CHAPTER 3 APPLICATIONS OF DERIVATIVES

3.1 EXTREME VALUES OF FUNCTIONS

1. An absolute minimum at $x = c_2$, an absolute maximum at $x = b$. Theorem 1 guarantees the existence of such extreme values because h is continuous on $[a, b]$.

2. An absolute minimum at $x = b$, an absolute maximum at $x = c$. Theorem 1 guarantees the existence of such extreme values because f is continuous on $[a, b]$.

3. No absolute minimum. An absolute maximum at $x = c$. Since the function's domain is an open interval, the function does not satisfy the hypotheses of Theorem 1 and need not have absolute extreme values.

4. No absolute extrema. The function is neither continuous nor defined on a closed interval, so it need not fulfill the conclusions of Theorem 1.

5. An absolute minimum at $x = a$ and an absolute maximum at $x = c$. Note that $y = g(x)$ is not continuous but still has extrema. When the hypothesis of Theorem 1 is satisfied then extrema are guaranteed, but when the hypothesis is not satisfied, absolute extrema may or may not occur.

6. Absolute minimum at $x = c$ and an absolute maximum at $x = a$. Note that $y = g(x)$ is not continuous but still has absolute extrema. When the hypothesis of Theorem 1 is satisfied then extrema are guaranteed, but when the hypothesis is not satisfied, absolute extrema may or may not occur.

7. Local minimum at $(-1, 0)$, local maximum at $(1, 0)$

8. Minima at $(-2, 0)$ and $(2, 0)$, maximum at $(0, 2)$

9. Maximum at $(0, 5)$. Note that there is no minimum since the endpoint $(2, 0)$ is excluded from the graph.

10. Local maximum at $(-3, 0)$, local minimum at $(2, 0)$, maximum at $(1, 2)$, minimum at $(0, -1)$

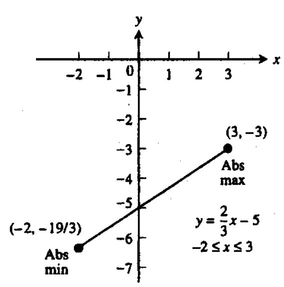
11. Graph (c), since this is the only graph that has positive slope at c.

12. Graph (b), since this is the only graph that represents a differentiable function at a and b and has negative slope at c.

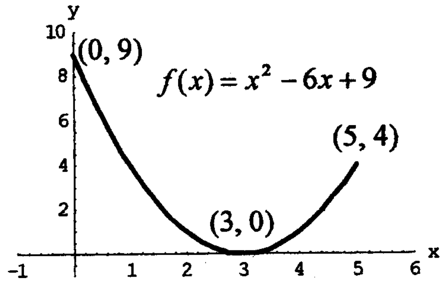
13. Graph (d), since this is the only graph representing a function that is differentiable at b but not at a.

14. Graph (a), since this is the only graph that represents a function that is not differentiable at a or b.

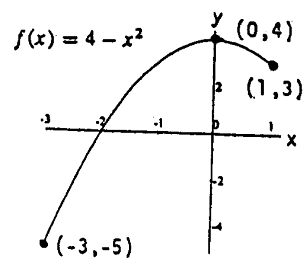
15. $f(x) = \frac{2}{3}x - 5 \Rightarrow f'(x) = \frac{2}{3} \Rightarrow$ no critical points; $f(-2) = -\frac{19}{3}$, $f(3) = -3 \Rightarrow$ the absolute maximum is -3 at $x = 3$ and the absolute minimum is $-\frac{19}{3}$ at $x = -2$



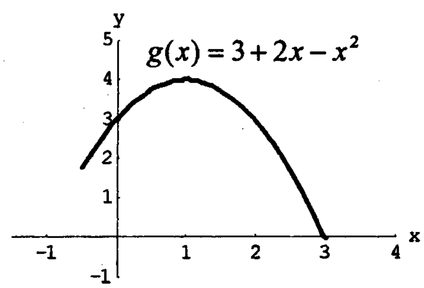
16. $f(x) = x^2 - 6x + 9 \Rightarrow f'(x) = 2x - 6 \Rightarrow$ a critical point at $x = 3$; $f(0) = 9$, $f(3) = 0$, and $f(5) = 4$ $\Rightarrow$ the absolute maximum is 9 at $x = 0$ and the absolute minimum is 0 at $x = 3$.

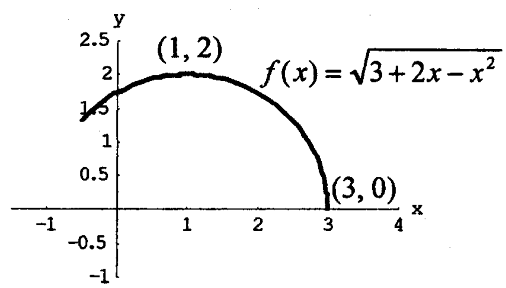


17. $f(x) = 4 - x^2 \Rightarrow f'(x) = -2x \Rightarrow$ a critical point at $x = 0$; $f(-3) = -5$, $f(0) = 4$, $f(1) = 3 \Rightarrow$ the absolute maximum is 4 at $x = 0$ and the absolute minimum is -5 at $x = -3$



18.

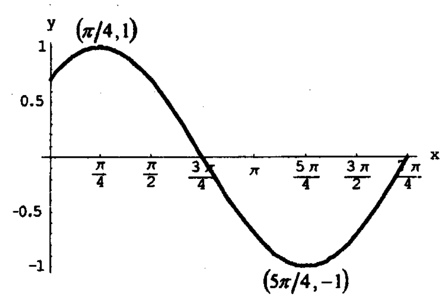




The extreme values of $f(x) = \sqrt{3 + 2x - x^2}$ occur at the extreme values of $g(x) = 3 + 2x - x^2$. Therefore, $g'(x) = 2 - 2x \Rightarrow x = 1$ is a critical value; $f(-0.5) = \sqrt{1.75} \approx 1.32288$, $f(1) = 2$, $f(3) = 0 \Rightarrow$ the absolute maximum is 2 at $x = 1$ and the absolute minimum is 0 at $x = 3$.

19. The first derivative of $f'(x) = \cos\left(x + \frac{\pi}{4}\right)$, has zeros at $x = \frac{\pi}{4}$, $x = \frac{5\pi}{4}$.

