(\frac{3}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{3\pi}{8}\right)\right)^4 = 0.97855$ ,  $f(m_3) = 1 - \left(\cos\left(\frac{\pi\left(\frac{5}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{5\pi}{8}\right)\right)^4$

$= 0.97855$ , and  $f(m_4) = 1 - \left(\cos\left(\frac{\pi\left(\frac{7}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{7\pi}{8}\right)\right)^4 = 0.27145$ . The width of each rectangle is

$\Delta x = 1$ . Thus, Area  $\approx (0.27145)(1) + (0.97855)(1) + (0.97855)(1) + (0.27145)(1) = 2.5 \Rightarrow$  average

value  $\approx \frac{\text{area}}{\text{length of } [0, 4]} = \frac{2.5}{4} = \frac{5}{8}$ .

21. Since the leakage is increasing, an upper estimate uses right end-points and a lower estimate uses left end-points:

- (a) upper estimate =  $(70)(1) + (97)(1) + (136)(1) + (190)(1) + (265)(1) = 758$  gal,  
 lower estimate =  $(50)(1) + (70)(1) + (97)(1) + (136)(1) + (190)(1) = 543$  gal.
- (b) upper estimate =  $(70 + 97 + 136 + 190 + 265 + 369 + 516 + 720) = 2363$  gal,  
 lower estimate =  $(50 + 70 + 97 + 136 + 190 + 265 + 369 + 516) = 1693$  gal.
- (c) worst case:  $2363 + 720t = 25,000 \Rightarrow t \approx 31.4$  hrs;  
 best case:  $1693 + 720t = 25,000 \Rightarrow t \approx 32.4$  hrs

22. Since the pollutant release increases over time, an upper estimate uses right end-points and a lower estimate uses left end-points:

- (a) upper estimate =  $(0.2)(30) + (0.25)(30) + (0.27)(30) + (0.34)(30) + (0.45)(30) + (0.52)(30) = 60.9$  tons  
 lower estimate =  $(0.05)(30) + (0.2)(30) + (0.25)(30) + (0.27)(30) + (0.34)(30) + (0.45)(30) = 46.8$  tons
- (b) Using the lower (best case) estimate:  $46.8 + (0.52)(30) + (0.63)(30) + (0.70)(30) + (0.81)(30) = 126.6$  tons,  
 so near the end of September 125 tons of pollutants will have been released.

23. (a) The diagonal of the square has length 2, so the side length is  $\sqrt{2}$ . Area =  $(\sqrt{2})^2 = 2$
- (b) Think of the octagon as a collection of 16 right triangles with a hypotenuse of length 1 and an acute angle measuring  $\frac{2\pi}{16} = \frac{\pi}{8}$ .  
 Area =  $16 \left(\frac{1}{2}\right) \left(\sin \frac{\pi}{8}\right) \left(\cos \frac{\pi}{8}\right) = 4 \sin \frac{\pi}{4} = 2\sqrt{2} \approx 2.828$
- (c) Think of the 16-gon as a collection of 32 right triangles with a hypotenuse of length 1 and an acute angle measuring  $\frac{2\pi}{32} = \frac{\pi}{16}$ .  
 Area =  $32 \left(\frac{1}{2}\right) \left(\sin \frac{\pi}{16}\right) \left(\cos \frac{\pi}{16}\right) = 8 \sin \frac{\pi}{8} \approx 3.061$
- (d) Each area is less than the area of the circle,  $\pi$ . As  $n$  increases, the area approaches  $\pi$ .

24. (a) Each of the isosceles triangles is made up of two right triangles having hypotenuse 1 and an acute angle measuring  $\frac{2\pi}{2n} = \frac{\pi}{n}$ . The area of each isosceles triangle is  $A_T = 2 \left(\frac{1}{2}\right) \left(\sin \frac{\pi}{n}\right) \left(\cos \frac{\pi}{n}\right) = \frac{1}{2} \sin \frac{2\pi}{n}$ .
- (b) The area of the polygon is  $A_P = nA_T = \frac{n}{2} \sin \frac{2\pi}{n}$ , so  $\lim_{n \rightarrow \infty} A_P = \lim_{n \rightarrow \infty} \frac{n}{2} \sin \frac{2\pi}{n} = \lim_{n \rightarrow \infty} \pi \cdot \frac{\sin \frac{2\pi}{n}}{\left(\frac{2\pi}{n}\right)} = \pi$
- (c) Multiply each area by  $r^2$ .  
 $A_T = \frac{1}{2} r^2 \sin \frac{2\pi}{n}$   
 $A_P = \frac{n}{2} r^2 \sin \frac{2\pi}{n}$   
 $\lim_{n \rightarrow \infty} A_P = \pi r^2$

25-28. Example CAS commands:

Maple:

```
with(student):
f:=x -> sin(x); a:= 0; b:= Pi;
plot(f(x),x=a..b);
n:= 1000;
```

```

middlebox(f(x),x=a..b,n);
middlesum(f(x),x=a..b,n);
average:= evalf(%)/(b-a);
fsolve(f(x)=average,x);

```

**Mathematica:**

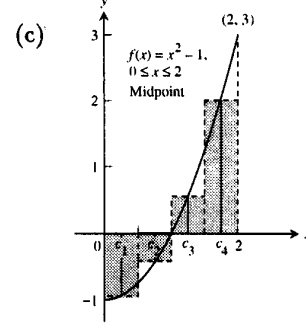
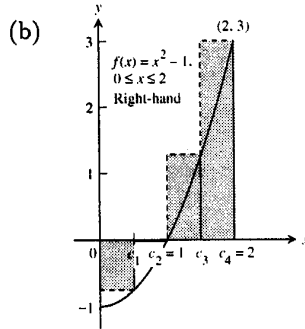
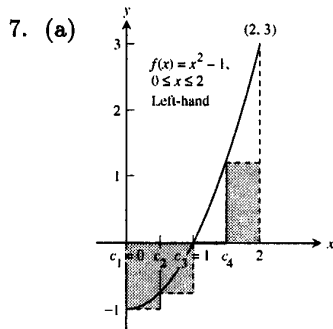
```

Clear[x]
f[x_] = Sin[x]
{a,b} = {0,Pi};
Plot[ f[x], {x,a,b} ]
n = 100; dx = (b-a)/n;
Table[ N[f[x]], {x,a+dx/2,b,dx} ];
fave = (Plus @@ %) / n
n = 200; dx = (b-a)/n;
Table[ N[f[x]], {x,a+dx/2,b,dx} ];
fave = (Plus @@ %) / n
n = 1000; dx = (b-a)/n;
Table[ N[f[x]], {x,a+dx/2,b,dx} ];
fave = (Plus @@ %) / n
FindRoot[ f[x] - fave, {x,a} ]

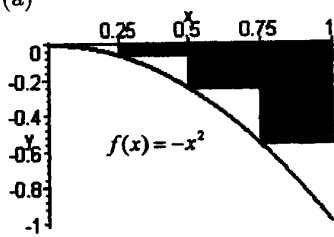
```

**4.4 RIEMANN SUMS AND DEFINITE INTEGRALS**

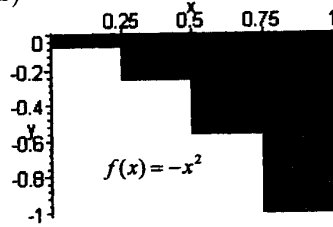
- $\sum_{k=1}^2 \frac{6k}{k+1} = \frac{6(1)}{1+1} + \frac{6(2)}{2+1} = \frac{6}{2} + \frac{12}{3} = 7$
- $\sum_{k=1}^3 \frac{k-1}{k} = \frac{1-1}{1} + \frac{2-1}{2} + \frac{3-1}{3} = 0 + \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
- $\sum_{k=1}^4 \cos k\pi = \cos(1\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) = -1 + 1 - 1 + 1 = 0$
- $\sum_{k=1}^5 \sin k\pi = \sin(1\pi) + \sin(2\pi) + \sin(3\pi) + \sin(4\pi) + \sin(5\pi) = 0 + 0 + 0 + 0 + 0 = 0$
- $\sum_{k=1}^3 (-1)^{k+1} \sin \frac{\pi}{k} = (-1)^{1+1} \sin \frac{\pi}{1} + (-1)^{2+1} \sin \frac{\pi}{2} + (-1)^{3+1} \sin \frac{\pi}{3} = 0 - 1 + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}-2}{2}$
- $\sum_{k=1}^4 (-1)^k \cos k\pi = (-1)^1 \cos(1\pi) + (-1)^2 \cos(2\pi) + (-1)^3 \cos(3\pi) + (-1)^4 \cos(4\pi)$   
 $= -(-1) + 1 - (-1) + 1 = 4$



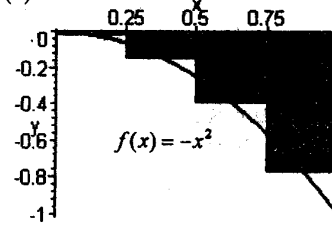
8. (a)



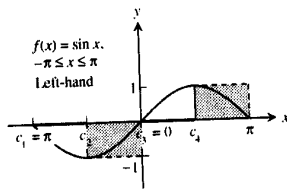
(b)



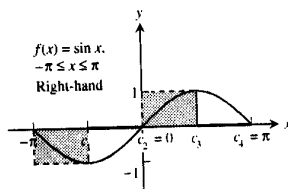
(c)



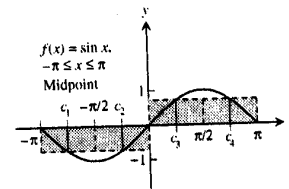
9. (a)



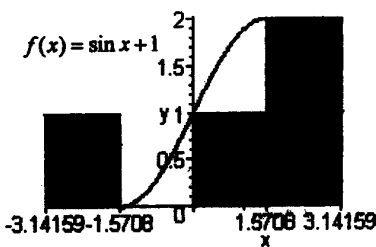
(b)



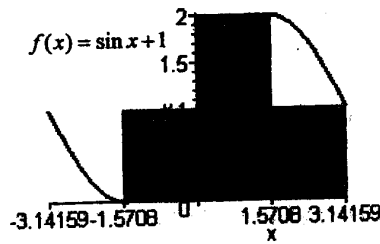
(c)



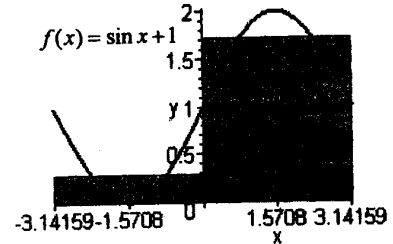
10. (a)



(b)



(c)



11.  $\int_0^2 x^2 dx$

12.  $\int_{-1}^0 2x^3 dx$

13.  $\int_{-7}^5 (x^2 - 3x) dx$

14.  $\int_2^3 \frac{1}{1-x} dx$

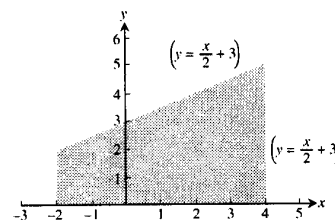
15.  $\int_0^1 \sqrt{4-x^2} dx$

16.  $\int_{-\pi/4}^0 (\sec x) dx$

17. The area of the trapezoid is  $A = \frac{1}{2}(B + b)h$

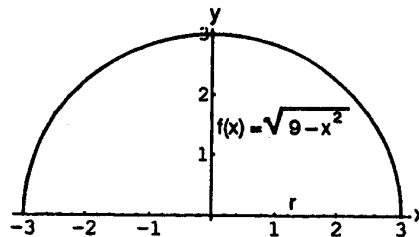
$$= \frac{1}{2}(5 + 2)(6) = 21 \Rightarrow \int_{-2}^4 \left(\frac{x}{2} + 3\right) dx$$

$$= 21 \text{ square units}$$



18. The area of the semicircle is  $A = \frac{1}{2} \pi r^2 = \frac{1}{2} \pi (3)^2$

$$= \frac{9}{2} \pi \Rightarrow \int_{-3}^3 \sqrt{9-x^2} dx = \frac{9}{2} \pi \text{ square units}$$

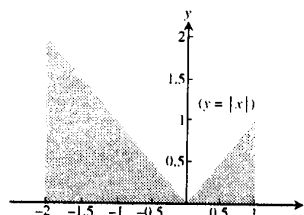


19. The area of the triangle on the left is  $A = \frac{1}{2} bh = \frac{1}{2} (2)(2)$

$= 2$ . The area of the triangle on the right is  $A = \frac{1}{2} bh$

$$= \frac{1}{2} (1)(1) = \frac{1}{2}. \text{ Then, the total area is } 2.5 \Rightarrow \int_{-2}^1 |x| dx$$

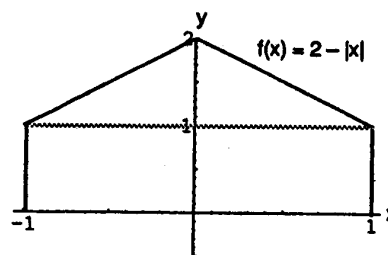
$= 2.5$  square units



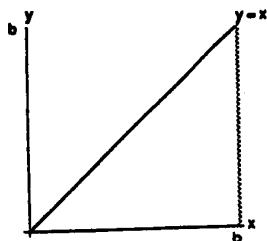
20. The area of the triangular peak is  $A = \frac{1}{2} bh = \frac{1}{2} (2)(1) = 1$ .

The area of the rectangular base is  $S = \ell w = (2)(1) = 2$ .

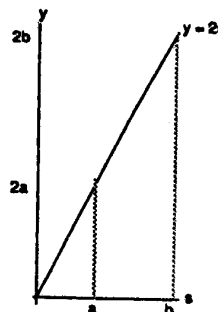
$$\text{Then the total area is } 3 \Rightarrow \int_{-1}^1 (2 - |x|) dx = 3 \text{ square units}$$



21.  $\int_0^b x dx = \frac{1}{2} (b)(b) = \frac{b^2}{2}$  square units



22.  $\int_a^b 2s ds = \frac{1}{2} b(2b) - \frac{1}{2} a(2a) = b^2 - a^2$  square units



23. The graph of  $f(x) = 1 - x$  on the interval  $[0, 1]$  forms a right isosceles triangle in the first quadrant with its two legs, each of length one, lying on the coordinate axes. The area of the triangle is  $A = \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$ , which

is also the value of the integral  $\int_0^1 (1-x) dx = \frac{1}{2}$ , therefore,  $\text{av}(f) = \frac{1}{1-0} \int_0^1 (1-x) dx = (1)\left(\frac{1}{2}\right) = \frac{1}{2}$ .

24. The graph of  $f(x) = |x|$  on the interval  $[-1, 1]$  forms two congruent isosceles right triangles one in the first and the other in the second quadrant. The total area of these two triangles is  $A = 2\left(\frac{1}{2}bh\right) = 2\left(\frac{1}{2} \cdot 1 \cdot 1\right) = 1$ ,

which is also the value of the integral  $\int_{-1}^1 |x| dx$ , therefore,  $\text{av}(f) = \frac{1}{1-(-1)} \int_{-1}^1 |x| dx = \frac{1}{2}(1) = \frac{1}{2}$ .

25. The function  $f(x) = \sqrt{1-x^2}$  on the interval  $[0, 1]$  forms a quarter-circular area of radius 1 lying in the first quadrant with its center on the origin. The area of this quarter-circle is  $A = \frac{\pi}{4}r^2 = \frac{\pi}{4}$ , which is also the value

of the integral  $\int_0^1 \sqrt{1-x^2} dx$ , therefore,  $\text{av}(f) = \frac{1}{1-0} \int_0^1 \sqrt{1-x^2} dx = (1)\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$ .

26. The function  $f(x) = \sqrt{1-(x-2)^2}$  on the interval  $[1, 2]$  forms a quarter-circular area of radius 1, lying in the first quadrant with the center of the circle on the point  $(2, 0)$ . The area of this quarter circle is  $A = \frac{\pi}{4}r^2 = \frac{\pi}{4}$ ,

which is also the value of the integral  $\int_1^2 \sqrt{1-(x-2)^2} dx$ , therefore,

$$\text{av}(f) = \frac{1}{2-1} \int_1^2 \sqrt{1-(x-2)^2} dx = (1)\left(\frac{\pi}{4}\right) = \frac{\pi}{4}.$$

$$27. (a) \int_2^2 g(x) dx = 0$$

$$(b) \int_5^1 g(x) dx = - \int_1^5 g(x) dx = -8$$

$$(c) \int_1^2 3f(x) dx = 3 \int_1^2 f(x) dx = 3(-4) = -12$$

$$(d) \int_2^5 f(x) dx = \int_1^5 f(x) dx - \int_1^2 f(x) dx = 6 - (-4) = 10$$

$$(e) \int_1^5 [f(x) - g(x)] dx = \int_1^5 f(x) dx - \int_1^5 g(x) dx = 6 - 8 = -2$$

$$(f) \int_1^5 [4f(x) - g(x)] dx = 4 \int_1^5 f(x) dx - \int_1^5 g(x) dx = 4(6) - 8 = 16$$

$$28. (a) \int_1^9 -2f(x) dx = -2 \int_1^9 f(x) dx = -2(-1) = 2$$

$$(b) \int_7^9 [f(x) + h(x)] dx = \int_7^9 f(x) dx + \int_7^9 h(x) dx = 5 + 4 = 9$$

$$(c) \int_7^9 [2f(x) - 3h(x)] dx = 2 \int_7^9 f(x) dx - 3 \int_7^9 h(x) dx = 2(5) - 3(4) = -2$$

$$(d) \int_9^1 f(x) dx = - \int_1^9 f(x) dx = -(-1) = 1$$

$$(e) \int_1^7 f(x) dx = \int_1^9 f(x) dx - \int_7^9 f(x) dx = -1 - 5 = -6$$

$$(f) \int_9^7 [h(x) - f(x)] dx = \int_7^9 [f(x) - h(x)] dx = \int_7^9 f(x) dx - \int_7^9 h(x) dx = 5 - 4 = 1$$

$$29. (a) \int_1^2 f(u) du = \int_1^2 f(x) dx = 5$$

$$(b) \int_1^2 \sqrt{3}f(z) dz = \sqrt{3} \int_1^2 f(z) dz = 5\sqrt{3}$$

$$(c) \int_2^1 f(t) dt = - \int_1^2 f(t) dt = -5$$

$$(d) \int_1^2 [-f(x)] dx = - \int_1^2 f(x) dx = -5$$

$$30. (a) \int_0^{-3} g(t) dt = - \int_{-3}^0 g(t) dt = -\sqrt{2}$$

$$(b) \int_{-3}^0 g(u) du = \int_{-3}^0 g(t) dt = \sqrt{2}$$

$$(c) \int_{-3}^0 [-g(x)] dx = - \int_{-3}^0 g(x) dx = -\sqrt{2}$$

$$(d) \int_{-3}^0 \frac{g(r)}{\sqrt{2}} dr = \frac{1}{\sqrt{2}} \int_{-3}^0 g(t) dt = \left(\frac{1}{\sqrt{2}}\right)(\sqrt{2}) = 1$$

$$31. (a) \int_3^4 f(z) dz = \int_0^4 f(z) dz - \int_0^3 f(z) dz = 7 - 3 = 4$$

$$(b) \int_4^3 f(t) dt = - \int_3^4 f(t) dt = -4$$

$$32. (a) \int_1^3 h(r) dr = \int_{-1}^3 h(r) dr - \int_{-1}^1 h(r) dr = 6 - 0 = 6$$

$$(b) - \int_3^1 h(u) du = - \left( - \int_1^3 h(u) du \right) = \int_1^3 h(u) du = 6$$

33. To find where  $x - x^2 \geq 0$ , let  $x - x^2 = 0 \Rightarrow x(1 - x) = 0 \Rightarrow x = 0$  or  $x = 1$ . If  $0 < x < 1$ , then  $x^2 < x \Rightarrow 0 < x - x^2 \Rightarrow a = 0$  and  $b = 1$  maximize the integral.



34. To find where  $x^4 - 2x^2 \leq 0$ , let  $x^4 - 2x^2 = 0 \Rightarrow x^2(x^2 - 2) = 0 \Rightarrow x = 0$  or  $x = \pm\sqrt{2}$ . By the sign graph,   
 $+++++ \begin{matrix} 0 & - & 0 & - & 0 \\ -\sqrt{2} & & 0 & & \sqrt{2} \end{matrix} ++++++$ , we can see that  $x^4 - 2x^2 \leq 0$  on  $[-\sqrt{2}, \sqrt{2}] \Rightarrow a = -\sqrt{2}$  and  $b = \sqrt{2}$  minimize the integral.

35. By the constant multiple rule,  $\int_a^b k \, dx = k \int_a^b 1 \, dx$ . The Riemann sums definition of the definite integral gives

$$\int_a^b 1 \, dx = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \Delta x_k, \text{ and if } \Delta x_k = \frac{b-a}{n}, \text{ then } \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \Delta x_k = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \frac{b-a}{n}$$

$$= \lim_{\|P\| \rightarrow 0} \left( \frac{b-a}{n} \sum_{k=1}^n 1 \right) = \lim_{\|P\| \rightarrow 0} \left( \frac{b-a}{n} \cdot n \right) = \lim_{\|P\| \rightarrow 0} (b-a) = b-a. \text{ Therefore, } \int_a^b k \, dx = k(b-a),$$

for any  $k$ .

36. If  $f(x) \geq 0$  on  $[a, b]$ , then  $\min f \geq 0$  and  $\max f \geq 0$  on  $[a, b]$ . Now,  $(b-a) \min f \leq \int_a^b f(x) \, dx \leq (b-a) \max f$ .

$$\text{Then } b \geq a \Rightarrow b-a \geq 0 \Rightarrow (b-a) \min f \geq 0 \Rightarrow \int_a^b f(x) \, dx \geq 0.$$

37.  $f(x) = \frac{1}{1+x^2}$  is decreasing on  $[0, 1] \Rightarrow$  maximum value of  $f$  occurs at  $0 \Rightarrow \max f = f(0) = 1$ ; minimum value of  $f$

$$\text{occurs at } 1 \Rightarrow \min f = f(1) = \frac{1}{1+1^2} = \frac{1}{2}. \text{ Therefore, } (1-0) \min f \leq \int_0^1 \frac{1}{1+x^2} \, dx \leq (1-0) \max f$$

$$\Rightarrow \frac{1}{2} \leq \int_0^1 \frac{1}{1+x^2} \, dx \leq 1. \text{ That is, an upper bound } = 1 \text{ and a lower bound } = \frac{1}{2}.$$

38. See Exercise 37 above. On  $[0, 0.5]$ ,  $\max f = \frac{1}{1+0^2} = 1$ ,  $\min f = \frac{1}{1+(0.5)^2} = 0.8$ . Therefore

$$(0.5-0) \min f \leq \int_0^{0.5} f(x) \, dx \leq (0.5-0) \max f \Rightarrow 0.4 \leq \int_0^{0.5} \frac{1}{1+x^2} \, dx \leq 0.5. \text{ On } [0.5, 1], \max f = \frac{1}{1+(0.5)^2} = 0.8$$

$$\text{and } \min f = \frac{1}{1+1^2} = 0.5. \text{ Thus } (1-0.5) \min f \leq \int_{0.5}^1 \frac{1}{1+x^2} \, dx \leq (1-0.5) \max f \Rightarrow 0.25 \leq \int_{0.5}^1 \frac{1}{1+x^2} \, dx \leq 0.4.$$

$$\text{Then } 0.25 + 0.4 \leq \int_0^{0.5} \frac{1}{1+x^2} \, dx + \int_{0.5}^1 \frac{1}{1+x^2} \, dx \leq 0.5 + 0.4 \Rightarrow 0.65 \leq \int_0^1 \frac{1}{1+x^2} \, dx \leq 0.9.$$

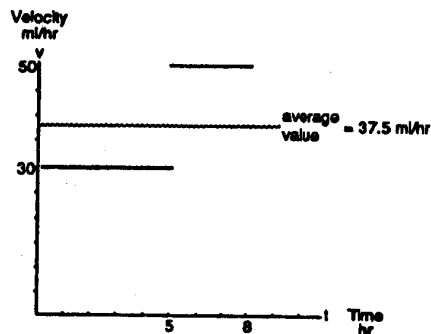
39. The car drove the first 150 miles in 5 hours and the second 150 miles in 3 hours, which means it drove 300 miles in 8 hours, for an average of  $\frac{300}{8}$  mi/hr = 37.5 mi/hr. In terms of average values of functions, the function whose average value we seek is

$$v(t) = \begin{cases} 30, & 0 \leq t \leq 5 \\ 50, & 5 < t \leq 8 \end{cases}, \text{ and the average value is}$$

$$\frac{(30)(5) + (50)(3)}{8} = 37.5 \text{ mph. It does not help to consider}$$

$$v(s) = \begin{cases} 30, & 0 \leq s \leq 150 \\ 50, & 150 < s \leq 300 \end{cases} \text{ whose average value is } \frac{(30)(150) + (50)(150)}{300} = 40 \text{ (mph)/mi}$$

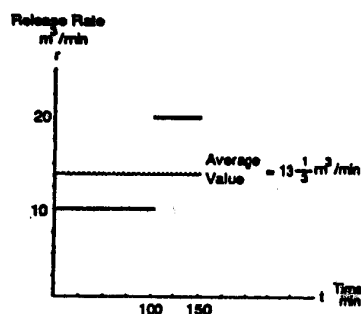
because we want the average speed with respect to time, not distance.