Critical point values:	$x = \frac{\pi}{4}$	$f(x) = 1$
	$x = \frac{5\pi}{4}$	$f(x) = -1$
Endpoint values:	$x = 0$	$f(x) = \frac{1}{\sqrt{2}}$
	$x = \frac{7\pi}{4}$	$f(x) = 0$



Maximum value is 1 at $x = \frac{\pi}{4}$;

minimum value is -1 at $x = \frac{5\pi}{4}$;

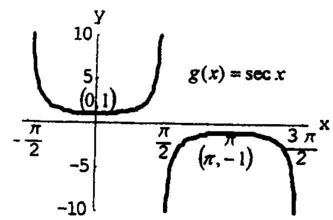
local minimum at $\left(0, \frac{1}{\sqrt{2}}\right)$;

local maximum at $\left(\frac{7\pi}{4}, 0\right)$

20. The first derivative $g'(x) = \sec x \tan x$ has zeros at $x = 0$ and $x = \pi$ and is undefined at $x = \frac{\pi}{2}$. Since $g(x) = \sec x$ is also undefined at $x = \frac{\pi}{2}$, the critical points occur only at $x = 0$ and $x = \pi$.

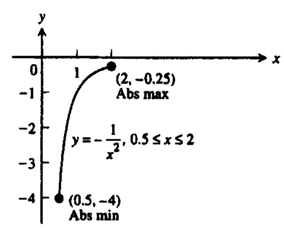
Critical point values: $x = 0$ $\quad g(x) = 1$

$x = \pi$ $\quad g(x) = -1$

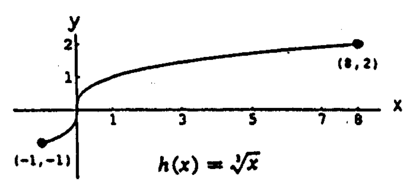


Since the range of $g(x)$ is $(-\infty, -1] \cup [1, \infty)$, these values must be a local minimum and local maximum, respectively. Local minimum at $(0, 1)$; local maximum at $(\pi, -1)$. There are no absolute extrema on the interval $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

21. $F(x) = -\frac{1}{x^2} = -x^{-2} \Rightarrow F'(x) = 2x^{-3} = \frac{2}{x^3}$, however $x = 0$ is not a critical point since 0 is not in the domain; $F(0.5) = -4$, $F(2) = -0.25 \Rightarrow$ the absolute maximum is -0.25 at $x = 2$ and the absolute minimum is -4 at $x = 0.5$



22. $h(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow h'(x) = \frac{1}{3}x^{-2/3} \Rightarrow$ a critical point at $x = 0$; $h(-1) = -1$, $h(0) = 0$, $h(8) = 2 \Rightarrow$ the absolute maximum is 2 at $x = 8$ and the absolute minimum is -1 at $x = -1$



23. The first derivative $f'(x) = -\frac{1}{x^2} + \frac{1}{x}$ has a zero at $x = 1$.

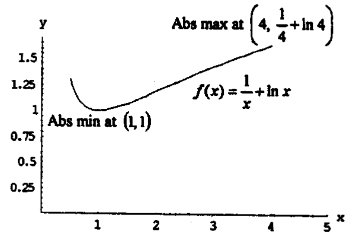
Critical point value: $f(1) = 1 + \ln 1 = 1$

Endpoint values: $f(0.5) = 2 + \ln 0.5 \approx 1.307$;

$$f(4) = \frac{1}{4} + \ln 4 \approx 1.636$$

Absolute maximum value is $\frac{1}{4} + \ln 4$ at $x = 4$;

absolute minimum value is 1 at $x = 1$; local maximum at $\left(\frac{1}{2}, 2 - \ln 2\right)$

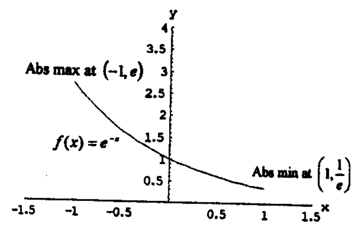


24. The first derivative $g'(x) = -e^{-x}$ has no zeros, so we need only consider the endpoints.

$g(-1) = e^{-(-1)} = e$; $g(1) = e^{-1} = \frac{1}{e}$

Maximum value is e at $x = -1$;

minimum value is $\frac{1}{e}$ at $x = 1$.



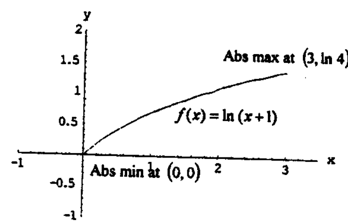
25. The first derivative $h'(x) = \frac{1}{x+1}$ has no zeros,

so we need only consider the endpoints.

$h(0) = \ln 1 = 0$; $h(3) = \ln 4$

Maximum value is $\ln 4$ at $x = 3$;

minimum value is 0 at $x = 0$.

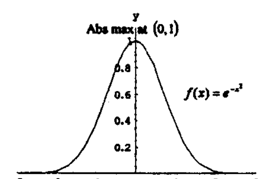


26. The first derivative $k'(x) = -2xe^{-x^2}$ has a zero

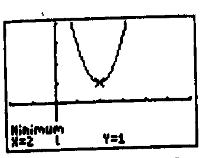
at $x = 0$. Since the domain has no endpoints, any extreme

value must occur at $x = 0$. Since $k(0) = e^{-0^2} = 1$ and

$\lim_{x \to \pm\infty} k(x) = 0$, the maximum value is 1 at $x = 0$.

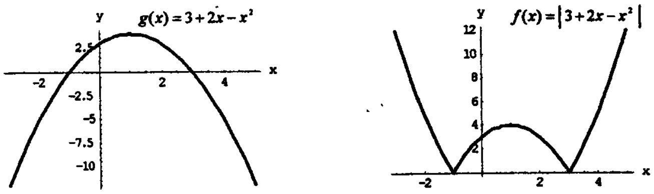


27.

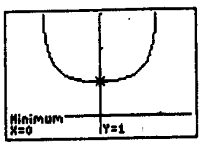


[−2, 6] by [−2, 4]

$y' = 4x - 8 = 0 \Rightarrow$ critical value at $x = 2$ and $y'' = 4 \Rightarrow$ minimum value is 1 at $x = 2$.

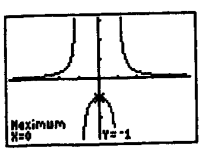
28. 

The minimum values of f(x) occur wherever $g(x) = 3 + 2x - x^2 = 0 \Rightarrow x = -1$ and $x = 3$. There is a relative maximum at the point where g(x) has a relative maximum $\Rightarrow g'(x) = 2 - 2x \Rightarrow$ there is a critical value at $x = 1$. There is no absolute maximum value of f(x), the absolute minimum value is 0 at $x = -1$ and $x = 3$. There is a relative maximum of 4 at $x = 1$. Note that $f'(x)$ is undefined at $x = -1$ and $x = 3$, and so these are critical points of f.

29. 

[−1.5, 1.5] by [−0.5, 3]

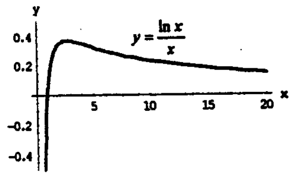
$y' = \dfrac{x}{(1-x^2)^{3/2}} \Rightarrow$ critical value at $x = 0$; $y'' = \dfrac{2x^2+1}{(1-x^2)^{5/2}}$; at $x = 0$, $y'' = 1 \Rightarrow$ minimum value is 1 at $x = 0$.