40. The dam released  $1000 \text{ m}^3$  of water in 100 min and then released another  $1000 \text{ m}^3$  of water in 50 min, for a total of  $2000 \text{ m}^3$  in 150 min, which averages to  $\frac{2000}{150} = \frac{40}{3} \text{ m}^3/\text{min}$ . In terms of average values of functions, the function whose average value we seek is

$$r(t) = \begin{cases} 10, & 0 \leq t \leq 100 \\ 20, & 100 < t \leq 150 \end{cases}, \text{ and the average value is}$$

$$\frac{(10)(100) + (20)(50)}{150} = \frac{40}{3} \text{ m}^3/\text{min}.$$



41-46. Example CAS commands:

**Maple:**

```
with(student):
f:= x -> x ^ 2 + 1; a:= 0; b:= 1;
n:=20;
leftbox(f(x),x=a..b,n);
leftsum(f(x),x=a..b,n);
evalf(%);
rightbox(f(x),x=a..b,n);
rightsum(f(x),x=a..b,n);
evalf(%);
middlebox(f(x),x=a..b,n);
middlesum(f(x),x=a..b,n);
evalf(%);
```

**Mathematica:**

This CAS does not have the leftbox, leftsum, etc. commands. Here are definitions of 3 functions that plot the boxes and also return the Riemann sum, using either left endpoints, right endpoints, or midpoints of each subinterval for the values of the function. The arguments to each are:

f: a pure function of one variable  
 {a,b}: the interval  
 n: the (positive integer) number of subintervals  
 plotopts: (optional) options for the plot

```

LeftSum[f_, {a_, b_}, n_, plotopts_____] := Module[
{x, dx = (b-a)/n, xvals, yvals, boxes},
xvals = Table[ N[x], {x, a, b-dx, dx} ];
yvals = Map[ f, xvals ] // N;
boxes = MapThread[
  Line[{{#1,0},{#1,#3},{#2,#3},{#2,0}}]&,
  {xvals,xvals+dx,yvals} ];
Plot[ f[x], {x,a,b}, Epilog -> boxes, plotopts ];
(Plus @@ yvals)*dx // N
]
RightSum[f_, {a_, b_}, n_, plotopts_____] := Module[
{x, dx = (b-a)/n, xvals, yvals, boxes},
xvals = Table[ N[x], {x, a+dx, b, dx} ];
yvals = Map[ f, xvals ] // N;
boxes = MapThread[
  Line[{{#1,0},{#1,#3},{#2,#3},{#2,0}}]&,
  {xvals-dx,xvals,yvals} ];
Plot[ f[x], {x,a,b}, Epilog -> boxes, plotopts ];
(Plus @@ yvals)*dx // N
]
MiddleSum[f_, {a_, b_}, n_, plotopts_____] := Module[
{x, dx = (b-a)/n, xvals, yvals, boxes},
xvals = Table[ N[x], {x, a+dx/2, b, dx} ];
yvals = Map[ f, xvals ] // N;
boxes = MapThread[
  Line[{{#1,0},{#1,#3},{#2,#3},{#2,0}}]&,
  {xvals-dx/2,xvals+dx/2,yvals} ];
Plot[ f[x], {x,a,b}, Epilog -> boxes, plotopts ];
(Plus @@ yvals)*dx // N
]
Clear[x]
f[x_] = x^2 + 1
{a,b} = {0,1};
n = 20;
LeftSum[ f, {a,b}, n ]
RightSum[ f, {a,b}, n ]
MiddleSum[ f, {a,b}, n ]

```

#### 4.5 THE MEAN VALUE AND FUNDAMENTAL THEOREMS

- $$\int_{-2}^0 (2x + 5) dx = [x^2 + 5x]_{-2}^0 = (0^2 + 5(0)) - ((-2)^2 + 5(-2)) = 6$$
- $$\int_0^4 \left( 3x - \frac{x^3}{4} \right) dx = \left[ \frac{3x^2}{2} - \frac{x^4}{16} \right]_0^4 = \left( \frac{3(4)^2}{2} - \frac{4^4}{16} \right) - \left( \frac{3(0)^2}{2} - \frac{(0)^4}{16} \right) = 8$$

$$3. \int_0^1 (x^2 + \sqrt{x}) dx = \left[ \frac{x^3}{3} + \frac{2}{3}x^{3/2} \right]_0^1 = \left( \frac{1}{3} + \frac{2}{3} \right) - 0 = 1$$

$$4. \int_{-2}^{-1} \frac{2}{x^2} dx = \int_{-2}^{-1} 2x^{-2} dx = [-2x^{-1}]_{-2}^{-1} = \left( \frac{-2}{-1} \right) - \left( \frac{-2}{-2} \right) = 1$$

$$5. \int_0^{\pi} (1 + \cos x) dx = [x + \sin x]_0^{\pi} = (\pi + \sin \pi) - (0 + \sin 0) = \pi$$

$$6. \int_0^{\pi/3} 2 \sec^2 x dx = [2 \tan x]_0^{\pi/3} = \left( 2 \tan \left( \frac{\pi}{3} \right) \right) - (2 \tan 0) = 2\sqrt{3} - 0 = 2\sqrt{3}$$

$$7. \int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta d\theta = [-\csc \theta]_{\pi/4}^{3\pi/4} = \left( -\csc \left( \frac{3\pi}{4} \right) \right) - \left( -\csc \left( \frac{\pi}{4} \right) \right) = -\sqrt{2} - (-\sqrt{2}) = 0$$

$$8. \int_0^{\pi/2} \frac{1 + \cos 2t}{2} dt = \frac{1}{2} \left( t + \frac{\sin 2t}{2} \right) \Big|_0^{\pi/2} = \frac{1}{2} \left[ \left( \frac{\pi}{2} + \frac{\sin 2(\pi/2)}{2} \right) - \left( 0 + \frac{\sin 2(0)}{2} \right) \right] = \frac{\pi}{4}$$

$$9. \int_{-\pi/2}^{\pi/2} (8y^2 + \sin y) dy = \left[ \frac{8y^3}{3} - \cos y \right]_{-\pi/2}^{\pi/2} = \left( \frac{8 \left( \frac{\pi}{2} \right)^3}{3} - \cos \frac{\pi}{2} \right) - \left( \frac{8 \left( -\frac{\pi}{2} \right)^3}{3} - \cos \left( -\frac{\pi}{2} \right) \right) = \frac{2\pi^3}{3}$$

$$10. \int_{-1}^1 (r+1)^2 dr = \int_{-1}^1 (r^2 + 2r + 1) dr = \left[ \frac{r^3}{3} + r^2 + r \right]_{-1}^1 = \left( \frac{1^3}{3} + 1^2 + 1 \right) - \left( \frac{(-1)^3}{3} + (-1)^2 + (-1) \right) = \frac{8}{3}$$

$$11. \int_1^{\sqrt{2}} \left( \frac{u^2}{2} - \frac{1}{u^5} \right) du = \int_1^{\sqrt{2}} \left( \frac{u^2}{2} - u^{-5} \right) du = \left( \frac{u^3}{6} + \frac{u^{-4}}{4} \right) \Big|_1^{\sqrt{2}} = \left[ \left( \frac{(\sqrt{2})^3}{6} + \frac{1}{4(\sqrt{2})^4} \right) - \left( \frac{(1)^3}{6} + \frac{1}{4(1)^4} \right) \right]$$

$$= \frac{\sqrt{2}}{3} + \frac{1}{16} - \frac{1}{6} - \frac{1}{4} = \frac{16\sqrt{2} + 3 - 8 - 12}{48} = \frac{16\sqrt{2} - 17}{48}$$

$$12. \int_4^9 \frac{1 - \sqrt{u}}{\sqrt{u}} du = \int_4^9 (u^{-1/2} - 1) du = (2u^{1/2} - u) \Big|_4^9 = [(2\sqrt{9} - 9) - (2\sqrt{4} - 4)] = -3 - (0) = -3$$

$$13. \text{ Let } u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow \frac{du}{2} = x dx \Rightarrow \int x e^{x^2} dx = \int \frac{1}{2} e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C$$

$$\Rightarrow \int_0^1 x e^{x^2} dx = \frac{1}{2} e^{x^2} \Big|_0^1 = \frac{e^1}{2} - \frac{e^0}{2} = \frac{e-1}{2}$$

$$14. \int_0^{\ln 2} e^{3x} dx = \frac{e^{3x}}{3} \Big|_0^{\ln 2} = \frac{e^{3 \ln 2}}{3} - \frac{e^{3(0)}}{3} = \frac{e^{\ln(2^3)} - 1}{3} = \frac{2^3 - 1}{3} = \frac{7}{3} = 2\frac{1}{3}$$

$$15. (a) \int_0^{\sqrt{x}} \cos t dt = [\sin t]_0^{\sqrt{x}} = \sin \sqrt{x} - \sin 0 = \sin \sqrt{x} \Rightarrow \frac{d}{dx} \left( \int_0^{\sqrt{x}} \cos t dt \right) = \frac{d}{dx} (\sin \sqrt{x}) = \cos \sqrt{x} \left( \frac{1}{2} x^{-1/2} \right) \\ = \frac{\cos \sqrt{x}}{2\sqrt{x}}$$

$$(b) \frac{d}{dx} \left( \int_0^{\sqrt{x}} \cos t dt \right) = (\cos \sqrt{x}) \left( \frac{d}{dx} (\sqrt{x}) \right) = (\cos \sqrt{x}) \left( \frac{1}{2} x^{-1/2} \right) = \frac{\cos \sqrt{x}}{2\sqrt{x}}$$

$$16. (a) \int_1^{\sin x} 3t^2 dt = [t^3]_1^{\sin x} = \sin^3 x - 1 \Rightarrow \frac{d}{dx} \left( \int_1^{\sin x} 3t^2 dt \right) = \frac{d}{dx} (\sin^3 x - 1) = 3 \sin^2 x \cos x$$

$$(b) \frac{d}{dx} \left( \int_1^{\sin x} 3t^2 dt \right) = (3 \sin^2 x) \left( \frac{d}{dx} (\sin x) \right) = 3 \sin^2 x \cos x$$

$$17. (a) \int_0^{t^4} \sqrt{u} du = \int_0^{t^4} u^{1/2} du = \left[ \frac{2}{3} u^{3/2} \right]_0^{t^4} = \frac{2}{3} (t^4)^{3/2} - 0 = \frac{2}{3} t^6 \Rightarrow \frac{d}{dt} \left( \int_0^{t^4} \sqrt{u} du \right) = \frac{d}{dt} \left( \frac{2}{3} t^6 \right) = 4t^5$$

$$(b) \frac{d}{dt} \left( \int_0^{t^4} \sqrt{u} du \right) = \sqrt{t^4} \left( \frac{d}{dt} (t^4) \right) = t^2 (4t^3) = 4t^5$$

$$18. (a) \int_0^{\tan \theta} \sec^2 y dy = [\tan y]_0^{\tan \theta} = \tan(\tan \theta) - 0 = \tan(\tan \theta) \Rightarrow \frac{d}{d\theta} \left( \int_0^{\tan \theta} \sec^2 y dy \right) = \frac{d}{d\theta} (\tan(\tan \theta)) \\ = (\sec^2(\tan \theta)) \sec^2 \theta$$

$$(b) \frac{d}{d\theta} \left( \int_0^{\tan \theta} \sec^2 y dy \right) = (\sec^2(\tan \theta)) \left( \frac{d}{d\theta} (\tan \theta) \right) = (\sec^2(\tan \theta)) \sec^2 \theta$$

$$19. y = \int_0^x \sqrt{1+t^2} dt \Rightarrow \frac{dy}{dx} = \sqrt{1+x^2}$$

$$20. y = \int_1^x \frac{1}{t} dt \Rightarrow \frac{dy}{dx} = \frac{1}{x}, x > 0$$

$$21. y = \int_{\sqrt{x}}^0 \sin t^2 dt \Rightarrow y = - \int_0^{\sqrt{x}} \sin t^2 dt \Rightarrow \frac{dy}{dx} = -(\sin(\sqrt{x})^2) \left( \frac{d}{dx} (\sqrt{x}) \right) = -(\sin x) \left( \frac{1}{2} x^{-1/2} \right) = -\frac{\sin x}{2\sqrt{x}}$$

$$22. y = \int_0^{x^2} \cos \sqrt{t} \, dt \Rightarrow \frac{dy}{dx} = (\cos \sqrt{x^2}) \left( \frac{d}{dx}(x^2) \right) = 2x \cos x$$

$$23. y = \int_1^{x^{1/3}} e^{(t^3+1)} \, dt; \text{ let } u = x^{1/3} \Rightarrow \frac{du}{dx} = \frac{1}{3}x^{-2/3} \text{ and } y = \int_1^u e^{(t^3+1)} \, dt$$

From the Fundamental Theorem, Part 1,  $\frac{dy}{du} = e^{(u^3+1)}$  so that the Chain Rule gives

$$\frac{d}{dx} \int_1^{x^{1/3}} e^{(t^3+1)} \, dt = \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (e^{(u^3+1)}) \left( \frac{1}{3}x^{-2/3} \right) = \frac{e^{(x+1)}}{3x^{2/3}}.$$

$$24. y = \int_{e^x}^e \ln t \, dt, x > 1; \text{ let } u = e^x \Rightarrow \frac{du}{dx} = e^x \text{ and } y = \int_u^e \ln t \, dt = - \int_e^u \ln t \, dt.$$

From the Fundamental Theorem, Part 1,  $\frac{dy}{du} = -\ln u$  so that the Chain Rule gives

$$\frac{d}{dx} \int_{e^x}^e \ln t \, dt = \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (-\ln u) e^x = e^x (-\ln e^x) = e^x \ln e^{-x} = -xe^x.$$

$$25. \text{ Let } u = 1 - 2x \Rightarrow du = -2 \, dx$$

$$\int (1 - 2x)^3 \, dx = \int -\frac{1}{2}u^3 \, du = -\frac{1}{8}u^4 + C \Rightarrow \int_0^1 (1 - 2x)^3 \, dx = \left[ -\frac{1}{8}(1 - 2x)^4 \right]_0^1 = -\frac{1}{8}(-1)^4 - \left( -\frac{1}{8} \right)(1)^4 = 0$$

$$26. \text{ Let } u = t^2 + 1 \Rightarrow du = 2t \, dt$$

$$\begin{aligned} \int t\sqrt{t^2+1} \, dt &= \int \frac{1}{2}u^{1/2} \, du = \frac{1}{3}u^{3/2} + C \Rightarrow \int_0^1 t\sqrt{t^2+1} \, dt = \left[ \frac{1}{3}(t^2+1)^{3/2} \right]_0^1 \\ &= \frac{1}{3}(2\sqrt{2}-1) \end{aligned}$$

$$27. \text{ Let } u = 1 + \frac{\theta}{2} \Rightarrow du = \frac{1}{2} \, d\theta$$

$$\begin{aligned} \int \sin^2 \left( 1 + \frac{\theta}{2} \right) \, d\theta &= \int 2 \sin^2 u \, du = 2 \left( \frac{u}{2} - \frac{1}{4} \sin 2u \right) + C \Rightarrow \int_0^\pi \sin^2 \left( 1 + \frac{\theta}{2} \right) \, d\theta = \left[ \left( 1 + \frac{\theta}{2} \right) - \frac{1}{2} \sin(2 + \theta) \right]_0^\pi \\ &= \left[ \left( 1 + \frac{\pi}{2} \right) - \frac{1}{2} \sin(2 + \pi) \right] - \left( 1 - \frac{1}{2} \sin 2 \right) = \frac{\pi}{2} + \sin 2 \end{aligned}$$

$$28. \text{ Let } u = \sin \frac{x}{4} \Rightarrow du = \frac{1}{4} \cos \frac{x}{4} \, dx$$

$$\begin{aligned} \int \sin^2 \frac{x}{4} \cos \frac{x}{4} \, dx &= \int 4u^2 \, du = \frac{4}{3}u^3 + C \Rightarrow \int_0^\pi \sin^2 \frac{x}{4} \cos \frac{x}{4} \, dx = \left[ \frac{4}{3} \sin^3 \frac{x}{4} \right]_0^\pi = \frac{4}{3} \sin^3 \frac{\pi}{4} - \frac{4}{3} \sin^3 0 \\ &= \frac{4}{3} \left( \frac{\sqrt{2}}{2} \right)^3 = \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{3} \end{aligned}$$

29. Using the method in Example 6,  $y = \int_2^x \sec t \, dt + 3$ . Evaluating the integral gives  $y = \ln|\sec t + \tan t| \Big|_2^x + 3$

$$\text{(see Exercise 65 in Section 4.2)} \Rightarrow y = \ln \left| \frac{\sec x + \tan x}{\sec 2 + \tan 2} \right| + 3.$$

30. Using the method in Example 6,  $y = \int_1^x t\sqrt{1+t^2} \, dt - 2$ . Evaluating the integral gives  $y = \frac{1}{3}(1+t^2)^{3/2} \Big|_0^x - 2$

$$= \frac{1}{3}(1+x^2)^{3/2} - \frac{2\sqrt{2}}{3} - 2 = \frac{1}{3}[(1+x^2)^{3/2} - 2\sqrt{2} - 6].$$

31.  $\frac{dy}{dt} = e^t \sin(e^t - 2) \Rightarrow y = \int e^t \sin(e^t - 2) \, dt;$

$$\text{let } u = e^t - 2 \Rightarrow du = e^t \, dt \Rightarrow y = \int \sin u \, du = -\cos u + C = -\cos(e^t - 2) + C; y(\ln 2) = 0$$

$$\Rightarrow -\cos(e^{\ln 2} - 2) + C = 0 \Rightarrow -\cos(2 - 2) + C = 0 \Rightarrow C = \cos 0 = 1; \text{ thus, } y = 1 - \cos(e^t - 2)$$

32.  $\frac{d^2y}{dt^2} = 1 - e^{2t} \Rightarrow \frac{dy}{dt} = t - \frac{1}{2}e^{2t} + C; t = 1 \text{ and } \frac{dy}{dt} = 0 \Rightarrow 0 = 1 - \frac{1}{2}e^2 + C \Rightarrow C = \frac{1}{2}e^2 - 1; \text{ thus}$

$$\frac{dy}{dt} = t - \frac{1}{2}e^{2t} + \frac{1}{2}e^2 - 1 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t + C_1; t = 1 \text{ and } y = -1 \Rightarrow -1 = \frac{1}{2} - \frac{1}{4}e^2 + \frac{1}{2}e^2 - 1 + C_1$$

$$\Rightarrow C_1 = -\frac{1}{2} - \frac{1}{4}e^2 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t - \left(\frac{1}{2} + \frac{1}{4}e^2\right)$$

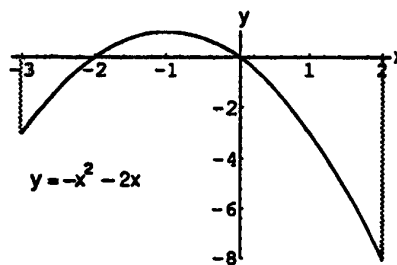
33.  $-x^2 - 2x = 0 \Rightarrow -x(x+2) = 0 \Rightarrow x = 0 \text{ or } x = -2; \text{ Area}$

$$= -\int_{-3}^{-2} (-x^2 - 2x) \, dx + \int_{-2}^0 (-x^2 - 2x) \, dx - \int_0^2 (-x^2 - 2x) \, dx$$

$$= -\left[-\frac{x^3}{3} - x^2\right]_{-3}^{-2} + \left[-\frac{x^3}{3} - x^2\right]_{-2}^0 - \left[-\frac{x^3}{3} - x^2\right]_0^2$$

$$= -\left[\left(-\frac{(-2)^3}{3} - (-2)^2\right) - \left(-\frac{(-3)^3}{3} - (-3)^2\right)\right]$$

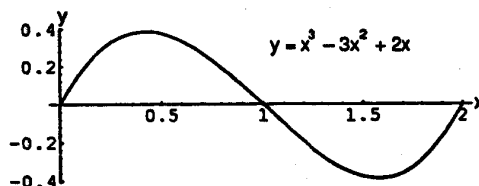
$$+ \left[\left(-\frac{0^3}{3} - 0^2\right) - \left(-\frac{(-2)^3}{3} - (-2)^2\right)\right] - \left[\left(-\frac{2^3}{3} - 2^2\right) - \left(-\frac{0^3}{3} - 0^2\right)\right] = \frac{28}{3}$$



34.  $x^3 - 3x^2 + 2x = 0 \Rightarrow x(x^2 - 3x + 2) = 0$

$$\Rightarrow x(x-2)(x-1) = 0, x = 0, 1, \text{ or } 2;$$

$$\text{Area} = \int_0^1 (x^3 - 3x^2 + 2x) \, dx - \int_1^2 (x^3 - 3x^2 + 2x) \, dx$$

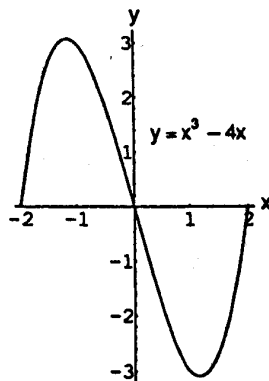


$$\begin{aligned}
 &= \left[ \frac{x^4}{4} - x^3 + x^2 \right]_0^1 - \left[ \frac{x^4}{4} - x^3 + x^2 \right]_1^2 \\
 &= \left( \frac{1^4}{4} - 1^3 + 1^2 \right) - \left( \frac{0^4}{4} - 0^3 + 0^2 \right) - \left[ \left( \frac{2^4}{4} - 2^3 + 2^2 \right) - \left( \frac{1^4}{4} - 1^3 + 1^2 \right) \right] = \frac{1}{2}
 \end{aligned}$$

35.  $x^3 - 4x = 0 \Rightarrow x(x^2 - 4) = 0 \Rightarrow x(x-2)(x+2) = 0$

$$\Rightarrow x = 0, 2, \text{ or } -2; \text{ Area} = \int_{-2}^0 (x^3 - 4x) dx - \int_0^2 (x^3 - 4x) dx$$

$$\begin{aligned}
 &= \left[ \frac{x^4}{4} - 2x^2 \right]_{-2}^0 - \left[ \frac{x^4}{4} - 2x^2 \right]_0^2 = \left( \frac{0^4}{4} - 2(0)^2 \right) \\
 &\quad - \left( \frac{(-2)^4}{4} - 2(-2)^2 \right) - \left[ \left( \frac{2^4}{4} - 2(2)^2 \right) - \left( \frac{0^4}{4} - 2(0)^2 \right) \right] = 8
 \end{aligned}$$



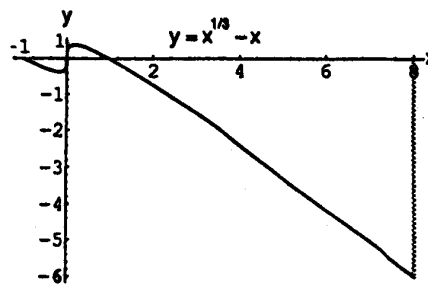
36.  $x^{1/3} - x = 0 \Rightarrow x^{1/3}(1 - x^{2/3}) = 0 \Rightarrow x^{1/3} = 0$  or

$$1 - x^{2/3} = 0 \Rightarrow x = 0 \text{ or } 1 = x^{2/3} \Rightarrow x = 0 \text{ or}$$

$$1 = x^2 \Rightarrow x = 0 \text{ or } \pm 1;$$

$$\text{Area} = - \int_{-1}^0 (x^{1/3} - x) dx + \int_0^1 (x^{1/3} - x) dx - \int_1^8 (x^{1/3} - x) dx$$

$$\begin{aligned}
 &= - \left[ \frac{3}{4} x^{4/3} - \frac{x^2}{2} \right]_{-1}^0 + \left[ \frac{3}{4} x^{4/3} - \frac{x^2}{2} \right]_0^1 - \left[ \frac{3}{4} x^{4/3} - \frac{x^2}{2} \right]_1^8 \\
 &= - \left[ \left( \frac{3}{4} (0)^{4/3} - \frac{0^2}{2} \right) - \left( \frac{3}{4} (-1)^{4/3} - \frac{(-1)^2}{2} \right) \right] + \left[ \left( \frac{3}{4} (1)^{4/3} - \frac{1^2}{2} \right) - \left( \frac{3}{4} (0)^{4/3} - \frac{0^2}{2} \right) \right] \\
 &\quad - \left[ \left( \frac{3}{4} (8)^{4/3} - \frac{8^2}{2} \right) - \left( \frac{3}{4} (1)^{4/3} - \frac{1^2}{2} \right) \right] = \frac{1}{4} + \frac{1}{4} - \left( 12 - \frac{64}{2} \right) + \frac{1}{4} = \frac{83}{4}
 \end{aligned}$$



37. The area of the rectangle bounded by the lines  $y = 2$ ,  $y = 0$ ,  $x = \pi$ , and  $x = 0$  is  $2\pi$ . The area under the curve  $y = 1 + \cos x$  on  $[0, \pi]$  is  $\int_0^\pi (1 + \cos x) dx = [x + \sin x]_0^\pi = (\pi + \sin \pi) - (0 + \sin 0) = \pi$ . Therefore the area of the shaded region is  $2\pi - \pi = \pi$ .

38. The area of the rectangle bounded by the lines  $y = 2$ ,  $y = 0$ ,  $t = -\frac{\pi}{4}$ , and  $t = 1$  is  $2 \left( 1 - \left( -\frac{\pi}{4} \right) \right) = 2 + \frac{\pi}{2}$ . The

area under the curve  $y = \sec^2 t$  on  $\left[ -\frac{\pi}{4}, 0 \right]$  is  $\int_{-\pi/4}^0 \sec^2 t dt = [\tan t]_{-\pi/4}^0 = \tan 0 - \tan \left( -\frac{\pi}{4} \right) = 1$ . The area

under the curve  $y = 1 - t^2$  on  $[0, 1]$  is  $\int_0^1 (1 - t^2) dt = \left[ t - \frac{t^3}{3} \right]_0^1 = \left( 1 - \frac{1^3}{3} \right) - \left( 0 - \frac{0^3}{3} \right) = \frac{2}{3}$ . Thus, the total

area under the curves on  $\left[ -\frac{\pi}{4}, 1 \right]$  is  $1 + \frac{2}{3} = \frac{5}{3}$ . Therefore the area of the shaded region is  $\left( 2 + \frac{\pi}{2} \right) - \frac{5}{3} = \frac{1}{3} + \frac{\pi}{2}$ .



$$39. \frac{dc}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2} \Rightarrow c = \int_0^x \frac{1}{2}t^{-1/2} dt = [t^{1/2}]_0^x = \sqrt{x}$$

$$(a) c(100) - c(1) = \sqrt{100} - \sqrt{1} = \$9.00$$

$$(b) c(400) - c(100) = \sqrt{400} - \sqrt{100} = \$10.00$$

$$40. r = \int_0^3 \left( 2 - \frac{2}{(x+1)^2} \right) dx = 2 \int_0^3 \left( 1 - \frac{1}{(x+1)^2} \right) dx = 2 \left[ x - \left( \frac{-1}{x+1} \right) \right]_0^3 = 2 \left[ \left( 3 + \frac{1}{3+1} \right) - \left( 0 + \frac{1}{0+1} \right) \right]$$

$$= 2 \left[ 3\frac{1}{4} - 1 \right] = 2 \left( 2\frac{1}{4} \right) = 4.5 \text{ or } \$4500$$

$$41. (a) v = \frac{ds}{dt} = \frac{d}{dt} \int_0^t f(x) dx = f(t) \Rightarrow v(5) = f(5) = 2 \text{ m/sec}$$

(b)  $a = \frac{dv}{dt}$  is negative since the slope of the tangent line at  $t = 5$  is negative.

(c)  $s = \int_0^3 f(x) dx = \frac{1}{2}(3)(3) = \frac{9}{2}$  m since the integral is the area of the triangle formed by  $y = f(x)$ , the  $x$ -axis,

and  $x = 3$ .

(d)  $t = 6$  since after  $t = 6$  to  $t = 9$ , the region lies below the  $x$ -axis.

(e) At  $t = 4$  and  $t = 7$ , since there are horizontal tangents there.

(f) Toward the origin between  $t = 6$  and  $t = 9$  since the velocity is negative on this interval. Away from the origin between  $t = 0$  and  $t = 6$  since the velocity is positive there.

(g) Right or positive side, because the integral of  $f$  from 0 to 9 is positive, there being more area above the  $x$ -axis than below it.

$$42. (a) v = \frac{ds}{dt} = \frac{d}{dt} \int_0^t f(x) dx = f(t) \Rightarrow v(3) = f(3) = 0 \text{ m/sec.}$$

(b)  $a = \frac{dv}{dt}$  is positive, since the slope of the tangent line at  $t = 3$  is positive.

(c) At  $t = 3$ , the particle's position is  $\int_0^3 f(x) dx = \frac{1}{2}(3)(-6) = -9$ .

(d) The particle passes through the origin at  $t = 6$  because  $s(6) = \int_0^6 f(x) dx = 0$ .

(e) At  $t = 7$ , since there is a horizontal tangent there.

(f) The particle starts at the origin and moves away to the left for  $0 < t < 3$ . It moves back toward the origin for  $3 < t < 6$ , passes through the origin at  $t = 6$ , and moves away to the right for  $t > 6$ .

(g) Right side, since its position at  $t = 9$  is positive, there being more area above the  $x$ -axis than below it.

$$43. \int_4^8 \pi(64 - x^2) dx = \pi \left[ 64x - \frac{x^3}{3} \right]_4^8 = \pi \left[ \left( 512 - \frac{512}{3} \right) - \left( 256 - \frac{64}{3} \right) \right] = \pi \left( 256 - \frac{448}{3} \right) = \frac{320\pi}{3}$$

$$44. \int_0^5 \pi(\sqrt{x})^2 dx = \pi \int_0^5 x dx = \pi \left[ \frac{x^2}{2} \right]_0^5 = \pi \left( \frac{25}{2} - \frac{0}{2} \right) = \frac{25\pi}{2}$$

$$45. \int_1^x f(t) dt = x^2 - 2x + 1 \Rightarrow f(x) = \frac{d}{dx} \int_1^x f(t) dt = \frac{d}{dx} (x^2 - 2x + 1) = 2x - 2$$

$$46. \int_0^x f(t) dt = x \cos \pi x \Rightarrow f(x) = \frac{d}{dx} \int_1^x f(t) dt = \cos \pi x - \pi x \sin \pi x \Rightarrow f(4) = \cos \pi(4) - \pi(4) \sin \pi(4) = 1$$

$$47. f(x) = 2 - \int_2^{x+1} \frac{9}{1+t} dt \Rightarrow f'(x) = -\frac{9}{1+(x+1)} = \frac{-9}{x+2} \Rightarrow f'(1) = -3; f(1) = 2 - \int_2^{1+1} \frac{9}{1+t} dt = 2 - 0 = 2;$$

$$L(x) = f'(1)(x-1) + f(1) = -3(x-1) + 2 = -3x + 5$$

$$48. g(x) = 3 + \int_1^{x^2} \sec(t-1) dt \Rightarrow g'(x) = (\sec(x^2-1))(2x) = 2x \sec(x^2-1) \Rightarrow g'(-1) = 2(-1) \sec((-1)^2-1)$$

$$= -2; g(-1) = 3 + \int_1^{(-1)^2} \sec(t-1) dt = 3 + \int_1^1 \sec(t-1) dt = 3 + 0 = 3; L(x) = g'(-1)(x - (-1)) + g(-1)$$

$$= -2(x+1) + 3 = -2x + 1$$

49. (a) True: since  $f$  is continuous,  $g$  is differentiable by Part 1 of the Fundamental Theorem of Calculus.

(b) True:  $g$  is continuous because it is differentiable.

(c) True, since  $g'(1) = f(1) = 0$ .

(d) False, since  $g''(1) = f'(1) > 0$ .

(e) True, since  $g'(1) = 0$  and  $g''(1) = f'(1) > 0$ .

(f) False:  $g''(x) = f'(x) > 0$ , so  $g''$  never changes sign.

(g) True, since  $g'(1) = f(1) = 0$  and  $g'(x) = f(x)$  is an increasing function of  $x$  (because  $f'(x) > 0$ ).

50. (a) True: by Part 1 of the Fundamental Theorem of Calculus,  $h'(x) = f(x)$ . Since  $f$  is differentiable for all  $x$ ,  $h$  has a second derivative for all  $x$ .

(b) True: they are continuous because they are differentiable.

(c) True, since  $h'(1) = f(1) = 0$ .

(d) True, since  $h'(1) = 0$  and  $h''(1) = f'(1) < 0$ .

(e) False, since  $h''(1) = f'(1) < 0$ .

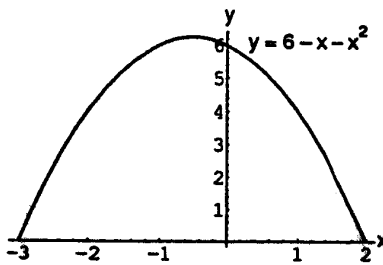
(f) False, since  $h''(x) = f'(x) < 0$  never changes sign.

(g) True, since  $h'(1) = f(1) = 0$  and  $h'(x) = f(x)$  is a decreasing function of  $x$  (because  $f'(x) < 0$ ).

51. (a)  $6 - x - x^2 = 0 \Rightarrow x^2 + x - 6 = 0$

$$\Rightarrow (x+3)(x-2) = 0 \Rightarrow x = -3 \text{ or } x = 2;$$

$$\begin{aligned} \text{Area} &= \int_{-3}^2 (6 - x - x^2) dx = \left[ 6x - \frac{x^2}{2} - \frac{x^3}{3} \right]_{-3}^2 \\ &= \left( 6(2) - \frac{2^2}{2} - \frac{2^3}{3} \right) - \left( 6(-3) - \frac{(-3)^2}{2} - \frac{(-3)^3}{3} \right) \\ &= \frac{125}{6} \end{aligned}$$

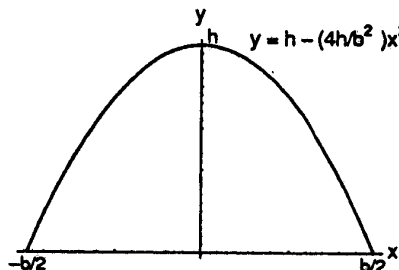


(b)  $y' = -1 - 2x = 0 \Rightarrow x = -\frac{1}{2}$ ;  $y' > 0$  for  $x < -\frac{1}{2}$  and  $y' < 0$  for  $x > -\frac{1}{2} \Rightarrow x = -\frac{1}{2}$  yields a local maximum

$$\Rightarrow \text{height} = y\left(-\frac{1}{2}\right) = \frac{25}{4}$$

(c) Base =  $2 - (-3) = 5$ , height =  $y\left(-\frac{1}{2}\right) = \frac{25}{4} \Rightarrow \text{Area} = \frac{2}{3}(\text{Base})(\text{Height}) = \frac{2}{3}(5)\left(\frac{25}{4}\right) = \frac{125}{6}$

(d) 
$$\begin{aligned} \text{Area} &= \int_{-b/2}^{b/2} \left( h - \left( \frac{4h}{b^2} \right) x^2 \right) dx = \left[ hx - \frac{4hx^3}{3b^2} \right]_{-b/2}^{b/2} \\ &= \left( h\left(\frac{b}{2}\right) - \frac{4h\left(\frac{b}{2}\right)^3}{3b^2} \right) - \left( h\left(-\frac{b}{2}\right) - \frac{4h\left(-\frac{b}{2}\right)^3}{3b^2} \right) \\ &= \left( \frac{bh}{2} - \frac{bh}{6} \right) - \left( -\frac{bh}{2} + \frac{bh}{6} \right) = bh - \frac{bh}{3} = \frac{2}{3}bh \end{aligned}$$

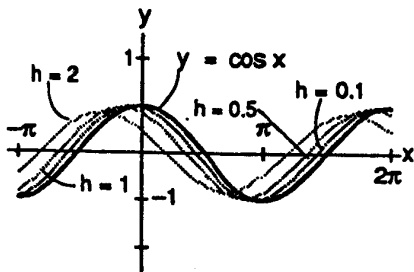


52. (a) 
$$\begin{aligned} \left( \frac{1}{60} - 0 \right) \int_0^{1/60} V_{\max} \sin 120\pi t dt &= 60 \left[ -V_{\max} \left( \frac{1}{120\pi} \right) \cos(120\pi t) \right]_0^{1/60} = -\frac{V_{\max}}{2\pi} [\cos 2\pi - \cos 0] \\ &= -\frac{V_{\max}}{2\pi} [1 - 1] = 0 \end{aligned}$$

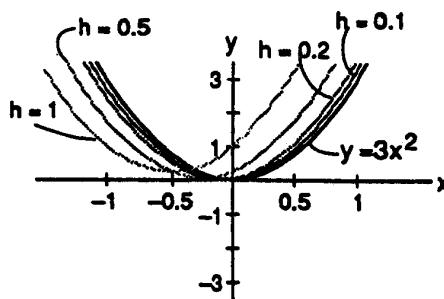
(b)  $V_{\max} = \sqrt{2} V_{\text{rms}} = \sqrt{2} (240) \approx 339$  volts

(c) 
$$\begin{aligned} \int_0^{1/60} (V_{\max})^2 \sin^2 120\pi t dt &= (V_{\max})^2 \int_0^{1/60} \left( \frac{1 - \cos 240\pi t}{2} \right) dt = \frac{(V_{\max})^2}{2} \int_0^{1/60} (1 - \cos 240\pi t) dt \\ &= \frac{(V_{\max})^2}{2} \left[ t - \left( \frac{1}{240\pi} \right) \sin 240\pi t \right]_0^{1/60} = \frac{(V_{\max})^2}{2} \left[ \left( \frac{1}{60} - \left( \frac{1}{240\pi} \right) \sin(2\pi) \right) - \left( 0 - \left( \frac{1}{240\pi} \right) \sin(0) \right) \right] = \frac{(V_{\max})^2}{120} \end{aligned}$$

53.



54. The limit is  $3x^2$



55-58. Example CAS commands:

Maple:

```
f:=x -> x^3 - 4*x^2 + 3*x;
F:=x -> int(f(t),t=0..x);
plot({F(x),f(x)},x=0..3.75);
solve(diff(F(x),x),x);
plot({diff(f(x),x),F(x)}, x=0..3);
map(evalf,[solve(diff(f(x),x)=0)]);
```

Mathematica:

```
Clear[x]
{a,b} = {0,2Pi}; f[x_] = Sin[2x] Cos[x/3]
F[x_] = Integrate[ f[t], {t,a,x} ]
Plot[ {f[x],F[x]}, {x,a,b} ]
x /. Map[
  FindRoot[ F' [x] == 0, {x,#} ]&,
  {2,3,5,6} ]
x /. Map[
  FindRoot[ f'[x] == 0, {x,#} ]&,
  {1,2,4,5,6} ]
```

59-62. Example CAS commands:

Maple:

```
f:=x -> sqrt(1 - x^2);
u:=x -> x^2;
F:=x -> int(f(t),t=1..u(x));
dFx:=diff(F(x),x);
simplify(%);
solve(dFx=0,x);
dFxx:=diff(F(x),x$2);
simplify(%);
solve(dFxx=0,x);
evalf(%);
plot (F(x),x=-1..1);
```

Mathematica:

```
a = 1; u[x_] = x^2; f[x_] = Sqrt[ 1 - x^2 ]
F[x_] = Integrate[ f[t], {t,a,u[x]} ]
F'[x]
x /. NSolve[ F'[x] == 0, x ]
F''[x]
x /. NSolve [ F''[x] == 0, x ]
Plot[ F[x], {x,-1,1} ]
```

63. In Maple type  $\text{diff}(\text{int}(f(x),x=a..u(x)),x)$ ; or, in Mathematica type  $\partial_x \int_a^{u[x]} f[t] dt$

64. In Maple type  $\text{diff}(\text{int}(f(x),x=a..u(x)),x,x)$ ; or in Mathematica type  $\partial_{x,x} \left( \int_a^{u[x]} f[t] dt \right)$

## 4.6 SUBSTITUTION IN DEFINITE INTEGRALS

1. (a) Let
- $u = y + 1 \Rightarrow du = dy$
- ;
- $y = 0 \Rightarrow u = 1$
- ,
- $y = 3 \Rightarrow u = 4$

$$\int_0^3 \sqrt{y+1} \, dy = \int_1^4 u^{1/2} \, du = \left[ \frac{2}{3} u^{3/2} \right]_1^4 = \left( \frac{2}{3} \right) (4)^{3/2} - \left( \frac{2}{3} \right) (1)^{3/2} = \left( \frac{2}{3} \right) (8) - \left( \frac{2}{3} \right) (1) = \frac{14}{3}$$

- (b) Use the same substitution for
- $u$
- as in part (a);
- $y = -1 \Rightarrow u = 0$
- ,
- $y = 0 \Rightarrow u = 1$

$$\int_{-1}^0 \sqrt{y+1} \, dy = \int_0^1 u^{1/2} \, du = \left[ \frac{2}{3} u^{3/2} \right]_0^1 = \left( \frac{2}{3} \right) (1)^{3/2} - 0 = \frac{2}{3}$$

2. (a) Let
- $u = \tan x \Rightarrow du = \sec^2 x \, dx$
- ;
- $x = 0 \Rightarrow u = 0$
- ,
- $x = \frac{\pi}{4} \Rightarrow u = 1$

$$\int_0^{\pi/4} \tan x \sec^2 x \, dx = \int_0^1 u \, du = \left[ \frac{u^2}{2} \right]_0^1 = \frac{1^2}{2} - 0 = \frac{1}{2}$$

- (b) Use the same substitution as in part (a);
- $x = -\frac{\pi}{4} \Rightarrow u = -1$
- ,
- $x = 0 \Rightarrow u = 0$

$$\int_{-\pi/4}^0 \tan x \sec^2 x \, dx = \int_{-1}^0 u \, du = \left[ \frac{u^2}{2} \right]_{-1}^0 = 0 - \frac{1}{2} = -\frac{1}{2}$$

3. (a) Let
- $u = \cos x \Rightarrow du = -\sin x \, dx \Rightarrow -du = \sin x \, dx$
- ;
- $x = 0 \Rightarrow u = 1$
- ,
- $x = \pi \Rightarrow u = -1$

$$\int_0^{\pi} 3 \cos^2 x \sin x \, dx = \int_1^{-1} -3u^2 \, du = \left[ -u^3 \right]_1^{-1} = -(-1)^3 - (-1)^3 = 2$$

- (b) Use the same substitution as in part (a);
- $x = 2\pi \Rightarrow u = 1$
- ,
- $x = 3\pi \Rightarrow u = -1$

$$\int_{2\pi}^{3\pi} 3 \cos^2 x \sin x \, dx = \int_1^{-1} -3u^2 \, du = 2$$

4. (a) Let
- $u = t^2 + 1 \Rightarrow du = 2t \, dt \Rightarrow \frac{1}{2} du = t \, dt$
- ;
- $t = 0 \Rightarrow u = 1$
- ,
- $t = \sqrt{7} \Rightarrow u = 8$

$$\int_0^{\sqrt{7}} t(t^2 + 1)^{1/3} \, dt = \int_1^8 \frac{1}{2} u^{1/3} \, du = \left[ \left( \frac{1}{2} \right) \left( \frac{3}{4} \right) u^{4/3} \right]_1^8 = \left( \frac{3}{8} \right) (8)^{4/3} - \left( \frac{3}{8} \right) (1)^{4/3} = \frac{45}{8}$$

- (b) Use the same substitution as in part (a);
- $t = -\sqrt{7} \Rightarrow u = 8$
- ,
- $t = 0 \Rightarrow u = 1$

$$\int_{-\sqrt{7}}^0 t(t^2 + 1)^{1/3} \, dt = \int_8^1 \frac{1}{2} u^{1/3} \, du = - \int_1^8 \frac{1}{2} u^{1/3} \, du = -\frac{45}{8}$$

5. (a) Let
- $u = 4 + r^2 \Rightarrow du = 2r \, dr \Rightarrow \frac{1}{2} du = r \, dr$
- ;
- $r = -1 \Rightarrow u = 5$
- ,
- $r = 1 \Rightarrow u = 5$

$$\int_{-1}^1 \frac{5r}{(4+r^2)^2} \, dr = 5 \int_5^5 \frac{1}{2} u^{-2} \, du = 0$$

- (b) Use the same substitution as in part (a);
- $r = 0 \Rightarrow u = 4$
- ,
- $r = 1 \Rightarrow u = 5$

$$\int_0^1 \frac{5r}{(4+r^2)^2} dr = 5 \int_4^5 \frac{1}{2} u^{-2} du = 5 \left[ -\frac{1}{2} u^{-1} \right]_4^5 = 5 \left( -\frac{1}{2} (5)^{-1} \right) - 5 \left( -\frac{1}{2} (4)^{-1} \right) = \frac{1}{8}$$

6. (a) Let
- $u = x^2 + 1 \Rightarrow du = 2x dx \Rightarrow 2 du = 4x dx$
- ;
- $x = 0 \Rightarrow u = 1$
- ,
- $x = \sqrt{3} \Rightarrow u = 4$

$$\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} dx = \int_1^4 \frac{2}{\sqrt{u}} du = \int_1^4 2u^{-1/2} du = [4u^{1/2}]_1^4 = 4(4)^{1/2} - 4(1)^{1/2} = 4$$

- (b) Use the same substitution as in part (a);
- $x = -\sqrt{3} \Rightarrow u = 4$
- ,
- $x = \sqrt{3} \Rightarrow u = 4$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} dx = \int_4^4 \frac{2}{\sqrt{u}} du = 0$$

7. (a) Let
- $u = 4 + 3 \sin z \Rightarrow du = 3 \cos z dz \Rightarrow \frac{1}{3} du = \cos z dz$
- ;
- $z = 0 \Rightarrow u = 4$
- ,
- $z = 2\pi \Rightarrow u = 4$

$$\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3 \sin z}} dz = \int_4^4 \frac{1}{\sqrt{u}} \left( \frac{1}{3} du \right) = 0$$

- (b) Use the same substitution as in part (a);
- $z = -\pi \Rightarrow u = 4 + 3 \sin(-\pi) = 4$
- ,
- $z = \pi \Rightarrow u = 4$

$$\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3 \sin z}} dz = \int_4^4 \frac{1}{\sqrt{u}} \left( \frac{1}{3} du \right) = 0$$

8. Let
- $u = t^5 + 2t \Rightarrow du = (5t^4 + 2) dt$
- ;
- $t = 0 \Rightarrow u = 0$
- ,
- $t = 1 \Rightarrow u = 3$

$$\int_0^1 \sqrt{t^5 + 2t} (5t^4 + 2) dt = \int_0^3 u^{1/2} du = \left[ \frac{2}{3} u^{3/2} \right]_0^3 = \frac{2}{3} (3)^{3/2} - \frac{2}{3} (0)^{3/2} = 2\sqrt{3}$$

9. Let
- $u = 1 + \sqrt{y} \Rightarrow du = \frac{dy}{2\sqrt{y}}$
- ;
- $y = 1 \Rightarrow u = 2$
- ,
- $y = 4 \Rightarrow u = 3$

$$\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2} = \int_2^3 \frac{1}{u^2} du = \int_2^3 u^{-2} du = [-u^{-1}]_2^3 = \left( -\frac{1}{3} \right) - \left( -\frac{1}{2} \right) = \frac{1}{6}$$

10. Let
- $u = \cos 2\theta \Rightarrow du = -2 \sin 2\theta d\theta \Rightarrow -\frac{1}{2} du = \sin 2\theta d\theta$
- ;
- $\theta = 0 \Rightarrow u = 1$
- ,
- $\theta = \frac{\pi}{6} \Rightarrow u = \cos 2\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$$\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta d\theta = \int_1^{1/2} u^{-3} \left( -\frac{1}{2} du \right) = -\frac{1}{2} \int_1^{1/2} u^{-3} du = \left[ -\frac{1}{2} \left( \frac{u^{-2}}{-2} \right) \right]_1^{1/2} = \frac{1}{4} \left( \frac{1}{2} \right)^2 - \frac{1}{4(1)^2} = \frac{3}{4}$$

11. Let
- $u = 1 - \sin 2t \Rightarrow du = -2 \cos 2t dt \Rightarrow -\frac{1}{2} du = \cos 2t dt$
- ;
- $t = 0 \Rightarrow u = 1$
- ,
- $t = \frac{\pi}{4} \Rightarrow u = 0$

$$\int_0^{\pi/4} (1 - \sin 2t)^{3/2} \cos 2t \, dt = \int_1^0 -\frac{1}{2} u^{3/2} \, du = \left[ -\frac{1}{2} \left( \frac{2}{5} u^{5/2} \right) \right]_1^0 = \left( -\frac{1}{5} (0)^{5/2} \right) - \left( -\frac{1}{5} (1)^{5/2} \right) = \frac{1}{5}$$

12. Let  $u = 4y - y^2 + 4y^3 + 1 \Rightarrow du = (4 - 2y + 12y^2) \, dy$ ;  $y = 0 \Rightarrow u = 1$ ,  $y = 1 \Rightarrow u = 4(1) - (1)^2 + 4(1)^3 + 1 = 8$

$$\int_0^1 (4y - y^2 + 4y^3 + 1)^{-2/3} (12y^2 - 2y + 4) \, dy = \int_1^8 u^{-2/3} \, du = \left[ 3u^{1/3} \right]_1^8 = 3(8)^{1/3} - 3(1)^{1/3} = 3$$

13. Let  $u = e^{\sin x} \Rightarrow du = e^{\sin x} \cos x \, dx$ ;  $x = 0 \Rightarrow u = 1$ ,  $x = \pi/2 \Rightarrow u = e$

$$\int_0^{\pi/2} e^{\sin x} \cos x \, dx = \int_1^e du = u \Big|_1^e = e - 1 \quad (\text{Note: Letting } u = \sin x \text{ also works.})$$

14. Let  $u = \tan \theta \Rightarrow du = \sec^2 \theta \, d\theta$ ;  $\theta = 0 \Rightarrow u = 0$ ,  $\theta = \frac{\pi}{4} \Rightarrow u = 1$ ;

$$\begin{aligned} \int_0^{\pi/4} (1 + e^{\tan \theta}) \sec^2 \theta \, d\theta &= \int_0^1 \sec^2 \theta \, d\theta + \int_0^1 e^u \, du = [\tan \theta]_0^{\pi/4} + [e^u]_0^1 = \left[ \tan\left(\frac{\pi}{4}\right) - \tan(0) \right] + (e^1 - e^0) \\ &= (1 - 0) + (e - 1) = e \end{aligned}$$

15. Let  $u = e^v \Rightarrow du = e^v \, dv \Rightarrow 2 \, du = 2e^v \, dv$ ;  $v = \ln \frac{\pi}{6} \Rightarrow u = \frac{\pi}{6}$ ,  $v = \ln \frac{\pi}{2} \Rightarrow u = \frac{\pi}{2}$ ;

$$\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^v \cos e^v \, dv = 2 \int_{\pi/6}^{\pi/2} \cos u \, du = \left[ 2 \sin u \right]_{\pi/6}^{\pi/2} = 2 \left[ \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right) \right] = 2 \left( 1 - \frac{1}{2} \right) = 1$$

16. Let  $u = e^{x^2} \Rightarrow du = 2xe^{x^2} \, dx$ ;  $x = 0 \Rightarrow u = 1$ ,  $x = \sqrt{\ln \pi} \Rightarrow u = e^{\ln \pi} = \pi$ ;

$$\int_0^{\sqrt{\ln \pi}} 2xe^{x^2} \cos(e^{x^2}) \, dx = \int_1^{\pi} \cos u \, du = [\sin u]_1^{\pi} = \sin(\pi) - \sin(1) = -\sin(1) \approx -0.84147$$

17.  $\frac{dy}{dt} = e^{-t} \sec^2(\pi e^{-t}) \Rightarrow y = \int e^{-t} \sec^2(\pi e^{-t}) \, dt$ ;

$$\begin{aligned} \text{let } u &= \pi e^{-t} \Rightarrow du = -\pi e^{-t} \, dt \Rightarrow -\frac{1}{\pi} \, du = e^{-t} \, dt \Rightarrow y = -\frac{1}{\pi} \int \sec^2 u \, du = -\frac{1}{\pi} \tan u + C \\ &= -\frac{1}{\pi} \tan(\pi e^{-t}) + C; y(\ln 4) = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi} \tan(\pi e^{-\ln 4}) + C = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi} \tan\left(\pi \cdot \frac{1}{4}\right) + C = \frac{2}{\pi} \\ &\Rightarrow -\frac{1}{\pi} (1) + C = \frac{2}{\pi} \Rightarrow C = \frac{3}{\pi}; \text{ thus, } y = \frac{3}{\pi} - \frac{1}{\pi} \tan(\pi e^{-t}) \end{aligned}$$

18.  $\frac{d^2y}{dx^2} = 2e^{-x} \Rightarrow \frac{dy}{dx} = -2e^{-x} + C$ ;  $x = 0$  and  $\frac{dy}{dx} = 0 \Rightarrow 0 = -2e^0 + C \Rightarrow C = 2$ ; thus  $\frac{dy}{dx} = -2e^{-x} + 2$

$$\Rightarrow y = 2e^{-x} + 2x + C_1; x = 0 \text{ and } y = 1 \Rightarrow 1 = 2e^0 + C_1 \Rightarrow C_1 = -1 \Rightarrow y = 2e^{-x} + 2x - 1 = 2(e^{-x} + x) - 1$$

19. For the sketch given,  $a = 0$ ,  $b = \pi$ ;  $f(x) - g(x) = 1 - \cos^2 x = \sin^2 x = \frac{1 - \cos 2x}{2}$ ,

$$A = \int_0^{\pi} \frac{(1 - \cos 2x)}{2} dx = \frac{1}{2} \int_0^{\pi} (1 - \cos 2x) dx = \frac{1}{2} \left[ x - \frac{\sin 2x}{2} \right]_0^{\pi} = \frac{1}{2} [(\pi - 0) - (0 - 0)] = \frac{\pi}{2}$$

20. For the sketch given,  $a = -\frac{\pi}{3}$ ,  $b = \frac{\pi}{3}$ ;  $f(t) - g(t) = \frac{1}{2} \sec^2 t - (-4 \sin^2 t) = \frac{1}{2} \sec^2 t + 4 \sin^2 t$ ;

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \left( \frac{1}{2} \sec^2 t + 4 \sin^2 t \right) dt = \frac{1}{2} \int_{-\pi/3}^{\pi/3} \sec^2 t dt + 4 \int_{-\pi/3}^{\pi/3} \sin^2 t dt = \frac{1}{2} \int_{-\pi/3}^{\pi/3} \sec^2 t dt + 4 \int_{-\pi/3}^{\pi/3} \frac{(1 - \cos 2t)}{2} dt \\ &= \frac{1}{2} \int_{-\pi/3}^{\pi/3} \sec^2 t dt + 2 \int_{-\pi/3}^{\pi/3} (1 - \cos 2t) dt = \frac{1}{2} [\tan t]_{-\pi/3}^{\pi/3} + 2 \left[ t - \frac{1}{2} \sin 2t \right]_{-\pi/3}^{\pi/3} = \sqrt{3} + 4 \cdot \frac{\pi}{3} - \sqrt{3} = \frac{4\pi}{3} \end{aligned}$$

21. For the sketch given,  $a = -2$ ,  $b = 2$ ;  $f(x) - g(x) = 2x^2 - (x^4 - 2x^2) = 4x^2 - x^4$ ;

$$A = \int_{-2}^2 (4x^2 - x^4) dx = \left[ \frac{4x^3}{3} - \frac{x^5}{5} \right]_{-2}^2 = \left( \frac{32}{3} - \frac{32}{5} \right) - \left[ -\frac{32}{3} - \left( -\frac{32}{5} \right) \right] = \frac{64}{3} - \frac{64}{5} = \frac{320 - 192}{15} = \frac{128}{15}$$

22. For the sketch given,  $a = -1$ ,  $b = 1$ ;  $f(x) - g(x) = x^2 - (-2x^4) = x^2 + 2x^4$ ;

$$A = \int_{-1}^1 (x^2 + 2x^4) dx = \left[ \frac{x^3}{3} + \frac{2x^5}{5} \right]_{-1}^1 = \left( \frac{1}{3} + \frac{2}{5} \right) - \left[ -\frac{1}{3} + \left( -\frac{2}{5} \right) \right] = \frac{2}{3} + \frac{4}{5} = \frac{10 + 12}{15} = \frac{22}{15}$$

23. AREA = A1 + A2

A1: For the sketch given,  $a = -3$  and we find  $b$  by solving the equations  $y = x^2 - 4$  and  $y = -x^2 - 2x$  simultaneously for  $x$ :  $x^2 - 4 = -x^2 - 2x \Rightarrow 2x^2 + 2x - 4 = 0 \Rightarrow 2(x+2)(x-1) \Rightarrow x = -2$  or  $x = 1$  so

$$\begin{aligned} b = -2: f(x) - g(x) &= (x^2 - 4) - (-x^2 - 2x) = 2x^2 + 2x - 4 \Rightarrow A1 = \int_{-3}^{-2} (2x^2 + 2x - 4) dx \\ &= \left[ \frac{2x^3}{3} + \frac{2x^2}{2} - 4x \right]_{-3}^{-2} = \left( -\frac{16}{3} + 4 + 8 \right) - (-18 + 9 + 12) = 9 - \frac{16}{3} = \frac{11}{3}; \end{aligned}$$

A2: For the sketch given,  $a = -2$  and  $b = 1$ :  $f(x) - g(x) = (-x^2 - 2x) - (x^2 - 4) = -2x^2 - 2x + 4$

$$\begin{aligned} \Rightarrow A2 &= - \int_{-2}^1 (2x^2 + 2x - 4) dx = - \left[ \frac{2x^3}{3} + x^2 - 4x \right]_{-2}^1 = - \left( \frac{2}{3} + 1 - 4 \right) + \left( -\frac{16}{3} + 4 + 8 \right) \\ &= -\frac{2}{3} - 1 + 4 - \frac{16}{3} + 4 + 8 = 9; \end{aligned}$$

$$\text{Therefore, AREA} = A1 + A2 = \frac{11}{3} + 9 = \frac{38}{3}$$

24. AREA = A1 + A2 + A3

A1: For the sketch given,  $a = -2$  and  $b = -1$ :  $f(x) - g(x) = (-x + 2) - (4 - x^2) = x^2 - x - 2$

$$\Rightarrow A1 = \int_{-2}^{-1} (x^2 - x - 2) dx = \left[ \frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_{-2}^{-1} = \left( -\frac{1}{3} - \frac{1}{2} + 2 \right) - \left( -\frac{8}{3} - \frac{4}{2} + 4 \right) = \frac{7}{3} - \frac{1}{2} = \frac{14 - 3}{6} = \frac{11}{6};$$



A2: For the sketch given,  $a = -1$  and  $b = 2$ :  $f(x) - g(x) = (4 - x^2) - (-x + 2) = -(x^2 - x - 2)$

$$\Rightarrow A_2 = - \int_{-1}^2 (x^2 - x - 2) dx = - \left[ \frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_{-1}^2 = - \left( \frac{8}{3} - \frac{4}{2} - 4 \right) + \left( -\frac{1}{3} - \frac{1}{2} + 2 \right) = -3 + 8 - \frac{1}{2} = \frac{9}{2};$$

A3: For the sketch given,  $a = 2$  and  $b = 3$ :  $f(x) - g(x) = (-x + 2) - (4 - x^2) = x^2 - x - 2$

$$\Rightarrow A_3 = \int_2^3 (x^2 - x - 2) dx = \left[ \frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_2^3 = \left( \frac{27}{3} - \frac{9}{2} - 6 \right) - \left( \frac{8}{3} - \frac{4}{2} - 4 \right) = 9 - \frac{9}{2} - \frac{8}{3};$$

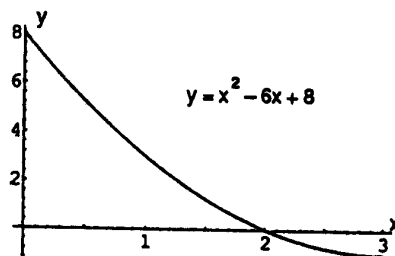
$$\text{Therefore, AREA} = A_1 + A_2 + A_3 = \frac{11}{6} + \frac{9}{2} + \left( 9 - \frac{9}{2} - \frac{8}{3} \right) = 9 - \frac{5}{6} = \frac{49}{6}$$

25.  $x^2 - 6x + 8 = 0 \Rightarrow (x - 4)(x - 2) = 0 \Rightarrow x = 4$  or

$x = 2$ , the  $x$ -intercepts.

$$\begin{aligned} \text{(a)} \quad \int_0^3 (x^2 - 6x + 8) dx &= \int_0^3 x^2 dx - 6 \int_0^3 x dx + \int_0^3 8 dx \\ &= \frac{3^3}{3} - 6 \left( \frac{3^2}{2} - \frac{0^2}{2} \right) + 8(3 - 0) = 6 \end{aligned}$$

$$\begin{aligned} \text{(b) Area} &= \int_0^2 (x^2 - 6x + 8) dx + \left( - \int_2^3 (x^2 - 6x + 8) dx \right) \\ &= \left( \int_0^2 x^2 dx - 6 \int_0^2 x dx + \int_0^2 8 dx \right) - \left( \int_2^3 x^2 dx - 6 \int_2^3 x dx + \int_2^3 8 dx \right) \\ &= \left[ \frac{2^3}{3} - 6 \left( \frac{2^2}{2} - \frac{0^2}{2} \right) + 8(2 - 0) \right] - \left( \int_0^3 x^2 dx - \int_0^2 x^2 dx - 6 \left( \frac{3^2}{2} - \frac{2^2}{2} \right) + 8(3 - 2) \right) \\ &= \left( \frac{8}{3} - 12 + 16 \right) - \left( \frac{3^3}{3} - \frac{2^3}{3} - 15 + 8 \right) = \frac{22}{3} = 7\frac{1}{3} \end{aligned}$$

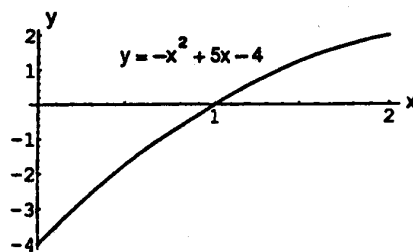


26.  $-x^2 + 5x - 4 = 0 \Rightarrow x^2 - 5x + 4 = 0 \Rightarrow (x - 4)(x - 1) = 0$

$\Rightarrow x = 4$  or  $x = 1$ , the  $x$ -intercepts.

$$\begin{aligned} \text{(a)} \quad \int_0^2 (-x^2 + 5x - 4) dx &= - \int_0^2 x^2 dx + 5 \int_0^2 x dx - \int_0^2 4 dx \\ &= -\frac{2^3}{3} + 5 \left( \frac{2^2}{2} - \frac{0^2}{2} \right) - 4(2 - 0) = -\frac{2}{3} \end{aligned}$$

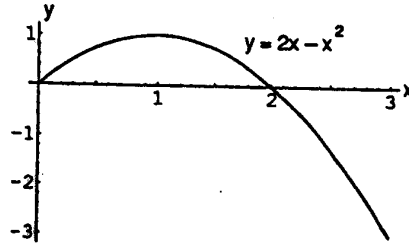
$$\begin{aligned} \text{(b) Area} &= - \int_0^1 (-x^2 + 5x - 4) dx + \int_1^2 (-x^2 + 5x - 4) dx \\ &= \int_0^1 x^2 dx - 5 \int_0^1 x dx + \int_0^1 4 dx + \int_1^2 -x^2 dx + 5 \int_1^2 x dx - \int_1^2 4 dx \end{aligned}$$



$$\begin{aligned}
 &= \frac{1^3}{3} - 5\left(\frac{1^2}{2} - \frac{0^2}{2}\right) + 4(1-0) + \left(\int_0^2 -x^2 dx - \int_0^1 -x^2 dx\right) + 5\left(\frac{2^2}{2} - \frac{1^2}{2}\right) - 4(2-1) \\
 &= \frac{1}{3} - \frac{5}{2} + 4 - \frac{2^3}{3} + \frac{1^3}{3} + \frac{15}{2} - 4 = 3
 \end{aligned}$$

27.  $2x - x^2 = 0 \Rightarrow x(2-x) = 0 \Rightarrow x = 0$  or  $x = 2$ ,  
the x-intercepts.