30. 

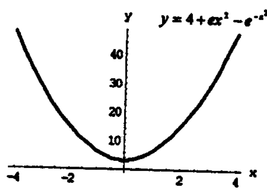
[−4.7, 4.7] by [−3.1, 3.1]

To confirm that there are no "hidden" extrema, note that $y' = -(x^2-1)^{-2}(2x) = \dfrac{-2x}{(x^2-1)^2}$ which is zero only at $x = 0$ and is undefined only where y is undefined. There is a local maximum at $(0, -1)$.

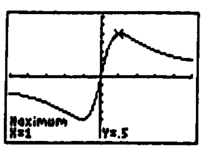
31. $y = \frac{\ln x}{x} \Rightarrow \dfrac{dy}{dx} = \dfrac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2} \Rightarrow$ there is a critical point where $\ln x = 1 \Rightarrow x = e$. The graph of the function shows a relative and absolute maximum near $x = e$ and the "maximum" function on the TI-89 calculator gives a maximum of $y = 0.36789 = \frac{1}{e}$ at $x = 2.71828 \approx e$. There is no absolute minimum.



32. $y = 4 + ex^2 - e^{-x^2} \Rightarrow \frac{dy}{dx} = 2ex + 2xe^{-x^2} = 2x\left(e + e^{-x^2}\right) \Rightarrow$ the only critical point is at $x = 0$, since $e + e^{-x^2} > 0$ for all real x. The graph of the function shows an absolute minimum value at $x = 0$, and the "minimum" function on the TI-89 calculator gives a minimum of $y = 3$ at $x = 0$, as expected. There is no absolute maximum.



33.

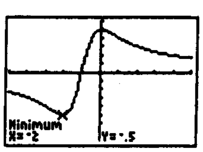


[−5, 5] by [−0.7, 0.7]

$y' = \frac{1-x^2}{\left(x^2+1\right)^2} \Rightarrow$ critical values at $x = \pm 1$; $y'' = \frac{2x\left(x^2-3\right)}{\left(x^2+1\right)^3}$; at $x = -1$, $y'' = \frac{1}{2}$ and at $x = 1$, $y'' = -\frac{1}{2}$

$\Rightarrow$ maximum value is $\frac{1}{2}$ at $x = 1$; minimum value is $-\frac{1}{2}$ at $x = -1$.

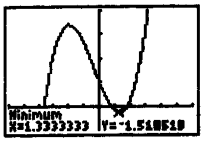
34.



[−5, 5] by [−0.8, 0.6]

$y' = \frac{-x(x+2)}{(x^2+2x+2)^2} \Rightarrow$ critical values at $x = 0$ and $x = -2$; $y'' = \frac{2(x+1)(x^2+2x-2)}{(x^2+2x+2)^3}$; at $x = 0$, $y'' = -\frac{1}{2}$ and at $x = -2$, $y'' = \frac{1}{2} \Rightarrow$ maximum value is $\frac{1}{2}$ at $x = 0$; minimum value is $-\frac{1}{2}$ at $x = -2$.

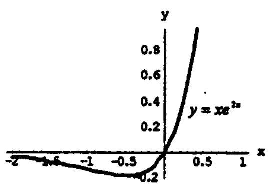
35.



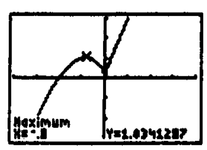
[−6, 6] by [−5, 20]

To find the exact values, note that $y' = 3x^2 + 2x - 8 = (3x-4)(x+2)$, which is zero when $x = -2$ or $x = \frac{4}{3}$. Local maximum at $(-2, 17)$; local minimum at $\left(\frac{4}{3}, -\frac{41}{27}\right)$

36. $y = xe^{2x} \Rightarrow \frac{dy}{dx} = (1+2x)e^{2x} \Rightarrow$ the only critical point is at $x = -\frac{1}{2}$, since $e^{2x} > 0$ for all real x. The graph of the function shows an absolute minimum value near $x = -\frac{1}{2}$, and the "minimum" function on the TI-89 calculator gives a minimum of $y = -0.18394 \approx \frac{e^{-1}}{2}$ at $x = -0.5$, as expected. There is no absolute maximum.



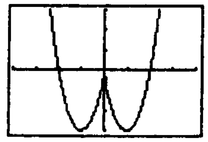
37. $y' = x^{2/3}(1) + \frac{2}{3}x^{-1/3}(x+2) = \frac{5x+4}{3\sqrt[3]{x}}$



[−4, 4] by [−3, 3]

crit. pt.	derivative	extremum	value
$x = -\frac{4}{5}$	0	local max	$\frac{12}{25}10^{1/3} = 1.034$
$x = 0$	undefined	local min	0

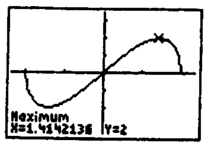
38. $y' = x^{2/3}(2x) + \frac{2}{3}x^{-1/3}(x^2 - 4) = \frac{8x^2 - 8}{3\sqrt[3]{x}}$


[−4, 4] by [−3, 3]

crit. pt.	derivative	extremum	value
$x = -1$	0	minimum	−3
$x = 0$	undefined	local max	0
$x = 1$	0	minimum	−3

39. $y' = x \cdot \frac{1}{2\sqrt{4 - x^2}}(-2x) + (1)\sqrt{4 - x^2}$

$= \frac{-x^2 + (4 - x^2)}{\sqrt{4 - x^2}} = \frac{4 - 2x^2}{\sqrt{4 - x^2}}$

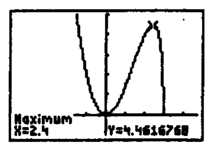


[−2.35, 2.35] by [−3.5, 3.5]

crit. pt.	derivative	extremum	value
$x = -2$	undefined	local max	0
$x = -\sqrt{2}$	0	minimum	−2
$x = \sqrt{2}$	0	maximum	2
$x = 2$	undefined	local min	0

40. $y' = x^2 \cdot \frac{1}{2\sqrt{3 - x}}(-1) + 2x\sqrt{3 - x}$

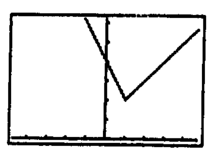
$= \frac{-x^2 + (4x)(3 - x)}{2\sqrt{3 - x}} = \frac{-5x^2 + 12x}{2\sqrt{3 - x}}$



[−4.7, 4.7] by [−1, 5]

crit. pt.	derivative	extremum	value
$x = 0$	0	minimum	0
$x = \frac{12}{5}$	0	local max	$\frac{144}{125}15^{1/2} \approx 4.462$
$x = 3$	undefined	minimum	0