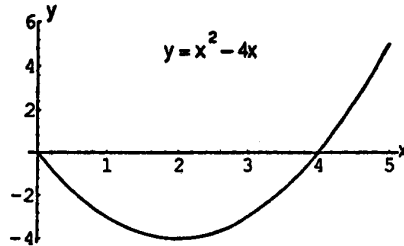
$$\begin{aligned}
 \text{(a)} \quad \int_0^3 (2x - x^2) dx &= 2 \int_0^3 x dx - \int_0^3 x^2 dx \\
 &= 2\left(\frac{3^2}{2} - \frac{0^2}{2}\right) - \frac{3^3}{3} = 0
 \end{aligned}$$



$$\begin{aligned}
 \text{(b)} \quad \text{Area} &= \int_0^2 (2x - x^2) dx - \int_2^3 (2x - x^2) dx = 2 \int_0^2 x dx - \int_0^2 x^2 dx - \left(2 \int_2^3 x dx - \int_2^3 x^2 dx\right) \\
 &= 2\left(\frac{2^2}{2} - \frac{0^2}{2}\right) - \frac{2^3}{3} - 2\left(\frac{3^2}{2} - \frac{2^2}{2}\right) + \left(\int_2^3 x^2 dx - \int_0^2 x^2 dx\right) = 4 - \frac{8}{3} - 5 + \frac{3^3}{3} - \frac{2^3}{3} = \frac{8}{3}
 \end{aligned}$$

28.  $x^2 - 4x = 0 \Rightarrow x(x-4) = 0 \Rightarrow x = 0$  or  $x = 4$ ,  
the x-intercepts.

$$\begin{aligned}
 \text{(a)} \quad \int_0^5 (x^2 - 4x) dx &= \int_0^5 x^2 dx - 4 \int_0^5 x dx \\
 &= \frac{5^3}{3} - 4\left(\frac{5^2}{2} - \frac{0^2}{2}\right) = -\frac{25}{3}
 \end{aligned}$$



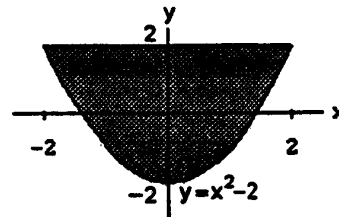
$$\begin{aligned}
 \text{(b)} \quad \text{Area} &= - \int_0^4 (x^2 - 4x) dx + \int_4^5 (x^2 - 4x) dx \\
 &= - \int_0^4 x^2 dx + 4 \int_0^4 x dx + \int_4^5 x^2 dx - 4 \int_4^5 x dx = -\frac{4^3}{3} + 4\left(\frac{4^2}{2} - \frac{0^2}{2}\right) + \left(\int_4^5 x^2 dx - \int_0^4 x^2 dx\right) - 4\left(\frac{5^2}{2} - \frac{4^2}{2}\right) \\
 &= -\frac{64}{3} + 32 + \frac{5^3}{3} - \frac{4^3}{3} - 18 = 13
 \end{aligned}$$

29.  $a = -2$ ,  $b = 2$ ;

$$f(x) - g(x) = 2 - (x^2 - 2) = 4 - x^2$$

$$\Rightarrow A = \int_{-2}^2 (4 - x^2) dx = \left[4x - \frac{x^3}{3}\right]_{-2}^2 = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$

$$= 2 \cdot \left(\frac{24}{3} - \frac{8}{3}\right) = \frac{32}{3}$$

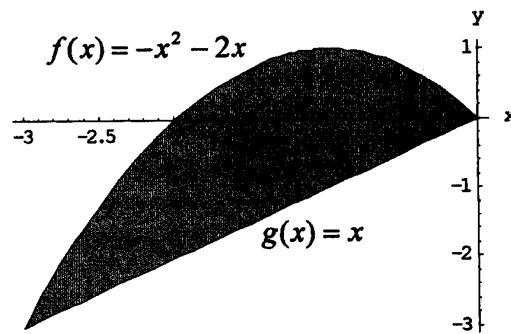


30. Limits of integration:  $-x^2 - 2x = x \Rightarrow x^2 = -3x$

$$\Rightarrow x(x+3) = 0 \Rightarrow a = -3 \text{ and } b = 0$$

$$f(x) - g(x) = (-x^2 - 2x) - x = -x^2 - 3x$$

$$\begin{aligned} \Rightarrow A &= \int_{-3}^0 (-x^2 - 3x) dx = \left[ -\frac{3x^2}{2} - \frac{x^3}{3} \right]_{-3}^0 \\ &= 0 - \left( -\frac{27}{2} + \frac{27}{3} \right) = \frac{9}{2} \end{aligned}$$

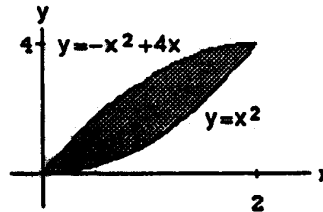


31. Limits of integration:  $x^2 = -x^2 + 4x \Rightarrow 2x^2 - 4x = 0$

$$\Rightarrow 2x(x-2) = 0 \Rightarrow a = 0 \text{ and } b = 2;$$

$$f(x) - g(x) = (-x^2 + 4x) - x^2 = -2x^2 + 4x$$

$$\begin{aligned} \Rightarrow A &= \int_0^2 (-2x^2 + 4x) dx = \left[ -\frac{2x^3}{3} + \frac{4x^2}{2} \right]_0^2 \\ &= -\frac{16}{3} + \frac{16}{2} = \frac{-32 + 48}{6} = \frac{8}{3} \end{aligned}$$

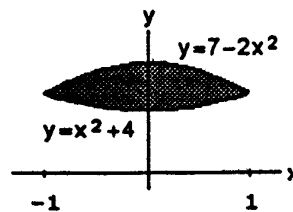


32. Limits of integration:  $7 - 2x^2 = x^2 + 4 \Rightarrow 3x^2 - 3 = 0$

$$\Rightarrow 3(x-1)(x+1) = 0 \Rightarrow a = -1 \text{ and } b = 1;$$

$$f(x) - g(x) = (7 - 2x^2) - (x^2 + 4) = 3 - 3x^2$$

$$\begin{aligned} \Rightarrow A &= \int_{-1}^1 (3 - 3x^2) dx = 3 \left[ x - \frac{x^3}{3} \right]_{-1}^1 \\ &= 3 \left[ \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) \right] = 6 \left( \frac{2}{3} \right) = 4 \end{aligned}$$



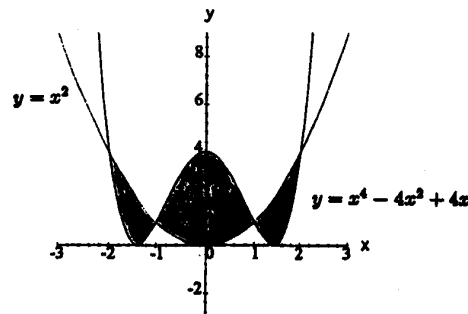
33. Limits of integration:  $x^4 - 4x^2 + 4 = x^2$

$$\Rightarrow x^4 - 5x^2 + 4 = 0 \Rightarrow (x^2 - 4)(x^2 - 1) = 0$$

$$\Rightarrow (x+2)(x-2)(x+1)(x-1) = 0 \Rightarrow x = -2, -1, 1, 2;$$

$$f(x) - g(x) = (x^4 - 4x^2 + 4) - x^2 = x^4 - 5x^2 + 4 \text{ and}$$

$$g(x) - f(x) = x^2 - (x^4 - 4x^2 + 4) = -x^4 + 5x^2 - 4$$



$$\begin{aligned} \Rightarrow A &= \int_{-2}^{-1} (-x^4 + 5x^2 - 4) dx + \int_{-1}^1 (x^4 - 5x^2 + 4) dx + \int_1^2 (-x^4 + 5x^2 - 4) dx \\ &= \left[ -\frac{x^5}{5} + \frac{5x^3}{3} - 4x \right]_{-2}^{-1} + \left[ \frac{x^5}{5} - \frac{5x^3}{3} + 4x \right]_{-1}^1 + \left[ -\frac{x^5}{5} + \frac{5x^3}{3} - 4x \right]_1^2 \\ &= \left( \frac{1}{5} - \frac{5}{3} + 4 \right) - \left( \frac{32}{5} - \frac{40}{3} + 8 \right) + \left( \frac{1}{5} - \frac{5}{3} + 4 \right) \\ &\quad - \left( -\frac{1}{5} + \frac{5}{3} - 4 \right) + \left( -\frac{32}{5} + \frac{40}{3} - 8 \right) - \left( -\frac{1}{5} + \frac{5}{3} - 4 \right) = -\frac{60}{5} + \frac{60}{3} = \frac{300 - 180}{15} = 8 \end{aligned}$$

$$34. \text{ Limits of integration: } y = |x^2 - 4| = \begin{cases} x^2 - 4, & x \leq -2 \text{ or } x \geq 2 \\ 4 - x^2, & -2 \leq x \leq 2 \end{cases}$$

$$\text{for } x \leq -2 \text{ and } x \geq 2: x^2 - 4 = \frac{x^2}{2} + 4$$

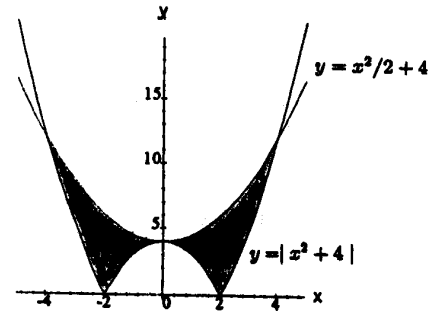
$$\Rightarrow 2x^2 - 8 = x^2 + 8 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4;$$

$$\text{for } -2 \leq x \leq 2: 4 - x^2 = \frac{x^2}{2} + 4 \Rightarrow 8 - 2x^2 = x^2 + 8$$

$$\Rightarrow x^2 = 0 \Rightarrow x = 0; \text{ by symmetry of the graph,}$$

$$A = 2 \int_0^2 \left[ \left( \frac{x^2}{2} + 4 \right) - (4 - x^2) \right] dx + 2 \int_2^4 \left[ \left( \frac{x^2}{2} + 4 \right) - (x^2 - 4) \right] dx$$

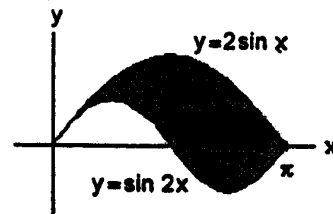
$$= 2 \left[ \frac{x^3}{2} \right]_0^2 + 2 \left[ 8x - \frac{x^3}{6} \right]_2^4 = 2 \left( \frac{8}{2} - 0 \right) + 2 \left( 32 - \frac{64}{6} - 16 + \frac{8}{6} \right) = 40 - \frac{56}{3} = \frac{64}{3}$$



$$35. a = 0, b = \pi; f(x) - g(x) = 2 \sin x - \sin 2x$$

$$\Rightarrow A = \int_0^\pi (2 \sin x - \sin 2x) dx = \left[ -2 \cos x + \frac{\cos 2x}{2} \right]_0^\pi$$

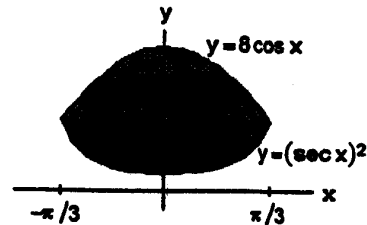
$$= \left[ -2(-1) + \frac{1}{2} \right] - \left[ -2 \cdot 1 + \frac{1}{2} \right] = 4$$



$$36. a = -\frac{\pi}{3}, b = \frac{\pi}{3}; f(x) - g(x) = 8 \cos x - \sec^2 x$$

$$\Rightarrow A = \int_{-\pi/3}^{\pi/3} (8 \cos x - \sec^2 x) dx = \left[ 8 \sin x - \tan x \right]_{-\pi/3}^{\pi/3}$$

$$= \left( 8 \cdot \frac{\sqrt{3}}{2} - \sqrt{3} \right) - \left( -8 \cdot \frac{\sqrt{3}}{2} + \sqrt{3} \right) = 6\sqrt{3}$$



$$37. A = A_1 + A_2$$

$$a_1 = -1, b_1 = 0 \text{ and } a_2 = 0, b_2 = 1;$$

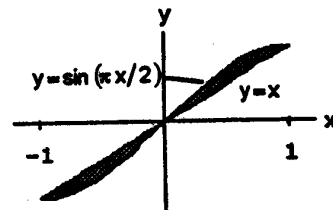
$$f_1(x) - g_1(x) = x - \sin\left(\frac{\pi x}{2}\right) \text{ and } f_2(x) - g_2(x) = \sin\left(\frac{\pi x}{2}\right) - x$$

$$\Rightarrow \text{by symmetry about the origin,}$$

$$A_1 + A_2 = 2A_1 \Rightarrow A = 2 \int_0^1 \left[ \sin\left(\frac{\pi x}{2}\right) - x \right] dx$$

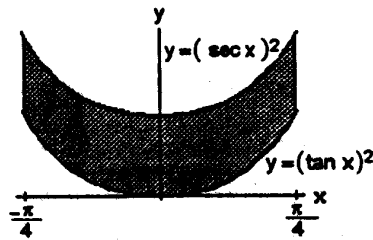
$$= 2 \left[ -\frac{2}{\pi} \cos\left(\frac{\pi x}{2}\right) - \frac{x^2}{2} \right]_0^1 = 2 \left[ \left( -\frac{2}{\pi} \cdot 0 - \frac{1}{2} \right) - \left( -\frac{2}{\pi} \cdot 1 - 0 \right) \right]$$

$$= 2 \left( \frac{2}{\pi} - \frac{1}{2} \right) = 2 \left( \frac{4 - \pi}{2\pi} \right) = \frac{4 - \pi}{\pi}$$



38.  $a = -\frac{\pi}{4}$ ,  $b = \frac{\pi}{4}$ ;  $f(x) - g(x) = \sec^2 x - \tan^2 x$

$$\begin{aligned} \Rightarrow A &= \int_{-\pi/4}^{\pi/4} (\sec^2 x - \tan^2 x) dx = \int_{-\pi/4}^{\pi/4} [\sec^2 x - (\sec^2 x - 1)] dx \\ &= \int_{-\pi/4}^{\pi/4} 1 \cdot dx = [x]_{-\pi/4}^{\pi/4} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2} \end{aligned}$$



39.  $A = A_1 + A_2$

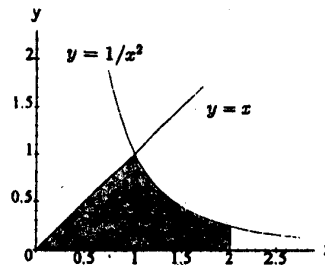
Limits of integration:  $y = x$  and  $y = \frac{1}{x^2} \Rightarrow x = \frac{1}{x^2}$ ,  $x \neq 0$

$\Rightarrow x^3 = 1 \Rightarrow x = 1$ ,  $f_1(x) - g_1(x) = x - 0 = x$

$\Rightarrow A_1 = \int_0^1 x dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$ ;  $f_2(x) - g_2(x) = \frac{1}{x^2} - 0$

$= x^{-2} \Rightarrow A_2 = \int_1^2 x^{-2} dx = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$ ;

$A = A_1 + A_2 = \frac{1}{2} + \frac{1}{2} = 1$

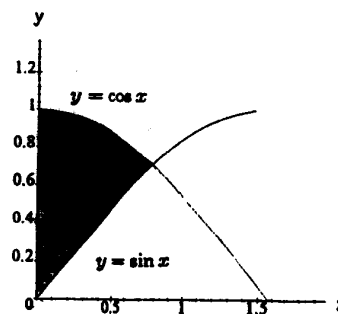


40. Limits of integration:  $\sin x = \cos x \Rightarrow x = \frac{\pi}{4} \Rightarrow a = 0$

and  $b = \frac{\pi}{4}$ ;  $f(x) - g(x) = \cos x - \sin x$

$\Rightarrow A = \int_0^{\pi/4} (\cos x - \sin x) dx = [\sin x + \cos x]_0^{\pi/4}$

$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) = \sqrt{2} - 1$



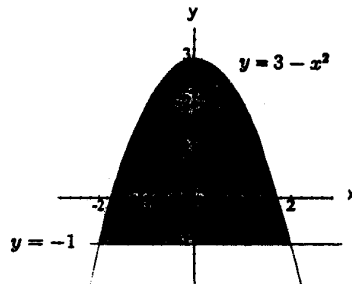
41. Limits of integration:  $y = 3 - x^2$  and  $y = -1$

$\Rightarrow 3 - x^2 = -1 \Rightarrow x^2 = 4 \Rightarrow a = -2$  and  $b = 2$ ;

$f(x) - g(x) = (3 - x^2) - (-1) = 4 - x^2$

$\Rightarrow A = \int_{-2}^2 (4 - x^2) dx = \left[4x - \frac{x^3}{3}\right]_{-2}^2$

$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = 16 - \frac{16}{3} = \frac{32}{3}$



42. Limits of integration:  $y = 1 + \sqrt{x}$  and  $y = \frac{2}{\sqrt{x}}$

$$\Rightarrow 1 + \sqrt{x} = \frac{2}{\sqrt{x}}, x \neq 0 \Rightarrow \sqrt{x} + x = 2 \Rightarrow x = (2 - x)^2$$

$$\Rightarrow x = 4 - 4x + x^2 \Rightarrow x^2 - 5x + 4 = 0$$

$$\Rightarrow (x - 4)(x - 1) = 0 \Rightarrow x = 1, 4 \text{ (but } x = 4 \text{ does not satisfy the equation); } y = \frac{2}{\sqrt{x}} \text{ and } y = \frac{x}{4} \Rightarrow \frac{2}{\sqrt{x}} = \frac{x}{4}$$

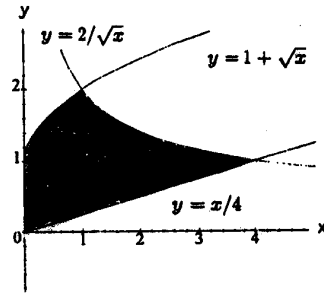
$$\Rightarrow 8 = x\sqrt{x} \Rightarrow 64 = x^3 \Rightarrow x = 4 \text{ since } x > 0;$$

$$\text{Therefore, AREA} = A_1 + A_2: f_1(x) - g_1(x) = (1 + x^{1/2}) - \frac{x}{4}$$

$$\Rightarrow A_1 = \int_0^1 \left(1 + x^{1/2} - \frac{x}{4}\right) dx = \left[x + \frac{2}{3}x^{3/2} - \frac{x^2}{8}\right]_0^1 = \left(1 + \frac{2}{3} - \frac{1}{8}\right) - 0 = \frac{37}{24}; f_2(x) - g_2(x) = 2x^{-1/2} - \frac{x}{4}$$

$$\Rightarrow A_2 = \int_1^4 \left(2x^{-1/2} - \frac{x}{4}\right) dx = \left[4x^{1/2} - \frac{x^2}{8}\right]_1^4 = \left(4 \cdot 2 - \frac{16}{8}\right) - \left(4 - \frac{1}{8}\right) = 4 - \frac{15}{8} = \frac{17}{8};$$

$$\text{Therefore, AREA} = A_1 + A_2 = \frac{37}{24} + \frac{17}{8} = \frac{37 + 51}{24} = \frac{88}{24} = \frac{11}{3}$$



43-46. Example CAS commands:

Maple:

```
p:=x^2*cos(x);
q:=x^3-x;
plot({p,q}, x=-2..2,-2..2);
intpt1:=fsolve(p=q,x=-2..0);
intpt2:=fsolve(p=q,x=0..2);
intone:=Int(q-p,x=intpt1..0);
inttwo:=Int(p-q,x=0..intpt2);
evalf(intone+inttwo);
```

Mathematica:

```
Clear[x]
f[x_] = x^2 Cos[x]
g[x_] = x^3 - x
Plot[ {f[x],g[x]}, {x,-4,4} ]
```

Here, need to use FindRoot for each crossing; can do all together using Map over initial guesses.

```
pts = x /. Map[
  FindRoot[ f[x] == g[x], {x,#} ]&,
  {-1,0,1} ]
i1 = NIntegrate[ f[x] - g[x], {x,pts[[1]],pts[[2]]} ]
i2 = NIntegrate[ g[x] - f[x], {x,pts[[2]],pts[[3]]} ]
i1 + i2
```

## 4.7 NUMERICAL INTEGRATION

1.  $\int_1^2 x \, dx$

I. (a) For  $n = 4$ ,  $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{h}{2} = \frac{1}{8}$ ;

$$\sum mf(x_i) = 12 \Rightarrow T = \frac{1}{8}(12) = \frac{3}{2};$$

$$f(x) = x \Rightarrow f'(x) = 1 \Rightarrow f'' = 0 \Rightarrow M = 0$$

$$\Rightarrow |E_T| = 0$$

(b)  $\int_1^2 x \, dx = \left[ \frac{x^2}{2} \right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2} \Rightarrow |E_T| = \int_1^2 x \, dx - T = 0$

(c)  $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

	$x_i$	$f(x_i)$	$m$	$mf(x_i)$
$x_0$	1	1	1	1
$x_1$	5/4	5/4	2	5/2
$x_2$	3/2	3/2	2	3
$x_3$	7/4	7/4	2	7/2
$x_4$	2	2	1	2

II. (a) For  $n = 4$ ,  $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{h}{3} = \frac{1}{12}$ ;

$$\sum mf(x_i) = 18 \Rightarrow S = \frac{1}{12}(18) = \frac{3}{2};$$

$$f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$$

(b)  $\int_1^2 x \, dx = \frac{3}{2} \Rightarrow |E_S| = \int_1^2 x \, dx - S = \frac{3}{2} - \frac{3}{2} = 0$

(c)  $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

	$x_i$	$f(x_i)$	$m$	$mf(x_i)$
$x_0$	1	1	1	1
$x_1$	5/4	5/4	4	5
$x_2$	3/2	3/2	2	3
$x_3$	7/4	7/4	4	7
$x_4$	2	2	1	2

2.  $\int_1^3 (2x-1) \, dx$

I. (a) For  $n = 4$ ,  $h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{2} = \frac{1}{4}$ ;

$$\sum mf(x_i) = 24 \Rightarrow T = \frac{1}{4}(24) = 6;$$

$$f(x) = 2x - 1 \Rightarrow f'(x) = 2 \Rightarrow f'' = 0 \Rightarrow M = 0$$

$$\Rightarrow |E_T| = 0$$

(b)  $\int_1^3 (2x-1) \, dx = [x^2 - x]_1^3 = (9-3) - (1-1) = 6 \Rightarrow |E_T| = \int_1^3 (2x-1) \, dx - T = 6 - 6 = 0$

	$x_i$	$f(x_i)$	$m$	$mf(x_i)$
$x_0$	1	1	1	1
$x_1$	3/2	2	2	4
$x_2$	2	3	2	6
$x_3$	5/2	4	2	8
$x_4$	3	5	1	5

$$(c) \frac{|E_T|}{\text{True Value}} \times 100 = 0\%$$

$$\text{II. (a) For } n = 4, h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{3} = \frac{1}{6};$$

$$\sum mf(x_i) = 36 \Rightarrow S = \frac{1}{6}(36) = 6;$$

$$f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$$

$$(b) \int_1^3 (2x-1) dx = 6 \Rightarrow |E_S| = \int_1^3 (2x-1) dx - S = 6 - 6 = 0$$

$$(c) \frac{|E_S|}{\text{True Value}} \times 100 = 0\%$$

$$3. \int_{-1}^1 (x^2+1) dx$$

$$\text{I. (a) For } n = 4, h = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{2} = \frac{1}{4};$$

$$\sum mf(x_i) = 11 \Rightarrow T = \frac{1}{4}(11) = 2.75;$$

$$f(x) = x^2 + 1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2 \Rightarrow M = 2$$

$$\Rightarrow |E_T| \leq \frac{1-(-1)}{12} \left(\frac{1}{2}\right)^2 (2) = \frac{1}{12} \text{ or } 0.08333$$

$$(b) \int_{-1}^1 (x^2+1) dx = \left[\frac{x^3}{3} + x\right]_{-1}^1 = \left(\frac{1}{3} + 1\right) - \left(-\frac{1}{3} - 1\right) = \frac{8}{3} \Rightarrow E_T = \int_{-1}^1 (x^2+1) dx - T = \frac{8}{3} - \frac{11}{4} = -\frac{1}{12}$$

$$\Rightarrow |E_T| = \left|-\frac{1}{12}\right| \approx 0.08333$$

$$(c) \frac{|E_T|}{\text{True Value}} \times 100 = \left(\frac{\frac{1}{12}}{\frac{8}{3}}\right) \times 100 \approx 3\%$$

$$\text{II. (a) For } n = 4, h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{3} = \frac{1}{6};$$

$$\sum mf(x_i) = 16 \Rightarrow S = \frac{1}{6}(16) = \frac{8}{3} = 2.66667;$$

$$f^{(3)}(x) = 0 \Rightarrow f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$$

	$x_i$	$f(x_i)$	$m$	$mf(x_i)$
$x_0$	1	1	1	1
$x_1$	3/2	2	4	8
$x_2$	2	3	2	6
$x_3$	5/2	4	4	16
$x_4$	3	5	1	5

	$x_i$	$f(x_i)$	$m$	$mf(x_i)$
$x_0$	-1	2	1	2
$x_1$	-1/2	5/4	2	5/2
$x_2$	0	1	2	2
$x_3$	1/2	5/4	2	5/2
$x_4$	1	2	1	2

	$x_i$	$f(x_i)$	$m$	$mf(x_i)$
$x_0$	-1	2	1	2
$x_1$	-1/2	5/4	4	5
$x_2$	0	1	2	2
$x_3$	1/2	5/4	4	5
$x_4$	1	2	1	2



$$(b) \int_{-1}^1 (x^2 + 1) dx = \left[ \frac{x^3}{3} + x \right]_{-1}^1 = \frac{8}{3} \Rightarrow |E_S| = \int_{-1}^1 (x^2 + 1) dx - S = \frac{8}{3} - \frac{8}{3} = 0$$

$$(c) \frac{|E_S|}{\text{True Value}} \times 100 = 0\%$$

$$4. \int_{-2}^0 (x^2 - 1) dx$$

$$I. (a) \text{ For } n = 4, h = \frac{b-a}{n} = \frac{0 - (-2)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{2} = \frac{1}{4};$$

$$\sum mf(x_i) = 3 \Rightarrow T = \frac{1}{4}(3) = \frac{3}{4};$$

$$f(x) = x^2 - 1 \Rightarrow f'(x) = 2x + 1 \Rightarrow f''(x) = 2 \Rightarrow M = 2$$

	$x_i$	$f(x_i)$	$m$	$mf(x_i)$
$x_0$	-2	3	1	3
$x_1$	-3/2	5/4	2	5/2
$x_2$	-1	0	2	0
$x_3$	-1/2	-3/4	2	-3/2
$x_4$	0	-1	1	-1

$$\Rightarrow |E_T| \leq \frac{0 - (-2)}{12} \left(\frac{1}{2}\right)^2 (2) = \frac{1}{12} = 0.08333$$

$$(b) \int_{-2}^0 (x^2 - 1) dx = \left[ \frac{x^3}{3} - x \right]_{-2}^0 = 0 - \left(-\frac{8}{3} + 2\right) = \frac{2}{3} \Rightarrow E_T = \int_{-2}^0 (x^2 - 1) dx - T = \frac{2}{3} - \frac{2}{4} = -\frac{1}{12}$$

$$\Rightarrow |E_T| = \frac{1}{12}$$

$$(c) \frac{|E_T|}{\text{True Value}} \times 100 = \left(\frac{\frac{1}{12}}{\frac{2}{3}}\right) \times 100 \approx 13\%$$

$$II. (a) \text{ For } n = 4, h = \frac{b-a}{n} = \frac{0 - (-2)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{3} = \frac{1}{6};$$

$$\sum mf(x_i) = 4 \Rightarrow S = \frac{1}{6}(4) = \frac{2}{3};$$

$$f^{(3)}(x) = 0 \Rightarrow f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$$

	$x_i$	$f(x_i)$	$m$	$mf(x_i)$
$x_0$	-2	3	1	3
$x_1$	-3/2	5/4	4	5
$x_2$	-1	0	2	0
$x_3$	-1/2	-3/4	4	-3
$x_4$	0	-1	1	-1

$$(b) \int_{-2}^0 (x^2 - 1) dx = \frac{2}{3} \Rightarrow |E_S| = \int_{-2}^0 (x^2 - 1) dx - S = \frac{2}{3} - \frac{2}{3} = 0$$

$$(c) \frac{|E_S|}{\text{True Value}} \times 100 = 0\%$$

$$5. \int_0^2 (t^3 + t) dt$$

I. (a) For  $n = 4$ ,  $h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{2} = \frac{1}{4}$ ;

$$\sum mf(t_i) = 25 \Rightarrow T = \frac{1}{4}(25) = \frac{25}{4};$$

$$f(t) = t^3 + t \Rightarrow f'(t) = 3t^2 + 1 \Rightarrow f''(t) = 6t \Rightarrow M = 12$$

$$= f''(2) \Rightarrow |E_T| \leq \frac{2-0}{12} \left(\frac{1}{2}\right)^2 (12) = \frac{1}{2}$$

(b)  $\int_0^2 (t^3 + t) dt = \left[ \frac{t^4}{4} + \frac{t^2}{2} \right]_0^2 = \left( \frac{2^4}{4} + \frac{2^2}{2} \right) - 0 = 6 \Rightarrow |E_T| = \left| \int_0^2 (t^3 + t) dt - T \right| = 6 - \frac{25}{4} = -\frac{1}{4} \Rightarrow |E_T| = \frac{1}{4}$

(c)  $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{\left| -\frac{1}{4} \right|}{6} \times 100 \approx 4\%$

	$t_i$	$f(t_i)$	$m$	$mf(t_i)$
$t_0$	0	0	1	0
$t_1$	1/2	5/8	2	5/4
$t_2$	1	2	2	4
$t_3$	3/2	39/8	2	39/4
$t_4$	2	10	1	10

II. (a) For  $n = 4$ ,  $h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{3} = \frac{1}{6}$ ;

$$\sum mf(t_i) = 36 \Rightarrow S = \frac{1}{6}(36) = 6;$$

$$f^{(3)}(t) = 6 \Rightarrow f^{(4)}(t) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$$

(b)  $\int_0^2 (t^3 + t) dt = 6 \Rightarrow |E_S| = \left| \int_0^2 (t^3 + t) dt - S \right| = 6 - 6 = 0$

(c)  $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

	$t_i$	$f(t_i)$	$m$	$mf(t_i)$
$t_0$	0	0	1	0
$t_1$	1/2	5/8	4	5/2
$t_2$	1	2	2	4
$t_3$	3/2	39/8	4	39/2
$t_4$	2	10	1	10

$$6. \int_{-1}^1 (t^3 + 1) dt$$

I. (a) For  $n = 4$ ,  $h = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{2} = \frac{1}{4}$ ;

$$\sum mf(t_i) = 8 \Rightarrow T = \frac{1}{4}(8) = 2;$$

$$f(t) = t^3 + 1 \Rightarrow f'(t) = 3t^2 \Rightarrow f''(t) = 6t \Rightarrow M = 6$$

$$= f''(1) \Rightarrow |E_T| \leq \frac{1-(-1)}{12} \left(\frac{1}{2}\right)^2 (6) = \frac{1}{4}$$

	$t_i$	$f(t_i)$	$m$	$mf(t_i)$
$t_0$	-1	0	1	0
$t_1$	-1/2	7/8	2	7/4
$t_2$	0	1	2	2
$t_3$	1/2	9/8	2	9/4
$t_4$	1	2	1	2

$$(b) \int_{-1}^1 (t^3 + 1) dt = \left[ \frac{t^4}{4} + t \right]_{-1}^1 = \left( \frac{1^4}{4} + 1 \right) - \left( \frac{(-1)^4}{4} + (-1) \right) = 2 \Rightarrow |E_T| = \left| \int_{-1}^1 (t^3 + 1) dt - T \right| = 2 - 2 = 0$$

$$(c) \frac{|E_T|}{\text{True Value}} \times 100 = 0\%$$

$$\text{II. (a) For } n = 4, h = \frac{b-a}{n} = \frac{1 - (-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{3} = \frac{1}{6};$$

$$\sum mf(t_i) = 12 \Rightarrow S = \frac{1}{6}(12) = 2;$$

$$f^{(3)}(t) = 6 \Rightarrow f^{(4)}(t) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$$

	$t_i$	$f(t_i)$	$m$	$mf(t_i)$
$t_0$	-1	0	1	0
$t_1$	-1/2	7/8	4	7/2
$t_2$	0	1	2	2
$t_3$	1/2	9/8	4	9/2
$t_4$	1	2	1	2

$$(b) \int_{-1}^1 (t^3 + 1) dt = 2 \Rightarrow |E_S| = \left| \int_{-1}^1 (t^3 + 1) dt - S \right| = 2 - 2 = 0$$

$$(c) \frac{|E_S|}{\text{True Value}} \times 100 = 0\%$$

$$7. \int_1^2 \frac{1}{s^2} ds$$

$$\text{I. (a) For } n = 4, h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{h}{2} = \frac{1}{8};$$

$$\sum mf(s_i) = \frac{179,573}{44,100} \Rightarrow T = \frac{1}{8} \left( \frac{179,573}{44,100} \right) = \frac{179,573}{352,800}$$

$$\approx 0.50899; f(s) = \frac{1}{s^2} \Rightarrow f'(s) = -\frac{2}{s^3} \Rightarrow f''(s) = \frac{6}{s^4}$$

$$\Rightarrow M = 6 = f''(1) \Rightarrow |E_T| \leq \frac{2-1}{12} \left( \frac{1}{4} \right)^2 (6) = \frac{1}{32} = 0.03125$$

$$(b) \int_1^2 \frac{1}{s^2} ds = \int_1^2 s^{-2} ds = \left[ -\frac{1}{s} \right]_1^2 = -\frac{1}{2} - \left( -\frac{1}{1} \right) = \frac{1}{2} \Rightarrow E_T = \int_1^2 \frac{1}{s^2} ds - T = \frac{1}{2} - 0.50899 = -0.00899$$

$$\Rightarrow |E_T| = 0.00899$$

$$(c) \frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.00899}{0.5} \times 100 \approx 2\%$$

	$s_i$	$f(s_i)$	$m$	$mf(s_i)$
$s_0$	1	1	1	1
$s_1$	5/4	16/25	2	32/25
$s_2$	3/2	4/9	2	8/9
$s_3$	7/4	16/49	2	32/49
$s_4$	2	1/4	1	1/4

$$\text{II. (a) For } n = 4, h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{h}{3} = \frac{1}{12};$$

$$\sum mf(s_i) = \frac{264,821}{44,100} \Rightarrow S = \frac{1}{12} \left( \frac{264,821}{44,100} \right) = \frac{264,821}{529,200}$$

$$\approx 0.50042; f^{(3)}(s) = -\frac{24}{s^5} \Rightarrow f^{(4)}(s) = \frac{120}{s^6}$$

	$s_i$	$f(s_i)$	$m$	$mf(s_i)$
$s_0$	1	1	1	1
$s_1$	5/4	16/25	4	64/25
$s_2$	3/2	4/9	2	8/9
$s_3$	7/4	16/49	4	64/49
$s_4$	2	1/4	1	1/4

$$\Rightarrow M = 120 = f^{(4)}(1) \Rightarrow |E_S| \leq \frac{2-1}{180} \left(\frac{1}{4}\right)^4 (120) = \frac{1}{384} \approx 0.002604$$

$$(b) \int_1^2 \frac{1}{s^2} ds = \frac{1}{2} \Rightarrow E_S = \int_1^2 \frac{1}{s^2} ds - S = \frac{1}{2} - 0.050041 = -0.00041 \Rightarrow |E_S| = 0.00041$$

$$(c) \frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.0004}{0.5} \times 100 \approx 0.08\%$$

$$8. \int_2^4 \frac{1}{(s-1)^2} ds$$

$$I. (a) \text{ For } n = 4, h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2} \Rightarrow \frac{h}{2} = \frac{1}{4};$$

$$\sum mf(s_i) = \frac{1269}{450} \Rightarrow T = \frac{1}{4} \left( \frac{1269}{450} \right) = \frac{1269}{1800} = 0.70500;$$

$$f(s) = (s-1)^{-2} \Rightarrow f'(s) = -\frac{2}{(s-1)^3}$$

$$\Rightarrow f''(s) = \frac{6}{(s-1)^4} \Rightarrow M = 6 = f''(2)$$

$$\Rightarrow |E_T| \leq \frac{4-2}{12} \left(\frac{1}{2}\right)^2 (6) = \frac{1}{4} = 0.25$$

$$(b) \int_2^4 \frac{1}{(s-1)^2} ds = \left[ \frac{-1}{(s-1)} \right]_2^4 = \left( \frac{-1}{4-1} \right) - \left( \frac{-1}{2-1} \right) = \frac{2}{3} \Rightarrow E_T = \int_2^4 \frac{1}{(s-1)^2} ds - T = \frac{2}{3} - 0.705 \approx -0.03833$$

$$\Rightarrow |E_T| \approx 0.03833$$

$$(c) \frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03833}{\left(\frac{2}{3}\right)} \times 100 \approx 6\%$$

	$s_i$	$f(s_i)$	$m$	$mf(s_i)$
$s_0$	2	1	1	1
$s_1$	5/2	4/9	2	8/9
$s_2$	3	1/4	2	1/2
$s_3$	7/2	4/25	2	8/25
$s_4$	4	1/9	1	1/9

$$II. (a) \text{ For } n = 4, h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2} \Rightarrow \frac{h}{3} = \frac{1}{6};$$

$$\sum mf(s_i) = \frac{1813}{450} \Rightarrow S = \frac{1}{6} \left( \frac{1813}{450} \right) = \frac{1813}{2700} \approx 0.67148;$$

$$f^{(3)}(s) = \frac{-24}{(s-1)^5} \Rightarrow f^{(4)}(s) = \frac{120}{(s-1)^6} \Rightarrow M = 120$$

$$= f^{(4)}(2) \Rightarrow |E_S| \leq \frac{4-2}{180} \left(\frac{1}{2}\right)^4 (120) = \frac{1}{12} \approx 0.08333$$

$$(b) \int_2^4 \frac{1}{(s-1)^2} ds = \frac{2}{3} \Rightarrow E_S = \int_2^4 \frac{1}{(s-1)^2} ds - S \approx \frac{2}{3} - 0.67148 = -0.00481 \Rightarrow |E_S| \approx 0.00481$$

$$(c) \frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00481}{\left(\frac{2}{3}\right)} \times 100 \approx 1\%$$

	$s_i$	$f(s_i)$	$m$	$mf(s_i)$
$s_0$	2	1	1	1
$s_1$	5/2	4/9	4	16/9
$s_2$	3	1/4	2	1/2
$s_3$	7/2	4/25	4	16/25
$s_4$	4	1/9	1	1/9

$$9. \int_0^{\pi} \sin t \, dt$$

I. (a) For  $n = 4$ ,  $h = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4} \Rightarrow \frac{h}{2} = \frac{\pi}{8}$ ;

$$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.8284 \Rightarrow T = \frac{\pi}{8}(2 + 2\sqrt{2})$$

$$\approx 1.89612; f(t) = \sin t \Rightarrow f'(t) = \cos t \Rightarrow f''(t) = -\sin t$$

$$\Rightarrow M = 1 = |f''(0)| \Rightarrow |E_T| \leq \frac{\pi-0}{12} \left(\frac{\pi}{4}\right)^2 (1) = \frac{\pi^3}{192} \approx 0.16149$$

(b)  $\int_0^{\pi} \sin t \, dt = [-\cos t]_0^{\pi} = (-\cos \pi) - (-\cos 0) = 2 \Rightarrow |E_T| = \left| \int_0^{\pi} \sin t \, dt - T \right| \approx 2 - 1.89612 = 0.10388$

(c)  $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.10388}{2} \times 100 \approx 5\%$

	$t_i$	$f(t_i)$	$m$	$mf(t_i)$
$t_0$	0	0	1	0
$t_1$	$\pi/4$	$\sqrt{2}/2$	2	$\sqrt{2}$
$t_2$	$\pi/2$	1	2	2
$t_3$	$3\pi/4$	$\sqrt{2}/2$	2	$\sqrt{2}$
$t_4$	$\pi$	0	1	0

II. (a) For  $n = 4$ ,  $h = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4} \Rightarrow \frac{h}{3} = \frac{\pi}{12}$ ;

$$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.6569 \Rightarrow S = \frac{\pi}{12}(2 + 4\sqrt{2})$$

$$\approx 2.00456; f^{(3)}(t) = -\cos t \Rightarrow f^{(4)}(t) = \sin t$$

$$\Rightarrow M = 1 = f^{(4)}(0) \Rightarrow |E_S| \leq \frac{\pi-0}{180} \left(\frac{\pi}{4}\right)^4 (1) \approx 0.00664$$

(b)  $\int_0^{\pi} \sin t \, dt = 2 \Rightarrow E_S = \int_0^{\pi} \sin t \, dt - S \approx 2 - 2.00456 = -0.00456 \Rightarrow |E_S| \approx 0.00456$

(c)  $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00456}{2} \times 100 \approx 0.23\%$

	$t_i$	$f(t_i)$	$m$	$mf(t_i)$
$t_0$	0	0	1	0
$t_1$	$\pi/4$	$\sqrt{2}/2$	4	$2\sqrt{2}$
$t_2$	$\pi/2$	1	2	2
$t_3$	$3\pi/4$	$\sqrt{2}/2$	4	$2\sqrt{2}$
$t_4$	$\pi$	0	1	0

$$10. \int_0^1 \sin \pi t \, dt$$

I. (a) For  $n = 4$ ,  $h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{h}{2} = \frac{1}{8}$ ;

$$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.828 \Rightarrow T = \frac{1}{8}(2 + 2\sqrt{2})$$

$$\approx 0.60355; f(t) = \sin \pi t \Rightarrow f'(t) = \pi \cos \pi t$$

$$\Rightarrow f''(t) = -\pi^2 \sin \pi t \Rightarrow M = \pi^2 = |f''(0)|$$

$$\Rightarrow |E_T| \leq \frac{1-0}{12} \left(\frac{1}{4}\right)^2 (\pi^2) \approx 0.05140$$

	$t_i$	$f(t_i)$	$m$	$mf(t_i)$
$t_0$	0	0	1	0
$t_1$	1/4	$\sqrt{2}/2$	2	$\sqrt{2}$
$t_2$	1/2	1	2	2
$t_3$	3/4	$\sqrt{2}/2$	2	$\sqrt{2}$
$t_4$	1	0	1	0

$$(b) \int_0^1 \sin \pi t \, dt = \left[-\frac{1}{\pi} \cos \pi t\right]_0^1 = \left(-\frac{1}{\pi} \cos \pi\right) - \left(-\frac{1}{\pi} \cos 0\right) = \frac{2}{\pi} \approx 0.63662 \Rightarrow |E_T| = \left| \int_0^1 \sin \pi t \, dt - T \right|$$

$$\approx \frac{2}{\pi} - 0.60355 = 0.03307$$

$$(c) \frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03307}{\left(\frac{2}{\pi}\right)} \times 100 \approx 5\%$$

II. (a) For  $n = 4$ ,  $h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{h}{3} = \frac{1}{12}$ ;

$$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.65685 \Rightarrow S = \frac{1}{12}(2 + 4\sqrt{2})$$

$$\approx 0.63807; f^{(3)}(t) = -\pi^3 \cos \pi t \Rightarrow f^{(4)}(t) = \pi^4 \sin \pi t$$

$$\Rightarrow M = \pi^4 = f^{(4)}(0) \Rightarrow |E_S| \leq \frac{1-0}{180} \left(\frac{1}{4}\right)^4 (\pi^4) \approx 0.00211$$

	$t_i$	$f(t_i)$	$m$	$mf(t_i)$
$t_0$	0	0	1	0
$t_1$	1/4	$\sqrt{2}/2$	4	$2\sqrt{2}$
$t_2$	1/2	1	2	2
$t_3$	3/4	$\sqrt{2}/2$	4	$2\sqrt{2}$
$t_4$	1	0	1	0

$$(b) \int_0^1 \sin \pi t \, dt = \frac{2}{\pi} \approx 0.63662 \Rightarrow E_S = \int_0^1 \sin \pi t \, dt - S \approx \frac{2}{\pi} - 0.63807 = -0.00145 \Rightarrow |E_S| \approx 0.00145$$

$$(c) \frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00145}{\left(\frac{2}{\pi}\right)} \times 100 \approx 0\%$$

11. (a)  $n = 8 \Rightarrow h = \frac{1}{8} \Rightarrow \frac{h}{2} = \frac{1}{16}$ ;

$$\sum mf(x_i) = 1(0.0) + 2(0.12402) + 2(0.24206) + 2(0.34763) + 2(0.43301) + 2(0.48789) + 2(0.49608) + 2(0.42361) + 1(0) = 5.1086 \Rightarrow T = \frac{1}{16}(5.1086) = 0.31929$$

(b)  $n = 8 \Rightarrow h = \frac{1}{8} \Rightarrow \frac{h}{3} = \frac{1}{24}$ ;

$$\sum mf(x_i) = 1(0.0) + 4(0.12402) + 2(0.24206) + 4(0.34763) + 2(0.43301) + 4(0.48789) + 2(0.49608) + 4(0.42361) + 1(0) = 7.8749 \Rightarrow S = \frac{1}{24}(7.8749) = 0.32812$$

(c) Let  $u = 1 - x^2 \Rightarrow du = -2x \, dx \Rightarrow -\frac{1}{2} du = x \, dx$ ;  $x = 0 \Rightarrow u = 1$ ,  $x = 1 \Rightarrow u = 0$

$$\int_0^1 x\sqrt{1-x^2} \, dx = \int_1^0 \sqrt{u} \left(-\frac{1}{2} du\right) = \frac{1}{2} \int_0^1 u^{1/2} \, du = \left[\frac{1}{2} \left(\frac{u^{3/2}}{3/2}\right)\right]_0^1 = \left[\frac{1}{3} u^{3/2}\right]_0^1 = \frac{1}{3}(\sqrt{1})^3 - \frac{1}{3}(\sqrt{0})^3 = \frac{1}{3};$$

$$E_T = \int_0^1 x\sqrt{1-x^2} \, dx - T \approx \frac{1}{3} - 0.31929 = 0.01404; E_S = \int_0^1 x\sqrt{1-x^2} \, dx - S \approx \frac{1}{3} - 0.32812 = 0.00521$$

12. (a)  $n = 8 \Rightarrow h = \frac{3}{8} \Rightarrow \frac{h}{2} = \frac{3}{16}$ ;

$$\sum mf(\theta_i) = 1(0) + 2(0.09334) + 2(0.18429) + 2(0.27075) + 2(0.35112) + 2(0.42443) + 2(0.49026) + 2(0.58466) + 1(0.6) = 5.3977 \Rightarrow T = \frac{3}{16}(5.3977) = 1.01207$$

$$(b) n = 8 \Rightarrow h = \frac{3}{8} \Rightarrow \frac{h}{3} = \frac{1}{8};$$

$$\sum mf(\theta_i) = 1(0) + 4(0.09334) + 2(0.18429) + 4(0.27075) + 2(0.35112) + 4(0.42443) + 2(0.49026) \\ + 4(0.58466) + 1(0.6) = 8.14406 \Rightarrow S = \frac{1}{8}(8.14406) = 1.01801$$

$$(c) \text{ Let } u = 16 + \theta^2 \Rightarrow du = 2\theta d\theta \Rightarrow \frac{1}{2} du = \theta d\theta; \theta = 0 \Rightarrow u = 16, \theta = 3 \Rightarrow u = 16 + 3^2 = 25$$

$$\int_0^3 \frac{\theta}{\sqrt{16 + \theta^2}} d\theta = \int_{16}^{25} \frac{1}{\sqrt{u}} \left(\frac{1}{2} du\right) = \frac{1}{2} \int_{16}^{25} u^{-1/2} du = \left[ \frac{1}{2} \left(\frac{u^{1/2}}{1/2}\right) \right]_{16}^{25} = \sqrt{25} - \sqrt{16} = 1;$$

$$E_T = \int_0^3 \frac{\theta}{\sqrt{16 + \theta^2}} d\theta - T \approx 1 - 1.01207 = -0.01207; E_S = \int_0^3 \frac{\theta}{\sqrt{16 + \theta^2}} d\theta - S \approx 1 - 0.01801 = -0.01801$$

$$13. (a) n = 8 \Rightarrow h = \frac{\pi}{8} \Rightarrow \frac{h}{2} = \frac{\pi}{16};$$

$$\sum mf(t_i) = 1(0.0) + 2(0.99138) + 2(1.26906) + 2(1.05961) + 2(0.75) + 2(0.48821) + 2(0.28946) + 2(0.13429) \\ + 1(0) = 9.96402 \Rightarrow T = \frac{\pi}{16}(9.96402) \approx 1.95643$$

$$(b) n = 8 \Rightarrow h = \frac{\pi}{8} \Rightarrow \frac{h}{3} = \frac{\pi}{24};$$

$$\sum mf(t_i) = 1(0.0) + 4(0.99138) + 2(1.26906) + 4(1.05961) + 2(0.75) + 4(0.48821) + 2(0.28946) + 4(0.13429) \\ + 1(0) = 15.311 \Rightarrow S \approx \frac{\pi}{24}(15.311) \approx 2.00421$$

$$(c) \text{ Let } u = 2 + \sin t \Rightarrow du = \cos t dt; t = -\frac{\pi}{2} \Rightarrow u = 2 + \sin\left(-\frac{\pi}{2}\right) = 1, t = \frac{\pi}{2} \Rightarrow u = 2 + \sin\frac{\pi}{2} = 3$$

$$\int_{-\pi/2}^{\pi/2} \frac{3 \cos t}{(2 + \sin t)^2} dt = \int_1^3 \frac{3}{u^2} du = 3 \int_1^3 u^{-2} du = \left[ 3 \left(\frac{u^{-1}}{-1}\right) \right]_1^3 = 3\left(-\frac{1}{3}\right) - 3\left(-\frac{1}{1}\right) = 2;$$

$$E_T = \int_{-\pi/2}^{\pi/2} \frac{3 \cos t}{(2 + \sin t)^2} dt - T \approx 2 - 1.95643 = 0.04357; E_S = \int_{-\pi/2}^{\pi/2} \frac{3 \cos t}{(2 + \sin t)^2} dt - S \\ \approx 2 - 2.00421 = -0.00421$$

$$14. (a) n = 8 \Rightarrow h = \frac{\pi}{32} \Rightarrow \frac{h}{2} = \frac{\pi}{64};$$

$$\sum mf(y_i) = 1(2.0) + 2(1.51606) + 2(1.18237) + 2(0.93998) + 2(0.75402) + 2(0.60145) + 2(0.46364) \\ + 2(0.31688) + 1(0) = 13.5488 \Rightarrow T \approx \frac{\pi}{64}(13.5488) = 0.66508$$

$$(b) n = 8 \Rightarrow h = \frac{\pi}{32} \Rightarrow \frac{h}{3} = \frac{\pi}{96};$$

$$\sum mf(y_i) = 1(2.0) + 4(1.51606) + 2(1.18237) + 4(0.93998) + 2(0.75402) + 4(0.60145) + 2(0.46364) \\ + 4(0.31688) + 1(0) = 20.29754 \Rightarrow S \approx \frac{\pi}{96}(20.29754) = 0.66424$$

$$(c) \text{ Let } u = \cot y \Rightarrow du = -\csc^2 y dy; y = \frac{\pi}{4} \Rightarrow u = 1, y = \frac{\pi}{2} \Rightarrow u = 0$$

$$\int_{\pi/4}^{\pi/2} (\csc^2 y) \sqrt{\cot y} dy = \int_1^0 \sqrt{u} (-du) = \int_0^1 u^{1/2} du = \left[ \frac{u^{3/2}}{3/2} \right]_0^1 = \frac{2}{3}(\sqrt{1})^3 - \frac{2}{3}(\sqrt{0})^3 = \frac{2}{3};$$

$$E_T = \int_{\pi/4}^{\pi/2} (\csc^2 y) \sqrt{\cot y} dy - T \approx \frac{2}{3} - 0.66508 = 0.00159; E_S = \int_{\pi/4}^{\pi/2} (\csc^2 y) \sqrt{\cot y} dy - S \approx \frac{2}{3} - 0.66424 = 0.00243$$

15.  $\frac{5}{2}(6.0 + 2(8.2) + 2(9.1 + \dots + 2(12.7) + 13.0)(30) = 15,990 \text{ ft}^3$

16. (a) Using the Trapezoid Rule,  $h = 200 \Rightarrow \frac{h}{2} = \frac{200}{2} = 100;$

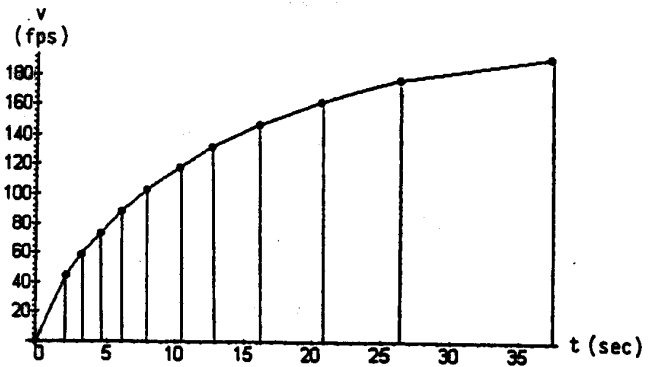
$$\sum mf(x_i) = 13,180 \Rightarrow \text{Area} \approx 100(13,180)$$

$= 1,318,000 \text{ ft}^2$ . Since the average depth  $= 20 \text{ ft}$  we obtain  $\text{Volume} \approx 20 (\text{Area}) \approx 26,360,000 \text{ ft}^3$ .