41.
$$y' = \begin{cases} -2, & x < 1 \\ 1, & x > 1 \end{cases}$$

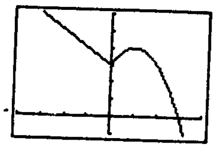

[−4.7, 4.7] by [0, 6.2]

crit. pt.	derivative	extremum	value
$x = 1$	undefined	minimum	2

42.

$$y' = \begin{cases} -1, & x < 0 \\ 2 - 2x, & x > 0 \end{cases}$$

crit. pt.	derivative	extremum	value
$x = 0$	undefined	local min	3
$x = 1$	0	local max	4

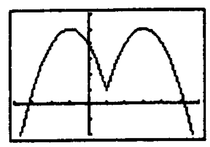


[−4, 4] by [−1, 6]

43.

$$y' = \begin{cases} -2x - 2, & x < 1 \\ -2x + 6, & x > 1 \end{cases}$$

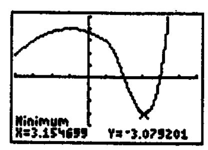
crit. pt.	derivative	extremum	value
$x = -1$	0	maximum	5
$x = 1$	undefined	local min	1
$x = 3$	0	maximum	5



[−4, 6] by [−2, 6]

44. We begin by determining whether f′(x) is defined at x = 1, where

$$f(x) = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$$



[−4, 6] by [−5, 5]

Left-hand derivative:

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0^-} \frac{-\frac{1}{4}(1+h)^2 - \frac{1}{2}(1+h) + \frac{15}{4} - 3}{h} = \lim_{h\to 0^-} \frac{-h^2 - 4h}{4h} = \lim_{h\to 0^-} \frac{1}{4}(-h-4) = -1$$

Right-hand derivative

$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{(1+h)^3 - 6(1+h)^2 + 8(1+h) - 3}{h} = \lim_{h\to 0^+} \frac{h^3 - 3h^2 - h}{4h} = \lim_{h\to 0^+} (h^2 - 3h - 1)$$

$$= -1$$

$$\text{Thus } f'(x) = \begin{cases} -\frac{1}{2}x - \frac{1}{2}, & x \le 1 \\ 3x^2 - 12x + 8, & x > 1 \end{cases}$$

Note that $-\frac{1}{2}x - \frac{1}{2} = 0$ when $x = -1$, and $3x^2 - 12x + 8 = 0$ when $x = \dfrac{12 \pm \sqrt{12^2 - 4(3)(8)}}{2(3)}$

$= \frac{12 \pm \sqrt{48}}{6} = 2 \pm \frac{2\sqrt{3}}{3}$. But $2 - \frac{2\sqrt{3}}{3} \approx 0.845 < 1$, so the only critical points occur at $x = -1$ and $x = 2 + \frac{2\sqrt{3}}{3} \approx 3.155$.

crit. pt.	derivative	extremum	value
$x = -1$	0	local max	4
$x \approx 3.155$	0	local max	≈ -3.079

45. (a) No, since $f'(x) = \frac{2}{3}(x-2)^{-1/3}$, which is undefined at $x = 2$.

(b) The derivative is defined and nonzero for all $x \neq 2$. Also, $f(2) = 0$ and $f(x) > 0$ for all $x \neq 2$.

(c) No, f(x) need not have a global maximum because its domain is all real numbers. Any restriction of f to a closed interval of the form [a, b] would have both a maximum value and a minimum value on the interval.

(d) The answers are the same as (a) and (b) with 2 replaced by a.

46. Note that $f(x) = \begin{cases} -x^3 + 9x, & x \leq -3 \text{ or } 0 \leq x < 3 \\ x^3 - 9x, & -3 < x < 0 \text{ or } x \geq 3 \end{cases}$

Therefore, $f'(x) = \begin{cases} -3x^2 + 9, & x < -3 \text{ or } 0 < x < 3 \\ 3x^2 - 9, & -3 < x < 0 \text{ or } x > 3 \end{cases}$

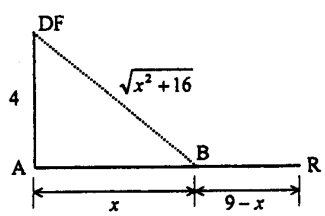
(a) No, since the left- and right-hand derivatives at $x = 0$ are -9 and 9, respectively.

(b) No, since the left- and right-hand derivatives at $x = 3$ are -18 and 18, respectively.

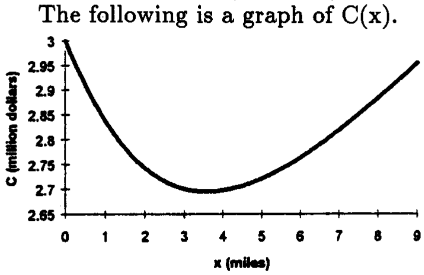
(c) No, since the left- and right-hand derivatives at $x = -3$ are -18 and 18, respectively.

(d) The critical points occur when $f'(x) = 0$ (at $x = \pm\sqrt{3}$) and when $f'(x)$ is undefined (at $x = 0$ and $x = \pm 3$). The minimum value is 0 at $x = -3$, at $x = 0$, and at $x = 3$; local maxima occur at $(-\sqrt{3}, 6\sqrt{3})$ and $(\sqrt{3}, 6\sqrt{3})$.

47.



(a) The construction cost is $C(x) = 0.3\sqrt{16 + x^2} + 0.2(9 - x)$ million dollars, where $0 \leq x \leq 9$ miles. The following is a graph of C(x).

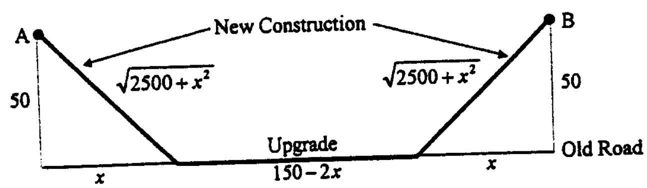


Solving $C'(x) = \frac{0.3x}{\sqrt{16+x^2}} - 0.2 = 0$ gives $x = \pm\frac{8\sqrt{5}}{5} \approx \pm 3.58$ miles, but only x = 3.58 miles is a critical point in the specified domain. Evaluating the costs at the critical and endpoints gives C(0) = \$3 million, $C(8\sqrt{5}/5) \approx$ \$2.694 million, and $C(9) \approx$ \$2.955 million. Therefore, to minimize the cost of construction, the pipeline should be placed from the docking facility to point B, 3.58 miles along the shore from point A, and then along the shore from B to the refinery.