(b) The number of fish  $= \frac{\text{Volume}}{1000} = 26,360$  (to the nearest fish)  $\Rightarrow$  Maximum to be caught  $= 75\%$  of  $26,360 = 19,770$   
 $\Rightarrow$  Number of licenses  $= \frac{19,770}{20} = 988$

	$x_i$	$f(x_i)$	$m$	$mf(x_i)$
	$x_0$	0	1	0
	$x_1$	200	2	1040
	$x_2$	400	2	1600
	$x_3$	600	2	2000
	$x_4$	800	2	2280
	$x_4$	1000	2	2320
	$x_6$	1200	2	2220
	$x_7$	1400	2	1720
	$x_8$	1600	1	0

17. Use the conversion  $30 \text{ mph} = 44 \text{ fps}$  (ft per sec) since time is measured in seconds. The distance traveled as the car accelerates from, say,  $40 \text{ mph} = 58.67 \text{ fps}$  to  $50 \text{ mph} = 73.33 \text{ fps}$  in  $(4.5 - 3.2) = 1.3 \text{ sec}$  is the area of the trapezoid (see figure) associated with that time interval:  $\frac{1}{2}(58.67 + 73.33)(1.3) = 85.8 \text{ ft}$ . The total distance traveled by the Ford Mustang Cobra is the sum of all these eleven trapezoids (using  $\frac{\Delta t}{2}$  and the table below):



$$s = (44)(1.1) + (102.67)(0.5) + (132)(0.65) + (161.33)(0.7) + (190.67)(0.95) + (220)(1.2) + (249.33)(1.25) + (278.67)(1.65) + (308)(2.3) + (337.33)(2.8) + (366.67)(5.45) = 5166.33 \text{ ft} \approx 0.978 \text{ mi}$$



v (mph)	0	30	40	50	60	70	80	90	100	110	120	130
v (fps)	0	44	58.67	73.33	88	102.67	117.33	132	146.67	161.33	176	190.67
t (sec)	0	2.2	3.2	4.5	5.9	7.8	10.2	12.7	16	20.6	26.2	37.1
$\Delta t/2$	0	1.1	0.5	0.65	0.7	0.95	1.2	1.25	1.65	2.3	2.8	5.45

18. Using Simpson's Rule,  $h = \frac{b-a}{n} = \frac{24-0}{6} = \frac{24}{6} = 4$ ;

$$\sum my_i = 350 \Rightarrow S = \frac{4}{3}(350) = \frac{1400}{3} \approx 466.7 \text{ in.}^2$$

	$x_i$	$y_i$	m	$my_i$
$x_0$	0	0	1	0
$x_1$	4	18.75	4	75
$x_2$	8	24	2	48
$x_3$	12	26	4	104
$x_4$	16	24	2	48
$x_5$	20	18.75	4	75
$x_6$	24	0	1	0

19. Using Simpson's Rule,  $h = 1 \Rightarrow \frac{h}{3} = \frac{1}{3}$ ;

$$\sum my_i = 33.6 \Rightarrow \text{Cross Section Area} \approx \frac{1}{3}(33.6) = 11.2 \text{ ft}^2.$$

Let  $x$  be the length of the tank. Then the Volume  $V$

= (Cross Sectional Area)  $x = 11.2x$ . Now 5000 lb of

gasoline at 42 lb/ft<sup>3</sup>  $\Rightarrow V = \frac{5000}{42} = 119.05 \text{ ft}^3$

$$\Rightarrow 119.05 = 11.2x \Rightarrow x \approx 10.63 \text{ ft}$$

	$x_i$	$y_i$	m	$my_i$
$x_0$	0	1.5	1	1.5
$x_1$	1	1.6	4	6.4
$x_2$	2	1.8	2	3.6
$x_3$	3	1.9	4	7.6
$x_4$	4	2.0	2	4.0
$x_5$	5	2.1	4	8.4
$x_6$	6	2.1	1	2.1

20. Using Simpson's Rule,  $h = 24 \Rightarrow \frac{h}{2} = 12$ ;  $12[0.019 + 2(0.020) + 2(0.021) + \dots + 2(0.031) + 0.035] = 4.2 \text{ L}$

21.  $n = 2 \Rightarrow h = \frac{2-0}{2} = 1 \Rightarrow \frac{h}{3} = \frac{1}{3}$ ;

$$\sum mf(x_i) = 12 \Rightarrow S = \frac{1}{3}(12) = 4;$$

$$\int_0^2 x^3 dx = \left[ \frac{x^4}{4} \right]_0^2 = \frac{2^4}{4} - \frac{0^4}{4} = 4$$

	$x_i$	$f(x_i)$	m	$mf(x_i)$
$x_0$	0	0	1	0
$x_1$	1	1	4	4
$x_2$	2	8	1	8

22. (a)  $|E_S| \leq \frac{b-a}{180}(h^4)M$ ;  $n = 4 \Rightarrow h = \frac{\pi-0}{4} = \frac{\pi}{8}$ ;  $|f^{(4)}| \leq 1 \Rightarrow M = 1 \Rightarrow |E_S| \leq \frac{(\pi-0)}{180} \left(\frac{\pi}{8}\right)^4 (1) \approx 0.00021$

$$(b) h = \frac{\pi}{8} \Rightarrow \frac{h}{3} = \frac{\pi}{24};$$

$$\sum mf(x_i) = 10.472087048$$

$$\Rightarrow S = \frac{\pi}{24}(10.472087048) \approx 1.37079$$

	$x_i$	$f(x_i)$	$m$	$mf(x_i)$
$x_0$	0	1	1	1
$x_1$	$\pi/8$	0.974495358	4	3.897981432
$x_2$	$\pi/4$	0.900316316	2	1.800632632
$x_3$	$3\pi/8$	0.784213303	4	3.136853212
$x_4$	$\pi/2$	0.636619772	1	0.636619772

$$(c) \approx \left(\frac{0.00021}{1.37079}\right) \times 100 \approx 0.015\%$$

$$23. (a) h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1 \Rightarrow \text{erf}(1) = \frac{2}{\sqrt{\pi}} \left(\frac{0.1}{3}\right) (y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 4y_9 + y_{10})$$

$$= \frac{2}{30\sqrt{\pi}} (e^0 + 4e^{-0.01} + 2e^{-0.04} + 4e^{-0.09} + \dots + 4e^{-0.81} + e^{-1}) \approx 0.8427$$

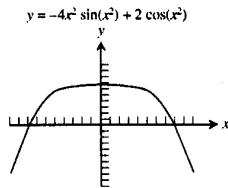
$$(b) |E_s| \leq \frac{1-0}{180} (0.1)^4 (12) \approx 6.7 \times 10^{-6}$$

24. The average of the 13 discrete temperatures gives equal weight to the low values at the end.

$$25. (a) f(x) = 2x \cos(x^2)$$

$$f'(x) = 2x \cdot -2x \sin(x^2) + 2 \cos(x^2) = -4x^2 \sin(x^2) + 2 \cos(x^2)$$

(b)



(c) The graph shows that  $-3 \leq f''(x) \leq 2$  so  $|f''(x)| \leq 3$  for  $-1 \leq x \leq 1$ .

$$(d) |E_T| \leq \frac{1-(-1)}{12} (h^2)(3) = \frac{h^2}{2}$$

$$(e) \text{ For } 0 < h \leq 0.1, |E_T| \leq \frac{h^2}{2} \leq \frac{0.1^2}{2} = 0.005 < 0.01$$

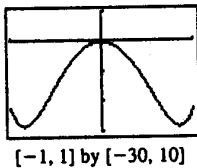
$$(f) n \geq \frac{1-(-1)}{h} \geq \frac{2}{0.1} = 20$$

$$26. (a) f'''(x) = -4x^2 \cdot 2x \cos(x^2) - 8x \sin(x^2) - 4x \sin(x^2) = -8x^3 \cos(x^2) - 12x \sin(x^2)$$

$$f^{(4)}(x) = -8x^3 \cdot -2x \sin(x^2) - 24x^2 \cos(x^2) - 12x \cdot 2x \cos(x^2) - 12 \sin(x^2)$$

$$= (16x^4 - 12) \sin(x^2) - 48x^2 \cos(x^2)$$

(b)

(c) The graph shows that  $-30 \leq f^{(4)}(x) \leq 0$  so  $|f^{(4)}(x)| \leq 30$  for  $-1 \leq x \leq 1$ .

(d)  $|E_T| \leq \frac{1 - (-1)}{180} (h^4)(30) = \frac{h^4}{3}$

(e) For  $0 < h \leq 0.4$ ,  $|E_S| \leq \frac{h^4}{3} \leq \frac{0.4^4}{3} \approx 0.00853 < 0.01$

(f)  $n \geq \frac{1 - (-1)}{h} \geq \frac{2}{0.4} = 5$

Exercises 27-30 were done using the `nInt( )` command on a TI-92 Plus calculator. Your answer may be different depending on your calculator.

27. 3.1415927

28. 1.0894294

29. 1.3707622

30. 0.82811633

31. (a)  $T_{10} \approx 1.983523538$ 

$T_{100} \approx 1.999835504$

$T_{1000} \approx 1.999998355$

n	$ E_T  = 2 - T_n$
10	$0.016476462 = 1.6476462 \times 10^{-2}$
100	$1.64496 \times 10^{-4}$
1000	$1.645 \times 10^{-6}$

(c)  $|E_{T_{10n}}| \approx 10^{-2} |E_{T_n}|$

(d)  $b - a = \pi$ ,  $h^2 = \frac{\pi^2}{n^2}$ ,  $M = 1 \Rightarrow |E_{T_n}| \leq \frac{\pi}{12} \left( \frac{\pi^2}{n^2} \right) = \frac{\pi^3}{12n^2}$  and  $|E_{T_{10n}}| \leq \frac{\pi^3}{12(10n)^2} = 10^{-2} |E_{T_n}|$

32. (a)  $S_{10} \approx 2.000109517$ 

$S_{100} \approx 2.000000011$

$S_{1000} \approx 2.000000000$

(b)	n	$ E_S  = 2 - S_n$
	10	$1.09517 \times 10^{-4}$
	100	$1.1 \times 10^{-8}$
	1000	0

(c)  $|E_{S_{10n}}| \approx 10^{-4} |E_{S_n}|$

(d)  $b - a = \pi$ ,  $h^4 = \frac{\pi^4}{n^4}$ ,  $M = 1 \Rightarrow |E_{S_n}| \leq \frac{\pi}{180} \left( \frac{\pi^4}{n^4} \right) = \frac{\pi^5}{180n^4}$  and  $|E_{S_{10n}}| \leq \frac{\pi^5}{180(10n)^4} = 10^{-4} |E_{S_n}|$

### CHAPTER 4 PRACTICE EXERCISES

1.  $\int (x^3 + 5x - 7) dx = \frac{x^4}{4} + \frac{5x^2}{2} - 7x + C$

2.  $\int \left( 8t^3 - \frac{t^2}{2} + t \right) dt = \frac{8t^4}{4} - \frac{t^3}{6} + \frac{t^2}{2} + C = 2t^4 - \frac{t^3}{6} + \frac{t^2}{2} + C$

3.  $\int \left( 3\sqrt{t} + \frac{4}{t^2} \right) dt = \int (3t^{1/2} + 4t^{-2}) dt = \frac{3t^{3/2}}{\left(\frac{3}{2}\right)} + \frac{4t^{-1}}{-1} + C = 2t^{3/2} - \frac{4}{t} + C$

4.  $\int \left( \frac{1}{2\sqrt{t}} - \frac{3}{t^4} \right) dt = \int \left( \frac{1}{2}t^{-1/2} - 3t^{-4} \right) dt = \frac{1}{2} \left( \frac{t^{1/2}}{\frac{1}{2}} \right) - \frac{3t^{-3}}{(-3)} + C = \sqrt{t} + \frac{1}{t^3} + C$

5. Let  $u = r^2 + 5 \Rightarrow du = 2r dr \Rightarrow \frac{1}{2} du = r dr$

$$\int \frac{r dr}{(r^2 + 5)^2} = \int \frac{\left(\frac{1}{2}\right) du}{u^2} = \frac{1}{2} \int u^{-2} du = \frac{1}{2} \left( \frac{u^{-1}}{-1} \right) + C = -\frac{1}{2} u^{-1} + C = -\frac{1}{2(r^2 + 5)} + C$$

6. Let  $u = r^3 - \sqrt{2} \Rightarrow du = 3r^2 dr \Rightarrow 2 du = 6r^2 dr$

$$\int \frac{6r^2 dr}{(r^3 - \sqrt{2})^3} = \int \frac{2 du}{u^3} = 2 \int u^{-3} du = 2 \left( \frac{u^{-2}}{-2} \right) + C = -u^{-2} + C = -\frac{1}{(r^3 - \sqrt{2})^2} + C$$

7. Let  $u = 2 - \theta^2 \Rightarrow du = -2\theta d\theta \Rightarrow -\frac{1}{2} du = \theta d\theta$

$$\int 3\theta\sqrt{2 - \theta^2} d\theta = \int \sqrt{u} \left( -\frac{3}{2} du \right) = -\frac{3}{2} \int u^{1/2} du = -\frac{3}{2} \left( \frac{u^{3/2}}{\frac{3}{2}} \right) + C = -u^{3/2} + C = -(2 - \theta^2)^{3/2} + C$$

$$8. \text{ Let } u = 73 + \theta^3 \Rightarrow du = 3\theta^2 d\theta \Rightarrow \frac{1}{27} du = \frac{\theta^2}{9} d\theta$$

$$\int \frac{\theta^2}{9\sqrt{73 + \theta^3}} d\theta = \int \frac{1}{\sqrt{u}} \left( \frac{1}{27} du \right) = \frac{1}{27} \int u^{-1/2} du = \frac{1}{27} \left( \frac{u^{1/2}}{\frac{1}{2}} \right) + C = \frac{2}{27} u^{1/2} + C = \frac{2}{27} \sqrt{73 + \theta^3} + C$$

$$9. \int e^x \sin(e^x) dx = \int \sin u du, \text{ where } u = e^x \text{ and } du = e^x dx$$

$$= -\cos u + C = -\cos(e^x) + C$$

$$10. \int e^t \cos(3e^t - 2) dt = \frac{1}{3} \int \cos u du, \text{ where } u = 3e^t - 2 \text{ and } du = 3e^t dt$$

$$= \frac{1}{3} \sin u + C = \frac{1}{3} \sin(3e^t - 2) + C$$

$$11. \int e^x \sec^2(e^x - 7) dx = \int \sec^2 u du, \text{ where } u = e^x - 7 \text{ and } du = e^x dx$$

$$= \tan u + C = \tan(e^x - 7) + C$$

$$12. \int e^y \csc(e^y + 1) \cot(e^y + 1) dy = \int \csc u \cot u du, \text{ where } u = e^y + 1 \text{ and } du = e^y dy$$

$$= -\csc u + C = -\csc(e^y + 1) + C$$

$$13. \int \frac{\tan(\ln v)}{v} dv = \int \tan u du = \int \frac{\sin u}{\cos u} du, \text{ where } u = \ln v \text{ and } du = \frac{1}{v} dv$$

$$= -\ln |\cos u| + C = -\ln |\cos(\ln v)| + C$$

$$14. \text{ Let } u = \frac{\theta}{3} \Rightarrow du = \frac{1}{3} d\theta \Rightarrow 3 du = d\theta$$

$$\int \sec \frac{\theta}{3} \tan \frac{\theta}{3} d\theta = \int (\sec u \tan u)(3 du) = 3 \sec u + C = 3 \sec \frac{\theta}{3} + C$$

$$15. \text{ Let } u = \frac{x}{4} \Rightarrow du = \frac{1}{4} dx \Rightarrow 4 du = dx$$

$$\int \sin^2 \frac{x}{4} dx = \int (\sin^2 u)(4 du) = \int 4 \left( \frac{1 - \cos 2u}{2} \right) du = 2 \int (1 - \cos 2u) du = 2 \left( u - \frac{\sin 2u}{2} \right) + C$$

$$= 2u - \sin 2u + C = 2 \left( \frac{x}{4} \right) - \sin 2 \left( \frac{x}{4} \right) + C = \frac{x}{2} - \sin \frac{x}{2} + C$$

$$16. \int \frac{(\ln x)^{-3}}{x} dx = \int u^{-3} du, \text{ where } u = \ln x \text{ and } du = \frac{1}{x} dx$$

$$= \frac{u^{-2}}{-2} + C = -\frac{1}{2} (\ln x)^{-2} + C$$

$$17. \int \frac{1}{r} \csc^2(1 + \ln r) dr = \int \csc^2 u du, \text{ where } u = 1 + \ln r \text{ and } du = \frac{1}{r} dr$$

$$= -\cot u + C = -\cot(1 + \ln r) + C$$

18.  $\int \frac{\cos(1 - \ln v)}{v} dv = - \int \cos u du$ , where  $u = 1 - \ln v$  and  $du = -\frac{1}{v} dv$   
 $= -\sin u + C = -\sin(1 - \ln v) + C$

19.  $\int x3^{x^2} dx = \frac{1}{2} \int 3^u du$ , where  $u = x^2$  and  $du = 2x dx$   
 $= \frac{1}{2 \ln 3} (3^u) + C = \frac{1}{2 \ln 3} (3^{x^2}) + C$

20.  $\int 2^{\tan x} \sec^2 x dx = \int 2^u du$ , where  $u = \tan x$  and  $du = \sec^2 x dx$   
 $= \frac{1}{\ln 2} (2^u) + C = \frac{2^{\tan x}}{\ln 2} + C$

21. (a) Each time subinterval is of length  $\Delta t = 0.4$  sec. The distance traveled over each subinterval, using the midpoint rule, is  $\Delta h = \frac{1}{2}(v_i + v_{i+1})\Delta t$ , where  $v_i$  is the velocity at the left, and  $v_{i+1}$  the velocity at the right, endpoint of the subinterval. We then add  $\Delta h$  to the height attained so far at the left endpoint  $v_i$  to arrive at the height associated with velocity  $v_{i+1}$  at the right endpoint. Using this methodology we build the following table based on the figure in the text:

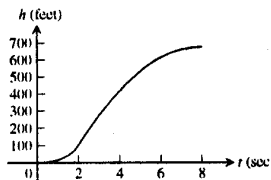
t (sec)	0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0	4.4	4.8	5.2	5.6	6.0
v (fps)	0	10	25	55	100	190	180	170	155	140	130	120	105	90	80	65
h (ft)	0	2	9	25	56	114	188	258	323	382	436	486	531	570	604	633

t (sec)	6.4	6.8	7.2	7.6	8.0
v (fps)	52	40	30	15	0
h (ft)	656	674	688	697	700

NOTE: Your table values may vary slightly from ours depending on the v-values you read from the graph. Remember that some shifting of the graph occurs in the printing process.

The total height attained is about 700 ft.

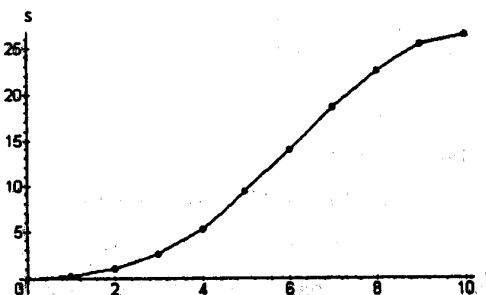
(b) The graph is based on the table in part (a).



22. (a) Each time subinterval is of length  $\Delta t = 1$  sec. The distance traveled over each subinterval, using the midpoint rule, is  $\Delta s = \frac{1}{2}(v_i + v_{i+1})\Delta t$ , where  $v_i$  is the velocity at the left, and  $v_{i+1}$  the velocity at the right, endpoint of the subinterval. We then add  $\Delta s$  to the distance attained so far at the left endpoint  $v_i$  to arrive at the distance associated with velocity  $v_{i+1}$  at the right endpoint. Using this methodology we build the table given below based on the figure in the text, obtaining approximately 26 m for the total distance traveled:

t (sec)	0	1	2	3	4	5	6	7	8	9	10
v (m/sec)	0	0.5	1.0	2	3.5	4.5	4.8	4.5	3.5	2	0
s (m)	0	0.25	1.00	2.5	5.25	9.25	13.9	18.55	22.55	25.3	26.3

- (b) The graph shows the distance traveled by the moving body as a function of time for  $0 \leq t \leq 10$ .



23. Let  $u = 2x - 1 \Rightarrow du = 2 dx \Rightarrow \frac{1}{2} du = dx$ ;  $x = 1 \Rightarrow u = 1$ ,  $x = 5 \Rightarrow u = 9$

$$\int_1^5 (2x - 1)^{-1/2} dx = \int_1^9 u^{-1/2} \left(\frac{1}{2} du\right) = \left[u^{1/2}\right]_1^9 = 3 - 1 = 2$$

24. Let  $u = x^2 - 1 \Rightarrow du = 2x dx \Rightarrow \frac{1}{2} du = x dx$ ;  $x = 1 \Rightarrow u = 0$ ,  $x = 3 \Rightarrow u = 8$

$$\int_1^3 x(x^2 - 1)^{1/3} dx = \int_0^8 u^{1/3} \left(\frac{1}{2} du\right) = \left[\frac{3}{8} u^{4/3}\right]_0^8 = \frac{3}{8}(16 - 0) = 6$$

25. Let  $u = \frac{x}{2} \Rightarrow 2 du = dx$ ;  $x = -\pi \Rightarrow u = -\frac{\pi}{2}$ ,  $x = 0 \Rightarrow u = 0$

$$\int_{-\pi}^0 \cos\left(\frac{x}{2}\right) dx = \int_{-\pi/2}^0 (\cos u)(2 du) = [2 \sin u]_{-\pi/2}^0 = 2 \sin 0 - 2 \sin\left(-\frac{\pi}{2}\right) = 2(0 - (-1)) = 2$$

26. Let  $u = \sin x \Rightarrow du = \cos x dx$ ;  $x = 0 \Rightarrow u = 0$ ,  $x = \frac{\pi}{2} \Rightarrow u = 1$

$$\int_0^{\pi/2} (\sin x)(\cos x) dx = \int_0^1 u du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$$

$$27. \text{ (a) } \int_{-2}^2 f(x) \, dx = \frac{1}{3} \int_{-2}^2 3 f(x) \, dx = \frac{1}{3}(12) = 4 \quad \text{(b) } \int_2^5 f(x) \, dx = \int_{-2}^5 f(x) \, dx - \int_{-2}^2 f(x) \, dx = 6 - 4 = 2$$

$$\text{(c) } \int_5^{-2} g(x) \, dx = - \int_{-2}^5 g(x) \, dx = -2 \quad \text{(d) } \int_{-2}^5 (-\pi g(x)) \, dx = -\pi \int_{-2}^5 g(x) \, dx = -\pi(2) = -2\pi$$

$$\text{(e) } \int_{-2}^5 \left( \frac{f(x) + g(x)}{5} \right) dx = \frac{1}{5} \int_{-2}^5 f(x) \, dx + \frac{1}{5} \int_{-2}^5 g(x) \, dx = \frac{1}{5}(6) + \frac{1}{5}(2) = \frac{8}{5}$$

$$28. \text{ (a) } \int_0^2 g(x) \, dx = \frac{1}{7} \int_0^2 7 g(x) \, dx = \frac{1}{7}(7) = 1 \quad \text{(b) } \int_1^2 g(x) \, dx = \int_0^2 g(x) \, dx - \int_0^1 g(x) \, dx = 1 - 2 = -1$$

$$\text{(c) } \int_2^0 f(x) \, dx = - \int_0^2 f(x) \, dx = -\pi \quad \text{(d) } \int_0^2 \sqrt{2} f(x) \, dx = \sqrt{2} \int_0^2 f(x) \, dx = \sqrt{2}(\pi) = \pi\sqrt{2}$$

$$\text{(e) } \int_0^2 [g(x) - 3f(x)] \, dx = \int_0^2 g(x) \, dx - 3 \int_0^2 f(x) \, dx = 1 - 3\pi$$

$$29. \int_{-1}^1 (3x^2 - 4x + 7) \, dx = [x^3 - 2x^2 + 7x]_{-1}^1 = [1^3 - 2(1)^2 + 7(1)] - [(-1)^3 - 2(-1)^2 + 7(-1)] = 6 - (-10) = 16$$

$$30. \int_0^1 (8s^3 - 12s^2 + 5) \, ds = [2s^4 - 4s^3 + 5s]_0^1 = [2(1)^4 - 4(1)^3 + 5(1)] - 0 = 3$$

$$31. \int_1^4 \left( \frac{x}{8} + \frac{1}{2x} \right) dx = \frac{1}{2} \int_1^4 \left( \frac{1}{4}x + \frac{1}{x} \right) dx = \frac{1}{2} \left[ \frac{1}{8}x^2 + \ln |x| \right]_1^4 = \frac{1}{2} \left[ \left( \frac{16}{8} + \ln 4 \right) - \left( \frac{1}{8} + \ln 1 \right) \right] = \frac{15}{16} + \frac{1}{2} \ln 4$$

$$= \frac{15}{16} + \ln \sqrt{4} = \frac{15}{16} + \ln 2$$

$$32. \int_1^8 \left( \frac{2}{3x} - \frac{8}{x^2} \right) dx = \frac{2}{3} \int_1^8 \left( \frac{1}{x} - 12x^{-2} \right) dx = \frac{2}{3} [\ln |x| + 12x^{-1}]_1^8 = \frac{2}{3} \left[ \left( \ln 8 + \frac{12}{8} \right) - \left( \ln 1 + 12 \right) \right]$$

$$= \frac{2}{3} \left( \ln 8 + \frac{3}{2} - 12 \right) = \frac{2}{3} \left( \ln 8 - \frac{21}{2} \right) = \frac{2}{3} (\ln 8) - 7 = \ln(8^{2/3}) - 7 = \ln 4 - 7$$

$$33. \int_1^4 \frac{dt}{t\sqrt{t}} = \int_1^4 \frac{dt}{t^{3/2}} = \int_1^4 t^{-3/2} dt = [-2t^{-1/2}]_1^4 = \frac{-2}{\sqrt{4}} - \frac{(-2)}{\sqrt{1}} = 1$$

$$34. \text{ Let } x = 1 + \sqrt{u} \Rightarrow dx = \frac{1}{2}u^{-1/2} du \Rightarrow 2 dx = \frac{du}{\sqrt{u}}; u = 1 \Rightarrow x = 2, u = 4 \Rightarrow x = 3$$



$$\int_1^4 \frac{(1 + \sqrt{u})^{1/2}}{\sqrt{u}} du = \int_2^3 x^{1/2}(2 dx) = \left[2\left(\frac{2}{3}\right)x^{3/2}\right]_2^3 = \frac{4}{3}(3^{3/2}) - \frac{4}{3}(2^{3/2}) = 4\sqrt{3} - \frac{8}{3}\sqrt{2} = \frac{4}{3}(3\sqrt{3} - 2\sqrt{2})$$

35. Let  $u = 2x + 1 \Rightarrow du = 2 dx \Rightarrow 18 du = 36 dx$ ;  $x = 0 \Rightarrow u = 1$ ,  $x = 1 \Rightarrow u = 3$

$$\int_0^1 \frac{36 dx}{(2x + 1)^3} = \int_1^3 18u^{-3} du = \left[\frac{18u^{-2}}{-2}\right]_1^3 = \left[\frac{-9}{u^2}\right]_1^3 = \left(\frac{-9}{3^2}\right) - \left(\frac{-9}{1^2}\right) = 8$$

36. Let  $u = 7 - 5r \Rightarrow du = -5 dr \Rightarrow -\frac{1}{5} du = dr$ ;  $r = 0 \Rightarrow u = 7$ ,  $r = 1 \Rightarrow u = 2$

$$\int_0^1 \frac{dr}{\sqrt[3]{(7-5r)^2}} = \int_0^1 (7-5r)^{-2/3} dr = \int_7^2 u^{-2/3} \left(-\frac{1}{5} du\right) = -\frac{1}{5} [3u^{1/3}]_7^2 = \frac{3}{5}(3\sqrt[3]{7} - 3\sqrt[3]{2})$$

37.  $\int_0^{\ln 5} e^r (3e^r + 1)^{-3/2} dr = \frac{1}{3} \int_4^{16} u^{-3/2} du$ , where  $u = 3e^r + 1$ ,  $du = 3e^r dr$ ;  $r = 0 \Rightarrow u = 4$ ,  $r = \ln 5 \Rightarrow u = 16$

$$= -\frac{2}{3} [u^{-1/2}]_4^{16} = -\frac{2}{3}(16^{-1/2} - 4^{-1/2}) = \left(-\frac{2}{3}\right)\left(\frac{1}{4} - \frac{1}{2}\right) = \left(-\frac{2}{3}\right)\left(-\frac{1}{4}\right) = \frac{1}{6}$$

38.  $\int_0^{\ln 9} e^\theta (e^\theta - 1)^{1/2} d\theta = \int_0^8 u^{1/2} du$ , where  $u = e^\theta - 1$ ,  $du = e^\theta d\theta$ ;  $\theta = 0 \Rightarrow u = 0$ ,  $\theta = \ln 9 \Rightarrow u = 8$

$$= \frac{2}{3} [u^{3/2}]_0^8 = \frac{2}{3}(8^{3/2} - 0^{3/2}) = \frac{2}{3}(2^{9/2} - 0) = \frac{2^{11/2}}{3} = \frac{32\sqrt{2}}{3}$$