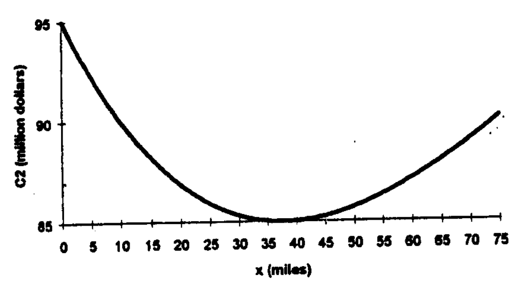
(b) If the per mile cost of underwater construction is p, then $C(x) = p\sqrt{16+x^2} + 0.2(9-x)$ and $C'(x) = \frac{px}{\sqrt{16+x^2}} - 0.2 = 0$ gives $x_c = \frac{0.8}{\sqrt{p^2 - 0.04}}$, which minimizes the construction cost provided $x_c \le 9$. The value of p that gives $x_c = 9$ miles is 0.218864. Consequently, if the underwater construction costs \$218,864 per mile or less, then running the pipeline along a straight line directly from the docking facility to the refinery will minimize the cost of construction.

In theory, p would have to be infinite to justify running the pipe directly from the docking facility to point A (i.e., for x_c to be zero). For all values of $p > 0.218864$ there is always an $x_c \in (0, 9)$ that will give a minimum value for C. This is proved by looking at $C''(x_c) = \frac{16p}{\left(16 + x_c^2\right)^{3/2}}$ which is always positive for $p > 0$.

48. There are two options to consider. The first is to build a new road straight from Village A to Village B. The second is to build a new highway segment from Village A to the Old Road, reconstruct a segment of Old Road, and build a new highway segment from Old Road to Village B, as shown in the figure. The cost of the first option is $C_1 = 0.5(150) = \$75$ million.

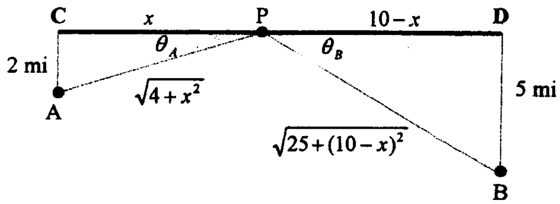


The construction cost for the second option is $C_2(x) = 0.5(2\sqrt{2500+x^2}) + 0.3(150-2x)$ million dollars for $0 \le x \le 75$ miles. The following is a graph of $C_2(x)$.

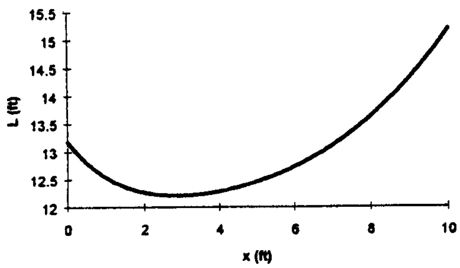


Solving $C_2'(x) = \frac{x}{\sqrt{2500+x^2}} - 0.6 = 0$ gives $x = \pm 37.5$ miles, but only x = 37.5 miles is in the specified domain. In summary, $C_1 = \$75$ million, $C_2(0) = \$95$ million, $C_2(37.5) = \$85$ million, and $C_2(75) = \$90.139$ million. Consequently, a new road straight from Village A to Village B is the least expensive option.

49.



The length of pipeline is $L(x) = \sqrt{4+x^2} + \sqrt{25+(10-x)^2}$ for $0 \le x \le 10$. The following is a graph of L(x).

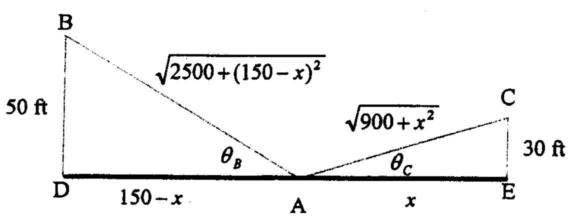


Setting the derivative of L(x) equal to zero gives $L'(x) = \frac{x}{\sqrt{4+x^2}} - \frac{(10-x)}{\sqrt{25+(10-x)^2}} = 0$. Note that $\frac{x}{\sqrt{4+x^2}} = \cos\theta_A$ and $\frac{10-x}{\sqrt{25-(10-x)^2}} = \cos\theta_B$, therefore, $L'(x) = 0$ when $\cos\theta_A = \cos\theta_B$, or $\theta_A = \theta_B$ and ΔACP is similar to ΔBDP. Use simple proportions to determine x as follows:

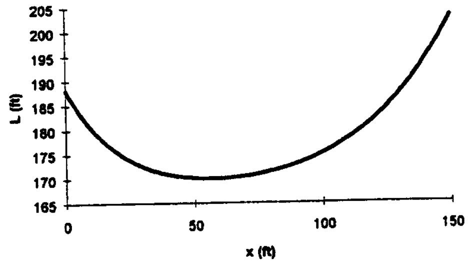
$\frac{x}{2} = \frac{10-x}{5} \Rightarrow x = \frac{20}{7} \approx 2.857$ miles along the coast from town A to town B.

If the two towns were on opposite sides of the river, the obvious solution would be to place the pump station on a straight line (the shortest distance) between the two towns, again forcing $\theta_A = \theta_B$. The shortest length of pipe is the same regardless of whether the towns are on the same or opposite sides of the river.

50.



(a) The length of guy wire is $L(x) = \sqrt{900+x^2} + \sqrt{2500+(150-x)^2}$ for $0 \le x \le 150$. The following is a graph of L(x):



Setting L′(x) equal to zero gives $L'(x) = \frac{x}{\sqrt{900 + x^2}} - \frac{(150 - x)}{\sqrt{2500 + (150 - x)^2}} = 0$. Note that $\frac{x}{\sqrt{900 + x^2}} = \cos\theta_C$ and $\frac{150 - x}{\sqrt{2500 - (150 - x)^2}} = \cos\theta_B$. Therefore, $L'(x) = 0$ when $\cos\theta_C = \cos\theta_B$, or $\theta_C = \theta_B$ and ΔACE is similar to ΔABD. Use simple proportions to determine x: $\frac{x}{30} = \frac{150 - x}{50}$

$\Rightarrow x = \frac{225}{4} = 56.25$ feet.

(b) If the heights of the towers are h_B and h_C, and the horizontal distance between them is s, then $L(x) = \sqrt{h_C^2 + x^2} + \sqrt{h_B^2 + (s - x)^2}$ and $L'(x) = \frac{x}{\sqrt{h_C^2 + x^2}} - \frac{(s - x)}{\sqrt{h_B^2 + (s - x)^2}}$. However, $\frac{x}{\sqrt{h_C^2 + x^2}} = \cos\theta_C$ and $\frac{(s - x)}{\sqrt{h_B^2 + (s - x)^2}} = \cos\theta_B$. Therefore, $L'(x) = 0$ when $\cos\theta_C = \cos\theta_B$, or $\theta_C = \theta_B$, and ΔACE is similar to ΔABD. Simple proportions can again be used to determine the optimum x: $\frac{x}{h_C} = \frac{s - x}{h_B}$

$\Rightarrow x = \left(\frac{h_C}{h_B + h_C}\right)s.$

51. (a) $V(x) = 160x - 52x^2 + 4x^3$

$V'(x) = 160 - 104x + 12x^2 = 4(x - 2)(3x - 20)$

The only critical point in the interval $(0, 5)$ is at $x = 2$. The maximum value of $V(x)$ is 144 at $x = 2$.