39.  $\int_e^{e^2} \frac{1}{x\sqrt{\ln x}} dx = \int_e^{e^2} (\ln x)^{-1/2} \frac{1}{x} dx = \int_1^2 u^{-1/2} du$ , where  $u = \ln x$ ,  $du = \frac{1}{x} dx$ ;  $x = e \Rightarrow u = 1$ ,  $x = e^2 \Rightarrow u = 2$

$$= 2[u^{1/2}]_1^2 = 2(\sqrt{2} - 1) = 2\sqrt{2} - 2$$

40. Let  $u = 4t - \frac{\pi}{4} \Rightarrow du = 4 dt \Rightarrow \frac{1}{4} du = dt$ ;  $t = 0 \Rightarrow u = -\frac{\pi}{4}$ ,  $t = \frac{\pi}{4} \Rightarrow u = \frac{3\pi}{4}$

$$\int_0^{\pi/4} \cos^2\left(4t - \frac{\pi}{4}\right) dt = \int_{-\pi/4}^{3\pi/4} (\cos^2 u) \left(\frac{1}{4} du\right) = \frac{1}{4} \left[\frac{u}{2} + \frac{\sin 2u}{4}\right]_{-\pi/4}^{3\pi/4} = \frac{1}{4} \left(\frac{3\pi}{8} + \frac{\sin\left(\frac{3\pi}{2}\right)}{4}\right) - \frac{1}{4} \left(-\frac{\pi}{8} + \frac{\sin\left(-\frac{\pi}{2}\right)}{4}\right)$$

$$= \frac{\pi}{8} - \frac{1}{16} + \frac{1}{16} = \frac{\pi}{8}$$

41.  $\int_1^e \frac{1}{x}(1 + 7 \ln x)^{-1/3} dx = \frac{1}{7} \int_1^8 u^{-1/3} du$ , where  $u = 1 + 7 \ln x$ ,  $du = \frac{7}{x} dx$ ,  $x = 1 \Rightarrow u = 1$ ,  $x = e \Rightarrow u = 8$

$$= \frac{3}{14} [u^{2/3}]_1^8 = \frac{3}{14}(8^{2/3} - 1^{2/3}) = \left(\frac{3}{14}\right)(4 - 1) = \frac{9}{14}$$

$$42. \int_{\pi/4}^{3\pi/4} \csc^2 x \, dx = [-\cot x]_{\pi/4}^{3\pi/4} = \left(-\cot \frac{3\pi}{4}\right) - \left(-\cot \frac{\pi}{4}\right) = 2$$

$$43. \int_{-2}^2 \frac{3}{4+3t^2} \, dt = \sqrt{3} \int_{-2}^2 \frac{\sqrt{3}}{2^2+(\sqrt{3}t)^2} \, dt = \sqrt{3} \int_{-2\sqrt{3}}^{2\sqrt{3}} \frac{1}{2^2+u^2} \, du, \text{ where } u = \sqrt{3}t, \, du = \sqrt{3} \, dt;$$

$$t = -2 \Rightarrow u = -2\sqrt{3}, \, t = 2 \Rightarrow u = 2\sqrt{3}$$

$$= \sqrt{3} \left[ \frac{1}{2} \tan^{-1} \left( \frac{u}{2} \right) \right]_{-2\sqrt{3}}^{2\sqrt{3}} = \frac{\sqrt{3}}{2} [\tan^{-1}(\sqrt{3}) - \tan^{-1}(-\sqrt{3})] = \frac{\sqrt{3}}{2} \left[ \frac{\pi}{3} - \left(-\frac{\pi}{3}\right) \right] = \frac{\pi}{\sqrt{3}}$$

$$44. \int_1^3 \frac{[\ln(v+1)]^2}{v+1} \, dv = \int_1^3 [\ln(v+1)]^2 \frac{1}{v+1} \, dv = \int_{\ln 2}^{\ln 4} u^2 \, du, \text{ where } u = \ln(v+1), \, du = \frac{1}{v+1} \, dv;$$

$$v = 1 \Rightarrow u = \ln 2, \, v = 3 \Rightarrow u = \ln 4;$$

$$= \frac{1}{3} [u^3]_{\ln 2}^{\ln 4} = \frac{1}{3} [(\ln 4)^3 - (\ln 2)^3] = \frac{1}{3} [(2 \ln 2)^3 - (\ln 2)^3] = \frac{(\ln 2)^3}{3} (8 - 1) = \frac{7}{3} (\ln 2)^3$$

$$45. \int_{-\pi/3}^0 \sec x \tan x \, dx = [\sec x]_{-\pi/3}^0 = \sec 0 - \sec \left(-\frac{\pi}{3}\right) = 1 - 2 = -1$$

$$46. \int_{\pi/4}^{3\pi/4} \csc z \cot z \, dz = [-\csc z]_{\pi/4}^{3\pi/4} = \left(-\csc \frac{3\pi}{4}\right) - \left(-\csc \frac{\pi}{4}\right) = -\sqrt{2} + \sqrt{2} = 0$$

$$47. \int_{\sqrt{2}/3}^{2/3} \frac{1}{|y|\sqrt{9y^2-1}} \, dy = \int_{\sqrt{2}/3}^{2/3} \frac{3}{|3y|\sqrt{(3y)^2-1}} \, dy = \int_{\sqrt{2}}^2 \frac{1}{|u|\sqrt{u^2-1}} \, du, \text{ where } u = 3y, \, du = 3 \, dy;$$

$$y = \frac{\sqrt{2}}{3} \Rightarrow u = \sqrt{2}, \, y = \frac{2}{3} \Rightarrow u = 2$$

$$= [\sec^{-1} u]_{\sqrt{2}}^2 = [\sec^{-1} 2 - \sec^{-1} \sqrt{2}] = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

$$48. \text{ Let } u = 1 - x^2 \Rightarrow du = -2x \, dx \Rightarrow -du = 2x \, dx; \, x = -1 \Rightarrow u = 0, \, x = 1 \Rightarrow u = 0$$

$$\int_{-1}^1 2x \sin(1-x^2) \, dx = \int_0^0 -\sin u \, du = 0$$

$$49. \text{ Let } u = 1 + 3 \sin^2 x \Rightarrow du = 6 \sin x \cos x \, dx \Rightarrow \frac{1}{2} du = 3 \sin x \cos x \, dx; \, x = 0 \Rightarrow u = 1, \, x = \frac{\pi}{2} \Rightarrow u = 1 + 3 \sin^2 \frac{\pi}{2} = 4$$

$$\int_0^{\pi/2} \frac{3 \sin x \cos x}{\sqrt{1+3 \sin^2 x}} \, dx = \int_1^4 \frac{1}{\sqrt{u}} \left(\frac{1}{2} du\right) = \int_1^4 \frac{1}{2} u^{-1/2} \, du = \left[ \frac{1}{2} \left( \frac{u^{1/2}}{1/2} \right) \right]_1^4 = [u^{1/2}]_1^4 = 4^{1/2} - 1^{1/2} = 1$$

$$50. \int_{-2}^{-1} \frac{2}{v^2 + 4v + 5} dv = 2 \int_{-2}^{-1} \frac{1}{1 + (v^2 + 4v + 4)} dv = 2 \int_{-2}^{-1} \frac{1}{1 + (v+2)^2} dv = 2 \int_0^1 \frac{1}{1+u^2} du,$$

where  $u = v + 2$ ,  $du = dv$ ;  $v = -2 \Rightarrow u = 0$ ,  $v = -1 \Rightarrow u = 1$

$$= 2[\tan^{-1} u]_0^1 = 2(\tan^{-1} 1 - \tan^{-1} 0) = 2\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{2}$$

$$51. \int_{-1}^1 \frac{3}{4v^2 + 4v + 4} dv = \frac{3}{4} \int_{-1}^1 \frac{1}{\frac{3}{4} + (v^2 + v + \frac{1}{4})} dv = \frac{3}{4} \int_{-1}^1 \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2} dv = \frac{3}{4} \int_{-1/2}^{3/2} \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2 + u^2} du$$

where  $u = v + \frac{1}{2}$ ,  $du = dv$ ;  $v = -1 \Rightarrow u = -\frac{1}{2}$ ,  $v = 1 \Rightarrow u = \frac{3}{2}$

$$= \frac{3}{4} \left[ \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{2u}{\sqrt{3}} \right) \right]_{-1/2}^{3/2} = \frac{\sqrt{3}}{2} \left[ \tan^{-1} \sqrt{3} - \tan^{-1} \left( -\frac{1}{\sqrt{3}} \right) \right] = \frac{\sqrt{3}}{2} \left[ \frac{\pi}{3} - \left( -\frac{\pi}{6} \right) \right] = \frac{\sqrt{3}}{2} \left( \frac{2\pi}{6} + \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\sqrt{3}\pi}{4}$$

$$52. \text{ Let } u = \sin \sqrt{t} \Rightarrow du = (\cos \sqrt{t}) \left( \frac{1}{2} t^{-1/2} \right) dt = \frac{\cos \sqrt{t}}{2\sqrt{t}} dt \Rightarrow 2 du = \frac{\cos \sqrt{t}}{\sqrt{t}} dt; t = \frac{\pi^2}{36} \Rightarrow u = \sin \frac{\pi}{6} = \frac{1}{2},$$

$$t = \frac{\pi^2}{4} \Rightarrow u = \sin \frac{\pi}{2} = 1$$

$$\int_{\pi^2/36}^{\pi^2/4} \frac{\cos \sqrt{t}}{\sqrt{t} \sin \sqrt{t}} dt = \int_{1/2}^1 \frac{1}{\sqrt{u}} (2 du) = 2 \int_{1/2}^1 u^{-1/2} du = [4\sqrt{u}]_{1/2}^1 = 4\sqrt{1} - 4\sqrt{1/2} = 2(2 - \sqrt{2})$$

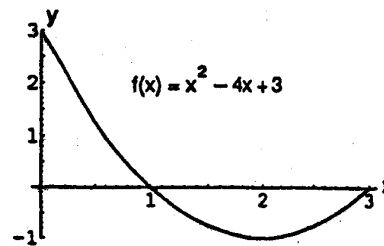
$$53. x^2 - 4x + 3 = 0 \Rightarrow (x-3)(x-1) = 0 \Rightarrow x = 3 \text{ or } x = 1;$$

$$\text{Area} = \int_0^1 (x^2 - 4x + 3) dx - \int_1^3 (x^2 - 4x + 3) dx$$

$$= \left[ \frac{x^3}{3} - 2x^2 + 3x \right]_0^1 - \left[ \frac{x^3}{3} - 2x^2 + 3x \right]_1^3$$

$$= \left[ \left( \frac{1^3}{3} - 2(1)^2 + 3(1) \right) - 0 \right]$$

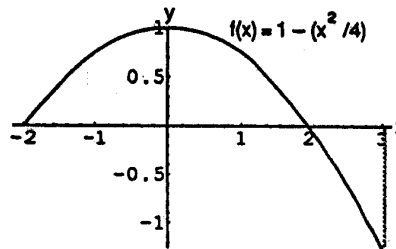
$$- \left[ \left( \frac{3^3}{3} - 2(3)^2 + 3(3) \right) - \left( \frac{1^3}{3} - 2(1)^2 + 3(1) \right) \right] = \left( \frac{1}{3} + 1 \right) - \left[ 0 - \left( \frac{1}{3} + 1 \right) \right] = \frac{8}{3}$$



$$54. 1 - \frac{x^2}{4} = 0 \Rightarrow 4 - x^2 = 0 \Rightarrow x = \pm 2;$$

$$\text{Area} = \int_{-2}^2 \left( 1 - \frac{x^2}{4} \right) dx - \int_2^3 \left( 1 - \frac{x^2}{4} \right) dx$$

$$= \left[ x - \frac{x^3}{12} \right]_{-2}^2 - \left[ x - \frac{x^3}{12} \right]_2^3$$



$$\begin{aligned}
 &= \left[ \left( 2 - \frac{2^3}{12} \right) - \left( -2 - \frac{(-2)^3}{12} \right) \right] \\
 &\quad - \left[ \left( 3 - \frac{3^3}{12} \right) - \left( 2 - \frac{2^3}{12} \right) \right] = \left[ \frac{4}{3} - \left( -\frac{4}{3} \right) \right] - \left[ \frac{3}{4} - \frac{4}{3} \right] = \frac{13}{4}
 \end{aligned}$$

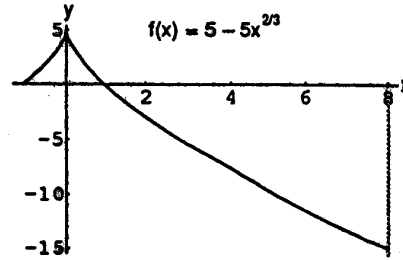
55.  $5 - 5x^{2/3} = 0 \Rightarrow 1 - x^{2/3} = 0 \Rightarrow x = \pm 1;$

$$\text{Area} = \int_{-1}^1 (5 - 5x^{2/3}) dx - \int_1^8 (5 - 5x^{2/3}) dx$$

$$= [5x - 3x^{5/3}]_{-1}^1 - [5x - 3x^{5/3}]_1^8$$

$$= [(5(1) - 3(1)^{5/3}) - (5(-1) - 3(-1)^{5/3})]$$

$$- [(5(8) - 3(8)^{5/3}) - (5(1) - 3(1)^{5/3})] = [2 - (-2)] - [(40 - 96) - 2] = 62$$



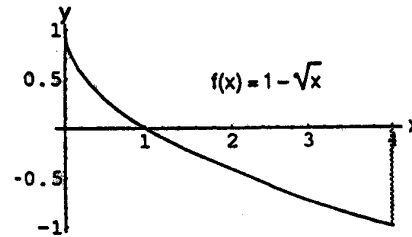
56.  $1 - \sqrt{x} = 0 \Rightarrow x = 1;$

$$\text{Area} = \int_0^1 (1 - \sqrt{x}) dx - \int_1^4 (1 - \sqrt{x}) dx$$

$$= \left[ x - \frac{2}{3}x^{3/2} \right]_0^1 - \left[ x - \frac{2}{3}x^{3/2} \right]_1^4$$

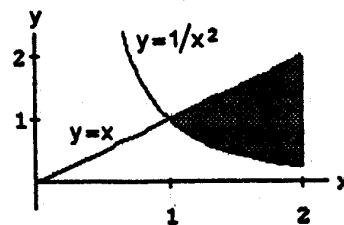
$$= \left[ \left( 1 - \frac{2}{3}(1)^{3/2} \right) - 0 \right] - \left[ \left( 4 - \frac{2}{3}(4)^{3/2} \right) - \left( 1 - \frac{2}{3}(1)^{3/2} \right) \right]$$

$$= \frac{1}{3} - \left[ \left( 4 - \frac{16}{3} \right) - \frac{1}{3} \right] = 2$$



57.  $f(x) = x, g(x) = \frac{1}{x^2}, a = 1, b = 2 \Rightarrow A = \int_a^b [f(x) - g(x)] dx$

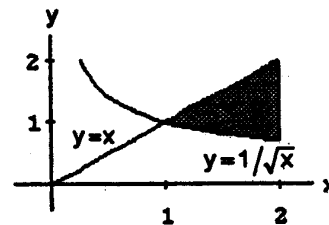
$$= \int_1^2 \left( x - \frac{1}{x^2} \right) dx = \left[ \frac{x^2}{2} + \frac{1}{x} \right]_1^2 = \left( \frac{4}{2} + \frac{1}{2} \right) - \left( \frac{1}{2} + 1 \right) = 1$$



58.  $f(x) = x, g(x) = \frac{1}{\sqrt{x}}, a = 1, b = 2 \Rightarrow A = \int_a^b [f(x) - g(x)] dx$

$$= \int_1^2 \left( x - \frac{1}{\sqrt{x}} \right) dx = \left[ \frac{x^2}{2} - 2\sqrt{x} \right]_1^2 = \left( \frac{4}{2} - 2\sqrt{2} \right) - \left( \frac{1}{2} - 2 \right)$$

$$= \frac{7 - 4\sqrt{2}}{2}$$



$$59. f(x) = (1 - \sqrt{x})^2, g(x) = 0, a = 0, b = 1 \Rightarrow A = \int_a^b [f(x) - g(x)] dx = \int_0^1 (1 - \sqrt{x})^2 dx = \int_0^1 (1 - 2\sqrt{x} + x) dx$$

$$= \int_0^1 (1 - 2x^{1/2} + x) dx = \left[ x - \frac{4}{3}x^{3/2} + \frac{x^2}{2} \right]_0^1 = 1 - \frac{4}{3} + \frac{1}{2} = \frac{1}{6}(6 - 8 + 3) = \frac{1}{6}$$

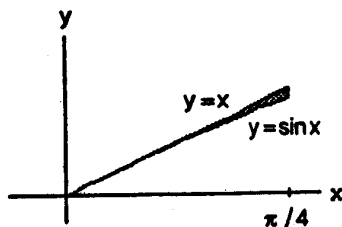
$$60. f(x) = (1 - x^3)^2, g(x) = 0, a = 0, b = 1 \Rightarrow A = \int_a^b [f(x) - g(x)] dx = \int_0^1 (1 - x^3)^2 dx = \int_0^1 (1 - 2x^3 + x^6) dx$$

$$= \left[ x - \frac{x^4}{2} + \frac{x^7}{7} \right]_0^1 = 1 - \frac{1}{2} + \frac{1}{7} = \frac{9}{14}$$

$$61. f(x) = x, g(x) = \sin x, a = 0, b = \frac{\pi}{4}$$

$$\Rightarrow A = \int_a^b [f(x) - g(x)] dx = \int_0^{\pi/4} (x - \sin x) dx$$

$$= \left[ \frac{x^2}{2} + \cos x \right]_0^{\pi/4} = \left( \frac{\pi^2}{32} + \frac{\sqrt{2}}{2} \right) - 1$$

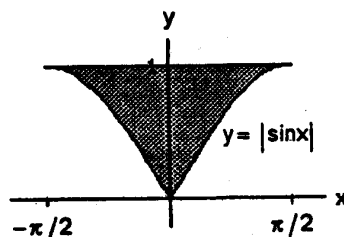


$$62. f(x) = 1, g(x) = |\sin x|, a = -\frac{\pi}{2}, b = \frac{\pi}{2}$$

$$\Rightarrow A = \int_a^b [f(x) - g(x)] dx = \int_{-\pi/2}^{\pi/2} (1 - |\sin x|) dx$$

$$= \int_{-\pi/2}^0 (1 + \sin x) dx + \int_0^{\pi/2} (1 - \sin x) dx$$

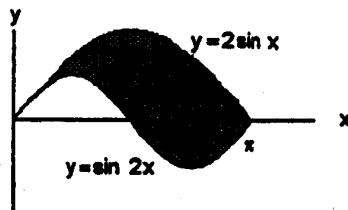
$$= 2 \int_0^{\pi/2} (1 - \sin x) dx = 2[x + \cos x]_0^{\pi/2} = 2\left(\frac{\pi}{2} - 1\right) = \pi - 2$$



$$63. a = 0, b = \pi, f(x) - g(x) = 2 \sin x - \sin 2x$$

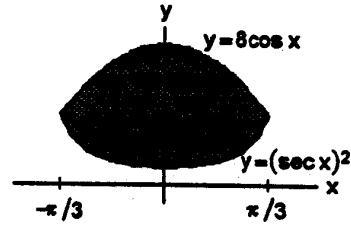
$$\Rightarrow A = \int_0^{\pi} (2 \sin x - \sin 2x) dx = \left[ -2 \cos x + \frac{\cos 2x}{2} \right]_0^{\pi}$$

$$= \left[ -2 \cdot (-1) + \frac{1}{2} \right] - \left[ -2 \cdot 1 + \frac{1}{2} \right] = 4$$



64.  $a = -\frac{\pi}{3}$ ,  $b = \frac{\pi}{3}$ ,  $f(x) - g(x) = 8 \cos x - \sec^2 x$

$$\begin{aligned} \Rightarrow A &= \int_{-\pi/3}^{\pi/3} (8 \cos x - \sec^2 x) dx = [8 \sin x - \tan x]_{-\pi/3}^{\pi/3} \\ &= \left(8 \cdot \frac{\sqrt{3}}{2} - \sqrt{3}\right) - \left(-8 \cdot \frac{\sqrt{3}}{2} + \sqrt{3}\right) = 6\sqrt{3} \end{aligned}$$



65.  $f(x) = x^3 - 3x^2 = x^2(x - 3) \Rightarrow f'(x) = 3x^2 - 6x = 3x(x - 2) \Rightarrow f' = +++ \mid \text{----} \mid +++$

$$\begin{aligned} \Rightarrow f(0) = 0 \text{ is a maximum and } f(2) = -4 \text{ is a minimum. Then } A &= - \int_0^3 (x^3 - 3x^2) dx = - \left[ \frac{x^4}{4} - x^3 \right]_0^3 \\ &= - \left( \frac{81}{4} - 27 \right) = \frac{27}{4} \end{aligned}$$

66.  $A = \int_0^1 (1 - x^{1/3})^3 dx = \int_0^1 (1 - 3x^{1/3} + 3x^{2/3} - x) dx = \left[ x - \frac{9}{4}x^{4/3} + \frac{9}{5}x^{5/3} - \frac{x^2}{2} \right]_0^1 = \frac{1}{20}$

67.  $y = \int \frac{x^2 + 1}{x^2} dx = \int (1 + x^{-2}) dx = x - x^{-1} + C = x - \frac{1}{x} + C$ ;  $y = -1$  when  $x = 1 \Rightarrow 1 - \frac{1}{1} + C = -1$   
 $\Rightarrow C = -1 \Rightarrow y = x - \frac{1}{x} - 1$

68.  $\frac{dy}{dx} = e^{-x-y-2} \Rightarrow \frac{dy}{dx} = e^{-x-2} \cdot e^{-y} \Rightarrow e^y dy = e^{-x-2} dx = \int e^y dy = \int e^{-x-2} dx \Rightarrow e^y = -e^{-x-2} + C$ ;  $x = 0$   
 and  $y = -2 \Rightarrow e^{-2} = -e^{-2} + C \Rightarrow C = 2e^{-2} \Rightarrow e^y = -e^{-x-2} + 2e^{-2} \Rightarrow \ln(e^y) = \ln(-e^{-x-2} + 2e^{-2})$   
 $\Rightarrow y = \ln(-e^{-x-2} + 2e^{-2})$

69.  $\frac{dr}{dt} = \int \left( 15\sqrt{t} + \frac{3}{\sqrt{t}} \right) dt = \int (15t^{1/2} + 3t^{-1/2}) dt = 10t^{3/2} + 6t^{1/2} + C$ ;  $\frac{dr}{dt} = 8$  when  $t = 1$   
 $\Rightarrow 10(1)^{3/2} + 6(1)^{1/2} + C = 8 \Rightarrow C = -8$ . Thus  $\frac{dr}{dt} = 10t^{3/2} + 6t^{1/2} - 8 \Rightarrow r = \int (10t^{3/2} + 6t^{1/2} - 8) dt$   
 $= 4t^{5/2} + 4t^{3/2} - 8t + C$ ;  $r = 0$  when  $t = 1 \Rightarrow 4(1)^{5/2} + 4(1)^{3/2} - 8(1) + C_1 = 0 \Rightarrow C_1 = 0$ . Therefore,  
 $r = 4t^{5/2} + 4t^{3/2} - 8t$

70.  $\frac{d^2r}{dt^2} = \int -\cos t dt = -\sin t + C$ ;  $r'' = 0$  when  $t = 0 \Rightarrow -\sin 0 + C = 0 \Rightarrow C = 0$ . Thus,  $\frac{d^2r}{dt^2} = -\sin t$   
 $\Rightarrow \frac{dr}{dt} = \int -\sin t dt = \cos t + C_1$ ;  $r' = 0$  when  $t = 0 \Rightarrow 1 + C_1 = 0 \Rightarrow C_1 = -1$ . Then  $\frac{dr}{dt} = \cos t - 1$   
 $\Rightarrow r = \int (\cos t - 1) dt = \sin t - t + C_2$ ;  $r = -1$  when  $t = 0 \Rightarrow 0 - 0 + C_2 = -1$ . Therefore,  $r = \sin t - t - 1$

$$71. y = x^2 + \int_1^x \frac{1}{t} dt \Rightarrow \frac{dy}{dx} = 2x + \frac{1}{x} \Rightarrow \frac{d^2y}{dx^2} = 2 - \frac{1}{x^2}; y(1) = 1 + \int_1^1 \frac{1}{t} dt = 1 \text{ and } y'(1) = 2 + 1 = 3$$

$$72. y = \int_0^x (1 + 2\sqrt{\sec t}) dt \Rightarrow \frac{dy}{dx} = 1 + 2\sqrt{\sec x} \Rightarrow \frac{d^2y}{dx^2} = 2\left(\frac{1}{2}\right)(\sec x)^{-1/2}(\sec x \tan x) = \sqrt{\sec x}(\tan x);$$

$$x = 0 \Rightarrow y = \int_0^0 (1 + 2\sqrt{\sec t}) dt = 0 \text{ and } x = 0 \Rightarrow \frac{dy}{dx} = 1 + 2\sqrt{\sec 0} = 3$$

$$73. y = \int_5^x \frac{\sin t}{t} dt - 3 \Rightarrow \frac{dy}{dx} = \frac{\sin x}{x}; x = 5 \Rightarrow y = \int_5^5 \frac{\sin t}{t} dt - 3 = -3$$

$$74. y = \int_{-1}^x \sqrt{2 - \sin^2 t} dt + 2 \text{ so that } \frac{dy}{dx} = \sqrt{2 - \sin^2 x}; x = -1 \Rightarrow y = \int_{-1}^{-1} \sqrt{2 - \sin^2 t} dt + 2 = 2$$

$$75. (a) \text{ av}(f) = \frac{1}{1 - (-1)} \int_{-1}^1 (mx + b) dx = \frac{1}{2} \left[ \frac{mx^2}{2} + bx \right]_{-1}^1 = \frac{1}{2} \left[ \left( \frac{m(1)^2}{2} + b(1) \right) - \left( \frac{m(-1)^2}{2} + b(-1) \right) \right] = \frac{1}{2}(2b) = b$$

$$(b) \text{ av}(f) = \frac{1}{k - (-k)} \int_{-k}^k (mx + b) dx = \frac{1}{2k} \left[ \frac{mx^2}{2} + bx \right]_{-k}^k = \frac{1}{2k} \left[ \left( \frac{m(k)^2}{2} + b(k) \right) - \left( \frac{m(-k)^2}{2} + b(-k) \right) \right]$$

$$= \frac{1}{2k}(2bk) = b$$

$$76. (a) y_{\text{av}} = \frac{1}{3 - 0} \int_0^3 \sqrt{3x} dx = \frac{1}{3} \int_0^3 \sqrt{3} x^{1/2} dx = \frac{\sqrt{3}}{3} \left[ \frac{2}{3} x^{3/2} \right]_0^3 = \frac{\sqrt{3}}{3} \left[ \frac{2}{3} (3)^{3/2} - \frac{2}{3} (0)^{3/2} \right] = \frac{\sqrt{3}}{3} (2\sqrt{3}) = 2$$

$$(b) y_{\text{av}} = \frac{1}{a - 0} \int_0^a \sqrt{ax} dx = \frac{1}{a} \int_0^a \sqrt{a} x^{1/2} dx = \frac{\sqrt{a}}{a} \left[ \frac{2}{3} x^{3/2} \right]_0^a = \frac{\sqrt{a}}{a} \left( \frac{2}{3} (a)^{3/2} - \frac{2}{3} (0)^{3/2} \right) = \frac{\sqrt{a}}{a} \left( \frac{2}{3} a\sqrt{a} \right) = \frac{2}{3} a$$

$$77. f'_{\text{av}} = \frac{1}{b - a} \int_a^b f'(x) dx = \frac{1}{b - a} [f(x)]_a^b = \frac{1}{b - a} [f(b) - f(a)] = \frac{f(b) - f(a)}{b - a} \text{ so the average value of } f' \text{ over } [a, b] \text{ is the}$$

slope of the secant line joining the points  $(a, f(a))$  and  $(b, f(b))$ .

$$78. \text{ Yes, because the average value of } f \text{ on } [a, b] \text{ is } \frac{1}{b - a} \int_a^b f(x) dx. \text{ If the length of the interval is 2, then } b - a = 2$$

and the average value of the function is  $\frac{1}{2} \int_a^b f(x) dx$ .

$$79. \frac{dy}{dx} = \sqrt{2 + \cos^3 x}$$

$$80. \frac{dy}{dx} = \sqrt{2 + \cos^3(7x^2)} \cdot \frac{d}{dx}(7x^2) = 14x \sqrt{2 + \cos^3(7x^2)}$$

$$81. \frac{dy}{dx} = \frac{d}{dx} \left( - \int_1^x \frac{6}{3+t^4} dt \right) = - \frac{6}{3+x^4}$$

$$82. \frac{dy}{dx} = \frac{d}{dx} \left( \int_0^{e^{2x}} \frac{1}{1+t^2} dt - \int_0^x \frac{1}{1+t^2} dt \right) = \frac{1}{1+(e^{2x})^2} (2e^{2x}) - \frac{1}{1+x^2} = \frac{2e^{2x}}{1+e^{4x}} - \frac{1}{1+x^2}$$

$$83. h = \frac{b-a}{n} = \frac{\pi-0}{6} = \frac{\pi}{6} \Rightarrow \frac{h}{2} = \frac{\pi}{12};$$

$$\sum_{i=0}^6 mf(x_i) = 12 \Rightarrow T = \left(\frac{\pi}{12}\right)(12) = \pi;$$

	$x_i$	$f(x_i)$	$m$	$mf(x_i)$
$x_0$	0	0	1	0
$x_1$	$\pi/6$	1/2	2	1
$x_2$	$\pi/3$	3/2	2	3
$x_3$	$\pi/2$	2	2	4
$x_4$	$2\pi/3$	3/2	2	3
$x_5$	$5\pi/6$	1/2	2	1
$x_6$	$\pi$	0	1	0

$$\sum_{i=0}^6 mf(x_i) = 18 \text{ and } \frac{h}{3} = \frac{\pi}{18} \Rightarrow S = \left(\frac{\pi}{18}\right)(18) = \pi.$$

	$x_i$	$f(x_i)$	$m$	$mf(x_i)$
$x_0$	0	0	1	0
$x_1$	$\pi/6$	1/2	4	2
$x_2$	$\pi/3$	3/2	2	3
$x_3$	$\pi/2$	2	4	8
$x_4$	$2\pi/3$	3/2	2	3
$x_5$	$5\pi/6$	1/2	4	2
$x_6$	$\pi$	0	1	0

$$84. (a) \text{ Each interval is } 5 \text{ min} = \frac{1}{12} \text{ hour.}$$

$$\frac{1}{24} [2.5 + 2(2.4) + 2(2.3) + \dots + 2(2.4) + 2.3] = \frac{29}{12} \approx 2.42 \text{ gal}$$

$$(b) (60 \text{ mph}) \left( \frac{12}{29} \text{ hours/gal} \right) \approx 24.83 \text{ mi/gal}$$

$$\begin{aligned} 85. y_{av} &= \frac{1}{365-0} \int_0^{365} \left[ 37 \sin \left( \frac{2\pi}{365} (x-101) \right) + 25 \right] dx = \frac{1}{365} \left[ -37 \left( \frac{365}{2\pi} \cos \left( \frac{2\pi}{365} (x-101) \right) + 25x \right) \right]_0^{365} \\ &= \frac{1}{365} \left[ \left( -37 \left( \frac{365}{2\pi} \cos \left[ \frac{2\pi}{365} (365-101) \right] + 25(365) \right) - \left( -37 \left( \frac{365}{2\pi} \cos \left[ \frac{2\pi}{365} (0-101) \right] + 25(0) \right) \right) \right] \\ &= -\frac{37}{2\pi} \cos \left( \frac{2\pi}{365} (264) \right) + 25 + \frac{37}{2\pi} \cos \left( \frac{2\pi}{365} (-101) \right) = -\frac{37}{2\pi} \left( \cos \left( \frac{2\pi}{365} (264) \right) - \cos \left( \frac{2\pi}{365} (-101) \right) \right) + 25 \end{aligned}$$



$$\approx -\frac{37}{2\pi}(0.16705 - 0.16705) + 25 = 25^\circ\text{F}$$

$$86. \text{av}(C_v) = \frac{1}{675-20} \int_{20}^{675} [8.27 + 10^{-5}(26T - 1.87T^2)] dT = \frac{1}{655} \left[ 8.27T + \frac{13}{10^5} T^2 - \frac{0.62333}{10^5} T^3 \right]_{20}^{675}$$

$$\approx \frac{1}{655} [(5582.25 + 59.23125 - 1917.03194) - (165.4 + 0.052 - 0.04987)] \approx 5.434;$$

$$8.27 + 10^{-5}(26T - 1.87T^2) = 5.434 \Rightarrow 1.87T^2 - 26T - 283,600 = 0 \Rightarrow T \approx \frac{26 + \sqrt{676 + 4(1.87)(283,600)}}{2(1.87)}$$

$$\approx 396.45^\circ\text{C}$$

87. Using the trapezoidal rule,  $h = 15 \Rightarrow \frac{h}{2} = 7.5$ ;

$$\sum mf(x_i) = 794.8 \Rightarrow \text{Area} \approx (794.8)(7.5) = 5961 \text{ ft}^2;$$

The cost is  $\text{Area} \cdot (\$2.10/\text{ft}^2) \approx (5961 \text{ ft}^2)(\$2.10/\text{ft}^2)$   
 $= \$12,518.10 \Rightarrow$  the job cannot be done for \$11,000.