(b) The largest possible volume of the box is 144 cubic units, and it occurs when $x = 2$.

52. (a) $P'(x) = 2 - 200x^{-2}$

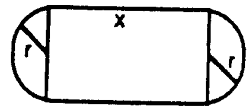
The only critical point in the interval $(0, \infty)$ is at $x = 10$. The minimum value of $P(x)$ is 40 at $x = 10$.

(b) The smallest possible perimeter of the rectangle is 40 units and it occurs at $x = 10$, which makes the rectangle a 10 by 10 square.

53. Let x represent the length of the base and $\sqrt{25 - x^2}$ the height of the triangle. The area of the triangle is represented by $A(x) = \frac{x}{2}\sqrt{25 - x^2}$ where $0 \le x \le 5$. Consequently, solving $A'(x) = 0 \Rightarrow \frac{25 - 2x^2}{2\sqrt{25 - x^2}} = 0$

$\Rightarrow x = \frac{5}{\sqrt{2}}$. Since $A(0) = A(5) = 0$, $A(x)$ is maximized at $x = \frac{5}{\sqrt{2}}$. The largest possible area is $A\left(\frac{5}{\sqrt{2}}\right) = \frac{25}{4}\ \text{cm}^2$.

54. (a) From the diagram the perimeter $P = 2x + 2\pi r = 400$ $\Rightarrow x = 200 - \pi r$. We wish to maximize the area $A = 2rx$ $\Rightarrow A(r) = 400r - 2\pi r^2$



(b) $A'(r) = 400 - 4\pi r$ and $A''(r) = -4\pi$. The critical point is $r = \frac{100}{\pi}$ and $A''\left(\frac{100}{\pi}\right) = -4\pi < 0$. There is a maximum at $r = \frac{100}{\pi}$. The values $x = 100$ m and $r = \frac{100}{\pi} \approx 31.83$ m maximize the area of the rectangle.

55. $s = -\frac{1}{2}gt^2 + v_0 t + s_0 \Rightarrow \frac{ds}{dt} = -gt + v_0 = 0 \Rightarrow t = \frac{v_0}{g}$. Then $s\left(\frac{v_0}{g}\right) = -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\left(\frac{v_0}{g}\right) + s_0$ $= \frac{v_0^2}{2g} + s_0$ is the maximum height since $\frac{d^2s}{dt^2} = -g < 0$.

56. $\frac{di}{dt} = -2 \sin t + 2 \cos t$, solving $\frac{di}{dt} = 0 \Rightarrow \tan t = 1 \Rightarrow t = \frac{\pi}{4} + n\pi$ where n is a nonnegative integer (in this exercise t is never negative) $\Rightarrow$ the peak current is $2\sqrt{2}$ amps

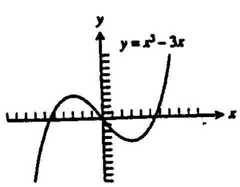
57. Yes, since $f(x) = |x| = \sqrt{x^2} = \left(x^2\right)^{1/2} \Rightarrow f'(x) = \frac{1}{2}\left(x^2\right)^{-1/2}(2x) = \frac{x}{\left(x^2\right)^{1/2}} = \frac{x}{|x|}$ is not defined at $x = 0$. Thus it is not required that f' be zero at a local extreme point since f' may be undefined there.

58. If f(c) is a local maximum value of f, then $f(x) \le f(c)$ for all x in some open interval (a, b) containing c. Since f is even, $f(-x) = f(x) \le f(c) = f(-c)$ for all $-x$ in the open interval $(-b, -a)$ containing $-c$. That is, f assumes a local maximum at the point $-c$. This is also clear from the graph of f because the graph of an even function is symmetric about the y-axis.

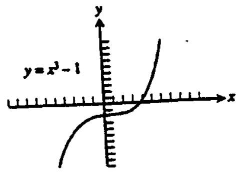
59. If g(c) is a local minimum value of g, then $g(x) \ge g(c)$ for all x in some open interval (a, b) containing c. Since g is odd, $g(-x) = -g(x) \le -g(c) = g(-c)$ for all $-x$ in the open interval $(-b, -a)$ containing $-c$. That is, g assumes a local maximum at the point $-c$. This is also clear from the graph of g because the graph of an odd function is symmetric about the origin.

60. If there are no boundary points or critical points the function will have no extreme values in its domain. Such functions do indeed exist, for example $f(x) = x$ for $-\infty < x < \infty$. (Any other linear function $f(x) = mx + b$ with $m \ne 0$ will do as well.)

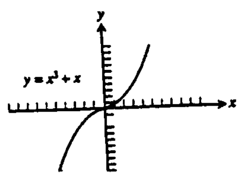
61. (a) $f'(x) = 3ax^2 + 2bx + c$ is a quadratic, so it can have 0, 1, or 2 zeros, which would be the critical points of f. Examples:



The function $f(x) = x^3 - 3x$ has two critical points at $x = -1$ and $x = 1$.



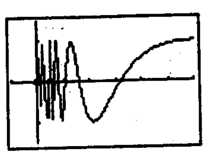
The function $f(x) = x^3 - 1$ has one critical point at $x = 0$.



The function $f(x) = x^3 + x$ has no critical points.

(b) The function can have either two local extreme values or no extreme values. (If there is only one critical point, the cubic function has no extreme values.)

62. (a)



[−0.1, 0.6] by [−1.5, 1.5]

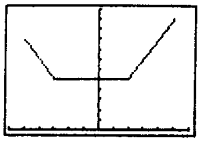
$f(0) = 0$ is not a local extreme value because in any open interval containing $x = 0$, there are infinitely many points where $f(x) = 1$ and where $f(x) = -1$.

(b) One possible answer, on the interval $[0, 1]$:

$$f(x) = \begin{cases} (1-x)\cos\frac{1}{1-x}, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

This function has no local extreme value at $x = 1$. Note that it is continuous on $[0, 1]$.

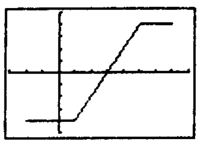
63.



[−6, 6] by [0, 12]

Maximum value is 11 at x = 5; minimum value is 5 on the interval [−3, 2]; local maximum is at (−5, 9)

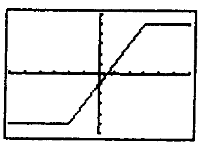
64.



[−3, 8] by [−5, 5]

Maximum value is 4 on the interval [5, 7]; minimum value is −4 on the interval [−2, 1].

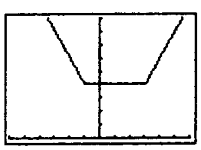
65.



[−6, 6] by [−6, 6]

Maximum value is 5 on the interval $[3, \infty)$; minimum value is −5 on the interval $(-\infty, -2]$.

66.



[−6, 6] by [0, 9]

Minimum value is 4 on the interval [−1, 3]

67-74. Example CAS commands:

Maple:

```
f:=x -> 2 + 2*x - 3*(x ∧ 2) ∧ (1/3);
plot(f(x), x=-1..10/3);
fp:=diff(f(x),x);
solve(fp=0,x);
simplify(fp);
```

```
den:=denom(%);
solve(denom(fp)=0,x);
evalf([f(−1),f(0),f(1),f(10/3)]);
```

Mathematica:

```
Note: Here, use (x ∧ 2) ∧ (1/3) instead of x ∧ (2/3), to avoid complex roots for negative x
a = −1; b = 10/3; f[x_] = 2 + 2 x − 3 (x ∧ 2) ∧ (1/3)
f'[x]
Plot[ {f[x], f'[x]}, {x,a,b} ]
NSolve[f'[x]==0]
Note: include critical point x=0
{f[a], f[0], f[x] /. %, f[b]} // N
```

3.2 THE MEAN VALUE THEOREM AND DIFFERENTIAL EQUATIONS

1. (a) f is continuous on $[0,1]$ and differentiable on $(0,1)$.

   (b) $f'(c) = \frac{f(1)-f(0)}{1-0} \Rightarrow 2c + 2 = \frac{2-(-1)}{1} \Rightarrow 2c = 1 \Rightarrow c = \frac{1}{2}$

2. (a) f is continuous on $[0,1]$ and differentiable on $(0,1)$.

   (b) $f'(c) = \frac{f(1)-f(0)}{1-0} \Rightarrow \frac{2}{3}c^{-1/3} = \frac{1-0}{1} \Rightarrow c^{-1/3} = \frac{3}{2} \Rightarrow c = \left(\frac{3}{2}\right)^{-3} \Rightarrow c = \frac{8}{27}$

3. (a) f is continuous on $[-1,1]$ and differentiable on $(-1,1)$.

   (b) $f'(c) = \frac{f(1)-f(-1)}{1-(-1)} \Rightarrow \frac{1}{\sqrt{1-c^2}} = \frac{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)}{2} \Rightarrow \sqrt{1-c^2} = \frac{2}{\pi} \Rightarrow 1-c^2 = \frac{4}{\pi^2} \Rightarrow c^2 = 1-\frac{4}{\pi^2}$

   $\Rightarrow c = \pm\sqrt{1-\frac{4}{\pi^2}} \approx \pm 0.771$

4. (a) f is continuous on $[2,4]$ and differentiable on $(2,4)$.

   (b) $f'(c) = \frac{f(4)-f(2)}{4-2} \Rightarrow \frac{1}{c-1} = \frac{\ln 3 - \ln 1}{2} \Rightarrow c-1 = \frac{2}{\ln 3} \Rightarrow c = 1 + \frac{2}{\ln 3} \approx 2.820$

5. Since f(x) is not continuous on $0 \le x \le 1$, Rolle's Theorem does not apply because $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} x = 1$ $\ne 0 = f(1)$ and f(x) is not continuous at $x = 1$.

6. Since f(x) must be continuous at $x = 0$ and $x = 1$ we have $\lim_{x\to 0^+} f(x) = a = f(0) \Rightarrow a = 3$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) \Rightarrow -1 + 3 + a = m + b \Rightarrow 5 = m + b$. Since f(x) must also be differentiable at $x = 1$ we have $\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^+} f'(x) \Rightarrow -2x + 3\big|_{x=1} = m\big|_{x=1} \Rightarrow 1 = m$. Therefore, $a = 3$, $m = 1$ and $b = 4$.

7. By Corollary 1, $f'(x) = 0$ for all $x \Rightarrow f(x) = C$, where C is a constant. Since $f(-1) = 3$ we have $C = 3$ $\Rightarrow f(x) = 3$ for all x.

8. $g(t) = 2t + 5 \Rightarrow g'(t) = 2 = f'(t)$ for all t. By Corollary 2, $f(t) = g(t) + C$ for some constant C. Then $f(0) = g(0) + C \Rightarrow 5 = 5 + C \Rightarrow C = 0 \Rightarrow f(t) = g(t) = 2t + 5$ for all t.

9. (a) $y = \frac{x^2}{2} + C$ (b) $y = \frac{x^3}{3} + C$ (c) $y = \frac{x^4}{4} + C$

10. (a) $y = x^2 + C$ (b) $y = x^2 - x + C$ (c) $y = x^3 + x^2 - x + C$

11. (a) $y = \ln\theta + C$ if $\theta > 0$ and $y = \ln(-\theta) + C$ if $\theta < 0$, where C is a constant. (These functions can be combined as $y = \ln|\theta| + C$.)
    (b) $y = \theta - \ln\theta + C$ if $\theta > 0$ and $y = \theta - \ln(-\theta) + C$ if $\theta < 0$, where C is a constant. (These functions can be combined as $y = \theta - \ln|\theta| + C$.)
    (c) $y = 5\theta + \ln\theta + C$ if $\theta > 0$ and $y = 5\theta + \ln(-\theta) + C$ if $\theta < 0$, where C is a constant. (These functions can be combined as $y = 5\theta + \ln|\theta| + C$.)

12. (a) $y' = \frac{1}{2}t^{-1/2} \Rightarrow y = t^{1/2} + C \Rightarrow y = \sqrt{t} + C$ (b) $y = 2\sqrt{t} + C$
    (c) $y = 2t^2 - 2\sqrt{t} + C$

13. $f(x) = x^2 - x + C$; $0 = f(0) = 0^2 - 0 + C \Rightarrow C = 0 \Rightarrow f(x) = x^2 - x$

14. $g(x) = \left\{\begin{array}{ll} x^2 + \ln x + C & \text{if } x > 0 \\ x^2 + \ln(-x) + C & \text{if } x < 0 \end{array}\right\} = x^2 + \ln|x| + C$; $g(1) = -1 \Rightarrow 1^2 + \ln 1 + C = -1 \Rightarrow C = -2$
    $\Rightarrow g(x) = x^2 + \ln|x| - 2$

15. $f(x) = \frac{e^{2x}}{2} + C$; $f(0) = \frac{3}{2} \Rightarrow \frac{e^{2(0)}}{2} + C = \frac{3}{2} \Rightarrow C = 1 \Rightarrow f(x) = 1 + \frac{e^{2x}}{2}$

16. $r(t) = \sec t - t + C$; $0 = r(0) = \sec(0) - 0 + C \Rightarrow C = -1 \Rightarrow r(t) = \sec t - t - 1$