	$x_i$	$f(x_i)$	$m$	$mf(x_i)$
	$x_0$	0	1	0
	$x_1$	15	2	72
	$x_2$	30	2	108
	$x_3$	45	2	102
	$x_4$	60	2	99
	$x_5$	75	2	108
	$x_6$	90	2	128.8
	$x_7$	105	2	135
	$x_8$	120	1	42

88. (a) Upper estimate:

$$3(5.30 + 5.25 + 5.04 + \dots + 1.11) = 103.05 \text{ ft}$$

Lower estimate:

$$3(5.25 + 5.04 + 4.71 + \dots + 0) = 87.15 \text{ ft}$$

(b)  $\frac{3}{2}[5.30 + 2(5.25) + 2(5.04) + \dots + 2(1.11) + 0] = 95.1 \text{ ft}$

89. Yes. The function  $f$ , being differentiable on  $[a, b]$ , is then continuous on  $[a, b]$ . The Fundamental Theorem of Calculus says that every continuous function on  $[a, b]$  is the derivative of a function on  $[a, b]$ .

90. The second part of the Fundamental Theorem of Calculus states that if  $F(x)$  is an antiderivative of  $f(x)$  on

$[a, b]$ , then  $\int_a^b f(x) dx = F(b) - F(a)$ . In particular, if  $F(x)$  is an antiderivative of  $\sqrt{1+x^4}$  on  $[0, 1]$ , then

$$\int_0^1 \sqrt{1+x^4} dx = F(1) - F(0).$$

91.  $y(x) = \int_5^x \frac{\sin t}{t} dt + 3$

$$92. y' = \cos x - \frac{d}{dx} \int_{\pi}^x \cos 2t \, dt = \cos x - \cos 2x \Rightarrow y'' = -\sin x + 2 \sin 2x$$

Thus,  $y$  satisfies condition i.

$$y(\pi) = \sin \pi + \int_{\pi}^{\pi} \cos 2t \, dt + 1 = 1 \text{ and } y'(\pi) = \cos \pi - \cos 2\pi = -2$$

Thus,  $y$  satisfies condition ii.

$$93. (a) g(1) = \int_1^1 f(t) \, dt = 0$$

$$(b) g(3) = \int_1^3 f(t) \, dt = -\frac{1}{2}(2)(1) = -1$$

$$(c) g(-1) = \int_1^{-1} f(t) \, dt = -\int_{-1}^1 f(t) \, dt = -\frac{1}{4}\pi(2)^2 = -\pi$$

(d)  $g'(x) = f(x)$ ; Since  $f(x) > 0$  for  $-3 < x < 1$  and  $f(x) < 0$  for  $1 < x < 3$ ,  $g(x)$  has a relative maximum at  $x = 1$ .

$$(e) g'(-1) = f(-1) = 2$$

The equation of the tangent line is  $y - (-\pi) = 2(x + 1)$  or  $y = 2x + 2 - \pi$

(f)  $g''(x) = f'(x)$ ,  $f'(x) = 0$  at  $x = -1$  and  $f'$  is not defined at  $x = 2$ . The inflection points are at  $x = -1$  and  $x = 2$ . Note that  $g''(x) = f'(x)$  is undefined at  $x = 1$  as well, but since  $g''(x) = f'(x)$  is negative on both sides of  $x = 1$ ,  $x = 1$  is not an inflection point.

(g) Note that the absolute maximum is  $g(1) = 0$  and the absolute minimum is

$$g(-3) = \int_1^{-3} f(t) \, dt = -\int_{-3}^1 f(t) \, dt = -\frac{1}{2}\pi(2)^2 = -2\pi.$$

The range of  $g$  is  $[-2\pi, 0]$ .

94. (a) Before the chute opens for A,  $a = -32 \text{ ft/sec}^2$ . Since the helicopter is hovering,  $v_0 = 0 \text{ ft/sec}$

$$\Rightarrow v = \int -32 \, dt = -32t + v_0 = -32t. \text{ Then } s_0 = 6400 \text{ ft} \Rightarrow s = \int -32t \, dt = -16t^2 + s_0 = -16t^2 + 6400.$$

At  $t = 4 \text{ sec}$ ,  $s = -16(4)^2 + 6400 = 6144 \text{ ft}$  when A's chute opens;

$$(b) \text{ For B, } s_0 = 7000 \text{ ft, } v_0 = 0, a = -32 \text{ ft/sec}^2 \Rightarrow v = \int -32 \, dt = -32t + v_0 = -32t \Rightarrow s = \int -32t \, dt \\ = -16t^2 + s_0 = -16t^2 + 7000. \text{ At } t = 13 \text{ sec, } s = -16(13)^2 + 7000 = 4296 \text{ ft when B's chute opens;}$$

(c) After the chutes open,  $v = -16 \text{ ft/sec} \Rightarrow s = \int -16 \, dt = -16t + s_0$ . For A,  $s_0 = 6144 \text{ ft}$  and for B,  $s_0 = 4296 \text{ ft}$ . Therefore, for A,  $s = -16t + 6144$  and for B,  $s = -16t + 4296$ . When they hit the ground,

$$s = 0 \Rightarrow \text{for A, } 0 = -16t + 6144 \Rightarrow t = \frac{6144}{16} = 384 \text{ seconds, and for B, } 0 = -16t + 4296 \Rightarrow t = \frac{4296}{16}$$

= 268.5 seconds to hit the ground after the chutes open. Since B's chute opens 54 seconds after A's opens  $\Rightarrow$  B hits the ground first.

$$\begin{aligned} 95. \text{ av}(I) &= \frac{1}{30} \int_0^{30} (1200 - 40t) dt = \frac{1}{30} [1200t - 20t^2]_0^{30} = \frac{1}{30} [(1200(30) - 20(30)^2) - (1200(0) - 20(0)^2)] \\ &= \frac{1}{30} (18,000) = 600; \text{ Average Daily Holding Cost} = (600)(\$0.03) = \$18 \end{aligned}$$

$$\begin{aligned} 96. \text{ av}(I) &= \frac{1}{14} \int_0^{14} (600 + 600t) dt = \frac{1}{14} [600t + 300t^2]_0^{14} = \frac{1}{14} [600(14) + 300(14)^2 - 0] = 4800; \text{ Average Daily} \\ \text{Holding Cost} &= (4800)(\$0.04) = \$192 \end{aligned}$$

$$\begin{aligned} 97. \text{ av}(I) &= \frac{1}{30} \int_0^{30} \left(450 - \frac{t^2}{2}\right) dt = \frac{1}{30} \left[450t - \frac{t^3}{6}\right]_0^{30} = \frac{1}{30} \left[450(30) - \frac{30^3}{6} - 0\right] = 300; \text{ Average Daily Holding Cost} \\ &= (300)(\$0.02) = \$6 \end{aligned}$$

$$\begin{aligned} 98. \text{ av}(I) &= \frac{1}{60} \int_0^{60} (600 - 20\sqrt{15}t) dt = \frac{1}{60} \int_0^{60} \left(600 - 20\sqrt{15}t^{1/2}\right) dt = \frac{1}{60} \left[600t - 20\sqrt{15}\left(\frac{2}{3}\right)t^{3/2}\right]_0^{60} \\ &= \frac{1}{60} \left[600(60) - \frac{40\sqrt{15}}{3}(60)^{3/2} - 0\right] = \frac{1}{60} \left(36,000 - \left(\frac{320}{3}\right)15^2\right) = 200; \text{ Average Daily Holding Cost} \\ &= (200)(\$0.005) = \$1.00 \end{aligned}$$

#### CHAPTER 4 ADDITIONAL EXERCISES—THEORY, EXAMPLES, APPLICATIONS

$$1. \text{ (a) Yes, because } \int_0^1 f(x) dx = \frac{1}{7} \int_0^1 7f(x) dx = \frac{1}{7}(7) = 1$$

$$\begin{aligned} \text{(b) No. For example, } \int_0^1 8x dx &= [4x^2]_0^1 = 4, \text{ but } \int_0^1 \sqrt{8x} dx = \left[2\sqrt{2}\left(\frac{x^{3/2}}{3/2}\right)\right]_0^1 = \frac{4\sqrt{2}}{3}(1^{3/2} - 0^{3/2}) \\ &= \frac{4\sqrt{2}}{3} \neq \sqrt{4} \end{aligned}$$

$$2. \text{ (a) True: } \int_5^2 f(x) dx = - \int_2^5 f(x) dx = -3$$

$$\begin{aligned} \text{(b) True: } \int_{-2}^5 [f(x) + g(x)] dx &= \int_{-2}^5 f(x) dx + \int_{-2}^5 g(x) dx = \int_{-2}^2 f(x) dx + \int_2^5 f(x) dx + \int_{-2}^5 g(x) dx \\ &= 4 + 3 + 2 = 9 \end{aligned}$$

$$(c) \text{ False: } \int_{-2}^5 f(x) dx = 4 + 3 = 7 > 2 = \int_{-2}^5 g(x) dx \Rightarrow \int_{-2}^5 [f(x) - g(x)] dx > 0 \Rightarrow \int_{-2}^5 [g(x) - f(x)] dx < 0.$$

On the other hand,  $f(x) \leq g(x) \Rightarrow [g(x) - f(x)] \geq 0 \Rightarrow \int_{-2}^5 [g(x) - f(x)] dx \geq 0$  which would be a contradiction.

$$\begin{aligned} 3. \quad y &= \frac{1}{a} \int_0^x f(t) \sin a(x-t) dt = \frac{1}{a} \int_0^x f(t) \sin ax \cos at dt - \frac{1}{a} \int_0^x f(t) \cos ax \sin at dt \\ &= \frac{\sin ax}{a} \int_0^x f(t) \cos at dt - \frac{\cos ax}{a} \int_0^x f(t) \sin at dt \Rightarrow \frac{dy}{dx} = \cos ax \int_0^x f(t) \cos at dt \\ &\quad + \frac{\sin ax}{a} \left( \frac{d}{dx} \int_0^x f(t) \cos at dt \right) + \sin ax \int_0^x f(t) \sin at dt - \frac{\cos ax}{a} \left( \frac{d}{dx} \int_0^x f(t) \sin at dt \right) \\ &= \cos ax \int_0^x f(t) \cos at dt + \frac{\sin ax}{a} (f(x) \cos ax) + \sin ax \int_0^x f(t) \sin at dt - \frac{\cos ax}{a} (f(x) \sin ax) \\ &\Rightarrow \frac{dy}{dx} = \cos ax \int_0^x f(t) \cos at dt + \sin ax \int_0^x f(t) \sin at dt. \text{ Next,} \\ \frac{d^2y}{dx^2} &= -a \sin ax \int_0^x f(t) \cos at dt + (\cos ax) \left( \frac{d}{dx} \int_0^x f(t) \cos at dt \right) + a \cos ax \int_0^x f(t) \sin at dt \\ &\quad + (\sin ax) \left( \frac{d}{dx} \int_0^x f(t) \sin at dt \right) = -a \sin ax \int_0^x f(t) \cos at dt + (\cos ax) f(x) \cos ax \\ &\quad + a \cos ax \int_0^x f(t) \sin at dt + (\sin ax) f(x) \sin ax = -a \sin ax \int_0^x f(t) \cos at dt + a \cos ax \int_0^x f(t) \sin at dt + f(x). \\ \text{Therefore, } y'' + a^2y &= a \cos ax \int_0^x f(t) \sin at dt - a \sin ax \int_0^x f(t) \cos at dt + f(x) \\ &\quad + a^2 \left( \frac{\sin ax}{a} \int_0^x f(t) \cos at dt - \frac{\cos ax}{a} \int_0^x f(t) \sin at dt \right) = f(x). \text{ Note also that } y'(0) = y(0) = 0. \end{aligned}$$

$$\begin{aligned} 4. \quad x &= \int_0^y \frac{1}{\sqrt{1+4t^2}} dt \Rightarrow \frac{d}{dx}(x) = \frac{d}{dx} \int_0^y \frac{1}{\sqrt{1+4t^2}} dt = \frac{d}{dy} \left[ \int_0^y \frac{1}{\sqrt{1+4t^2}} dt \right] \left( \frac{dy}{dx} \right) \text{ from the chain rule} \\ &\Rightarrow 1 = \frac{1}{\sqrt{1+4y^2}} \left( \frac{dy}{dx} \right) \Rightarrow \frac{dy}{dx} = \sqrt{1+4y^2}. \text{ Then } \frac{d^2y}{dx^2} = \frac{d}{dx}(\sqrt{1+4y^2}) = \frac{d}{dy}(\sqrt{1+4y^2}) \left( \frac{dy}{dx} \right) \end{aligned}$$

$$= \frac{1}{2}(1+4y^2)^{-1/2} (8y) \left( \frac{dy}{dx} \right) = \frac{4y \left( \frac{dy}{dx} \right)}{\sqrt{1+4y^2}} = \frac{4y(\sqrt{1+4y^2})}{\sqrt{1+4y^2}} = 4y. \text{ Thus } \frac{d^2y}{dx^2} = 4y, \text{ and the constant of}$$

proportionality is 4.

$$5. \text{ (a) } \int_0^{x^2} f(t) dt = x \cos \pi x \Rightarrow \frac{d}{dx} \int_0^{x^2} f(t) dt = \cos \pi x - \pi x \sin \pi x \Rightarrow f(x^2)(2x) = \cos \pi x - \pi x \sin \pi x$$

$$\Rightarrow f(x^2) = \frac{\cos \pi x - \pi x \sin \pi x}{2x}. \text{ Thus, } x = 2 \Rightarrow f(4) = \frac{\cos 2\pi - 2\pi \sin 2\pi}{4} = \frac{1}{4}$$

$$\text{(b) } \int_0^{f(x)} t^2 dt = \left[ \frac{t^3}{3} \right]_0^{f(x)} = \frac{1}{3}(f(x))^3 \Rightarrow \frac{1}{3}(f(x))^3 = x \cos \pi x \Rightarrow (f(x))^3 = 3x \cos \pi x \Rightarrow f(x) = \sqrt[3]{3x \cos \pi x}$$

$$\Rightarrow f(4) = \sqrt[3]{3(4) \cos 4\pi} = \sqrt[3]{12}$$

$$6. \int_0^a f(x) dx = \frac{a^2}{2} + \frac{a}{2} \sin a + \frac{\pi}{2} \cos a \text{ and let } F(a) = \int_0^a f(t) dt \Rightarrow f(a) = F'(a). \text{ Now } F(a) = \frac{a^2}{2} + \frac{a}{2} \sin a + \frac{\pi}{2} \cos a$$

$$\Rightarrow f(a) = F'(a) = a + \frac{1}{2} \sin a + \frac{a}{2} \cos a - \frac{\pi}{2} \sin a \Rightarrow f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \frac{1}{2} \sin \frac{\pi}{2} + \frac{\pi}{2} \cos \frac{\pi}{2} - \frac{\pi}{2} \sin \frac{\pi}{2} = \frac{\pi}{2} + \frac{1}{2} - \frac{\pi}{2} = \frac{1}{2}$$

$$7. \int_1^b f(x) dx = \sqrt{b^2+1} - \sqrt{2} \Rightarrow f(b) = \frac{d}{db} \int_1^b f(x) dx = \frac{1}{2}(b^2+1)^{-1/2}(2b) = \frac{b}{\sqrt{b^2+1}} \Rightarrow f(x) = \frac{x}{\sqrt{x^2+1}}$$

$$8. \text{ The derivative of the left side of the equation is: } \frac{d}{dx} \left[ \int_0^x \left[ \int_0^u f(t) dt \right] du \right] = \int_0^x f(t) dt; \text{ the derivative of the right}$$

$$\text{side of the equation is: } \frac{d}{dx} \left[ \int_0^x f(u)(x-u) du \right] = \frac{d}{dx} \int_0^x f(u) x du - \frac{d}{dx} \int_0^x u f(u) du$$

$$= \frac{d}{dx} \left[ x \int_0^x f(u) du \right] - \frac{d}{dx} \int_0^x u f(u) du = \int_0^x f(u) du + x \left[ \frac{d}{dx} \int_0^x f(u) du \right] - x f(x) = \int_0^x f(u) du + x f(x) - x f(x)$$

$$= \int_0^x f(u) du. \text{ Since each side has the same derivative, they differ by a constant, and since both sides equal 0}$$

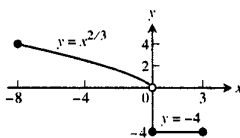
$$\text{when } x = 0, \text{ the constant must be 0. Therefore, } \int_0^x \left[ \int_0^u f(t) dt \right] du = \int_0^x f(u)(x-u) du.$$

$$9. \frac{dy}{dx} = 3x^2 + 2 \Rightarrow y = \int (3x^2 + 2) dx = x^3 + 2x + C. \text{ Then } (1, -1) \text{ on the curve } \Rightarrow 1^3 + 2(1) + C = -1 \Rightarrow C = -4$$

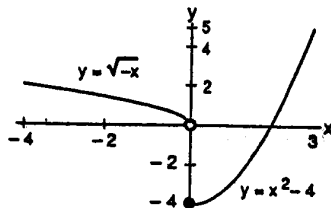
$$\Rightarrow y = x^3 + 2x - 4$$

10. The acceleration due to gravity downward is  $-32 \text{ ft/sec}^2 \Rightarrow v = \int -32 \, dt = -32t + v_0$ , where  $v_0$  is the initial velocity  $\Rightarrow v = -32t + 32 \Rightarrow s = \int (-32t + 32) \, dt = -16t^2 + 32t + C$ . If the release point is  $s = 0$ , then  $C = 0 \Rightarrow s = -16t^2 + 32t$ . Then  $s = 17 \Rightarrow 17 = -16t^2 + 32t \Rightarrow 16t^2 - 32t + 17 = 0$ . The discriminant of this quadratic equation is  $-64$  which says there is no real time when  $s = 17$  ft. You had better duck.

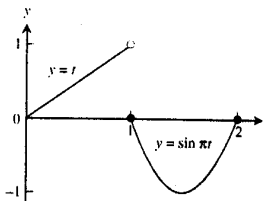
$$\begin{aligned} 11. \int_{-8}^3 f(x) \, dx &= \int_{-8}^0 x^{2/3} \, dx + \int_0^3 -4 \, dx \\ &= \left[ \frac{3}{5} x^{5/3} \right]_{-8}^0 + [-4x]_0^3 \\ &= \left( 0 - \frac{3}{5} (-8)^{5/3} \right) + (-4(3) - 0) = \frac{96}{5} - 12 \\ &= \frac{36}{5} \end{aligned}$$



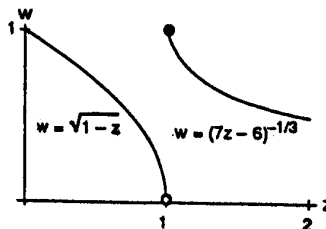
$$\begin{aligned} 12. \int_{-4}^3 f(x) \, dx &= \int_{-4}^0 \sqrt{-x} \, dx + \int_0^3 (x^2 - 4) \, dx \\ &= \left[ -\frac{2}{3} (-x)^{3/2} \right]_{-4}^0 + \left[ \frac{x^3}{3} - 4x \right]_0^3 \\ &= \left[ 0 - \left( -\frac{2}{3} (4)^{3/2} \right) \right] + \left( \frac{3^3}{3} - 4(3) \right) - 0 \\ &= \frac{16}{3} - 3 = \frac{7}{3} \end{aligned}$$



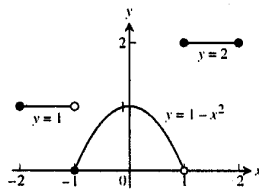
$$\begin{aligned} 13. \int_0^2 f(t) \, dt &= \int_0^1 t \, dt + \int_1^2 \sin \pi t \, dt \\ &= \left[ \frac{t^2}{2} \right]_0^1 + \left[ -\frac{1}{\pi} \cos \pi t \right]_1^2 \\ &= \left( \frac{1}{2} - 0 \right) + \left[ -\frac{1}{\pi} \cos 2\pi - \left( -\frac{1}{\pi} \cos \pi \right) \right] \\ &= \frac{1}{2} - \frac{2}{\pi} \end{aligned}$$



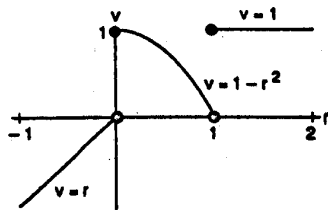
$$\begin{aligned} 14. \int_0^2 f(z) \, dz &= \int_0^1 \sqrt{1-z} \, dz + \int_1^2 (7z-6)^{-1/3} \, dz \\ &= \left[ -\frac{2}{3} (1-z)^{3/2} \right]_0^1 + \left[ \frac{3}{14} (7z-6)^{2/3} \right]_1^2 \\ &= \left[ -\frac{2}{3} (1-1)^{3/2} - \left( -\frac{2}{3} (1-0)^{3/2} \right) \right] \\ &\quad + \left[ \frac{3}{14} (7(2)-6)^{2/3} - \frac{3}{14} (7(1)-6)^{2/3} \right] \\ &= \frac{2}{3} + \left( \frac{6}{7} - \frac{3}{14} \right) = \frac{55}{42} \end{aligned}$$



$$\begin{aligned}
 15. \int_{-2}^2 f(x) dx &= \int_{-2}^{-1} dx + \int_{-1}^1 (1-x^2) dx + \int_1^2 2 dx \\
 &= [x]_{-2}^{-1} + \left[ x - \frac{x^3}{3} \right]_{-1}^1 + [2x]_1^2 \\
 &= -1 - (-2) + \left( 1 - \frac{1^3}{3} \right) - \left( -1 - \frac{(-1)^3}{3} \right) + 2(2) - 2(1) \\
 &= 1 + \frac{2}{3} - \left( -\frac{2}{3} \right) + 4 - 2 = \frac{13}{3}
 \end{aligned}$$



$$\begin{aligned}
 16. \int_{-1}^2 h(r) dr &= \int_{-1}^0 r dr + \int_0^1 (1-r^2) dr + \int_1^2 dr \\
 &= \left[ \frac{r^2}{2} \right]_{-1}^0 + \left[ r - \frac{r^3}{3} \right]_0^1 + [r]_1^2 \\
 &= 0 - \frac{(-1)^2}{2} + \left( 1 - \frac{1^3}{3} \right) - 0 + 2 - 1 \\
 &= -\frac{1}{2} + \frac{2}{3} + 1 = \frac{7}{6}
 \end{aligned}$$



$$\begin{aligned}
 17. \text{Ave. value} &= \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{2-0} \int_0^2 f(x) dx = \frac{1}{2} \left[ \int_0^1 x dx + \int_1^2 (x-1) dx \right] = \frac{1}{2} \left[ \frac{x^2}{2} \right]_0^1 + \frac{1}{2} \left[ \frac{x^2}{2} - x \right]_1^2 \\
 &= \frac{1}{2} \left[ \frac{1^2}{2} - 0 + \left( \frac{2^2}{2} - 2 \right) - \left( \frac{1^2}{2} - 1 \right) \right] = \frac{1}{2}
 \end{aligned}$$

$$18. \text{Ave. value} = \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{3-0} \int_0^3 f(x) dx = \frac{1}{3} \left[ \int_0^1 dx + \int_1^2 0 dx + \int_2^3 dx \right] = \frac{1}{3} [1 - 0 + 0 + 3 - 2] = \frac{2}{3}$$

$$19. f(x) = \int_{1/x}^x \frac{1}{t} dt \Rightarrow f'(x) = \frac{1}{x} \left( \frac{dx}{dx} \right) - \left( \frac{1}{x} \right) \left( \frac{d}{dx} \left( \frac{1}{x} \right) \right) = \frac{1}{x} - x \left( -\frac{1}{x^2} \right) = \frac{1}{x} + \frac{1}{x} = \frac{2}{x}$$

$$\begin{aligned}
 20. f(x) &= \int_{\cos x}^{\sin x} \frac{1}{1-t^2} dt \Rightarrow f'(x) = \left( \frac{1}{1-\sin^2 x} \right) \left( \frac{d}{dx} (\sin x) \right) - \left( \frac{1}{1-\cos^2 x} \right) \left( \frac{d}{dx} (\cos x) \right) = \frac{\cos x}{\cos^2 x} + \frac{\sin x}{\sin^2 x} \\
 &= \frac{1}{\cos x} + \frac{1}{\sin x}
 \end{aligned}$$

$$21. g(y) = \int_{\sqrt{y}}^{2\sqrt{y}} \sin t^2 dt \Rightarrow g'(y) = \left( \sin(2\sqrt{y})^2 \right) \left( \frac{d}{dy} (2\sqrt{y}) \right) - \left( \sin(\sqrt{y})^2 \right) \left( \frac{d}{dy} (\sqrt{y}) \right) = \frac{\sin 4y}{\sqrt{y}} - \frac{\sin y}{2\sqrt{y}}$$

22.  $f(x) = \int_x^{x+3} t(5-t) dt \Rightarrow f'(x) = (x+3)(5-(x+3))\left(\frac{d}{dx}(x+3)\right) - x(5-x)\left(\frac{dx}{dx}\right) = (x+3)(2-x) - x(5-x)$   
 $= 6 - x - x^2 - 5x + x^2 = 6 - 6x$ . Thus  $f'(x) = 0 \Rightarrow 6 - 6x = 0 \Rightarrow x = 1$ . Also,  $f''(x) = -6 < 0 \Rightarrow x = 1$  gives a maximum.

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**NOTES:**

**TECHNOLOGY NOTES:**