17. $v = \frac{ds}{dt} = 9.8t + 5 \Rightarrow s = 4.9t^2 + 5t + C$; at $s = 10$ and $t = 0$ we have $C = 10 \Rightarrow s = 4.9t^2 + 5t + 10$

18. $v = \frac{ds}{dt} = 32t - 2 \Rightarrow s = 16t^2 - 2t + C$; at $s = 4$ and $t = \frac{1}{2}$ we have $C = 1 \Rightarrow s = 16t^2 - 2t + 1$

19. $v = \frac{ds}{dt} = \sin(\pi t) \Rightarrow s = -\frac{1}{\pi}\cos(\pi t) + C$; at $s = 0$ and $t = 0$ we have $C = \frac{1}{\pi} \Rightarrow s = \frac{1 - \cos(\pi t)}{\pi}$

20. $v = \frac{ds}{dt} = \frac{1}{t+2} \Rightarrow s = \ln(t+2) + C$; at $s = \frac{1}{2}$ and $t = -1$ we have $C = \frac{1}{2} \Rightarrow s = \frac{1}{2} + \ln(t+2)$

21. $a = \frac{dv}{dt} = e^t \Rightarrow v = e^t + C$; at $v = 20$ and $t = 0$ we have $C = 19 \Rightarrow v = e^t + 19$
    $v = \frac{ds}{dt} = e^t + 19 \Rightarrow s = e^t + 19t + C$; at $s = 5$ and $t = 0$ we have $C = 4 \Rightarrow s = e^t + 19t + 4$

22. $a = 9.8 \Rightarrow v = 9.8t + C_1$; at $v = -3$ and $t = 0$ we have $C_1 = -3 \Rightarrow v = 9.8t - 3 \Rightarrow s = 4.9t^2 - 3t + C_2$; at $s = 0$ and $t = 0$ we have $C_2 = 0 \Rightarrow s = 4.9t^2 - 3t$

23. $a = -4\sin(2t) \Rightarrow v = 2\cos(2t) + C_1$; at $v = 2$ and $t = 0$ we have $C_1 = 0 \Rightarrow v = 2\cos(2t)$
    $\Rightarrow s = \sin(2t) + C_2$; at $s = -3$ and $t = 0$ we have $C_2 = -3 \Rightarrow s = \sin(2t) - 3$

24. $a = \frac{9}{\pi^2}\cos\left(\frac{3t}{\pi}\right) \Rightarrow v = \frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right) + C_1$; at $v = 0$ and $t = 0$ we have $C_1 = 0 \Rightarrow v = \frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right)$
$\Rightarrow s = -\cos\left(\frac{3t}{\pi}\right) + C_2$; at $s = -1$ and $t = 0$ we have $C_2 = 0 \Rightarrow s = -\cos\left(\frac{3t}{\pi}\right)$

25. $a(t) = v'(t) = 1.6 \Rightarrow v(t) = 1.6t + C$; at $(0,0)$ we have $C = 0 \Rightarrow v(t) = 1.6t$. When $t = 30$, then $v(30) = 48$ m/sec.

26. $a(t) = v'(t) = 20 \Rightarrow v(t) = 20t + C$; at $(0,0)$ we have $C = 0 \Rightarrow v(t) = 20t$. When $t = 60$, then $v(60) = 20(60)$ $= 1200$ m/sec.

27. $a(t) = v'(t) = 9.8 \Rightarrow v(t) = 9.8t + C_1$; at $(0,0)$ we have $C_1 = 0 \Rightarrow s'(t) = v(t) = 9.8t \Rightarrow s(t) = 4.9t^2 + C_2$; at $(0,0)$ we have $C_2 = 0 \Rightarrow s(t) = 4.9t^2$. Then $s(t) = 10 \Rightarrow t^2 = \frac{10}{4.9} \Rightarrow t = \sqrt{\frac{10}{4.9}}$, and $v\left(\sqrt{\frac{10}{4.9}}\right) = 9.8\sqrt{\frac{10}{4.9}}$
$= \frac{2(4.9)\sqrt{10}}{\sqrt{4.9}} = (2)\sqrt{4.9}\sqrt{10} = 14$ m/sec.

28. $a(t) = v'(t) = -3.72 \Rightarrow v(t) = -3.72t + C_1$; at $(0,93)$ we have $C_1 = 93 \Rightarrow s'(t) = v(t) = -3.72t + 93$
$\Rightarrow s(t) = -1.86t^2 + 93t + C_2$; at $(0,0)$ we have $C_2 = 0 \Rightarrow s(t) = -1.86t^2 + 93t$. Then $v(t) = 0 \Rightarrow t = \frac{93}{3.72} = 25$ so the maximum height of the rock is $s(25) = 1162.5$ m.

29. (a) $v = \int a\,dt = \int \left(15t^{1/2} - 3t^{-1/2}\right) dt = 10t^{3/2} - 6t^{1/2} + C$; $\frac{ds}{dt}(1) = 4 \Rightarrow 4 = 10(1)^{3/2} - 6(1)^{1/2} + C \Rightarrow C = 0$
$\Rightarrow v = 10t^{3/2} - 6t^{1/2}$

(b) $s = \int v\,dt = \int \left(10t^{3/2} - 6t^{1/2}\right) dt = 4t^{5/2} - 4t^{3/2} + C$; $s(1) = 0 \Rightarrow 0 = 4(1)^{5/2} - 4(1)^{3/2} + C \Rightarrow C = 0$
$\Rightarrow s = 4t^{5/2} - 4t^{3/2}$

30. (a) $\frac{ds}{dt} = 9.8t - 3 \Rightarrow s = 4.9t^2 - 3t + C$; i) at $s = 5$ and $t = 0$ we have $C = 5 \Rightarrow s = 4.9t^2 - 3t + 5$; displacement $= s(3) - s(1) = [(4.9)(9) - 9 + 5] - (4.9 - 3 + 5) = 33.2$ units; ii) at $s = -2$ and $t = 0$ we have $C = -2 \Rightarrow s = 4.9t^2 - 3t - 2$; displacement $= s(3) - s(1) = ((4.9)(9) - 9 - 2) - (4.9 - 3 - 2) = 33.2$ units; iii) at $s = s_0$ and $t = 0$ we have $C = s_0 \Rightarrow s = 4.9t^2 - 3t + s_0$; displacement $= s(3) - s(1)$ $= ((4.9)(9) - 9 + s_0) - (4.9 - 3 + s_0) = 33.2$ units

(b) True. Given an antiderivative $f(t)$ of the velocity function, we know that the body's position function is $s = f(t) + C$ for some constant C. Therefore, the displacement from $t = a$ to $t = b$ is $(f(b) + C) - (f(a) + C)$ $= f(b) - f(a)$. Thus we can find the displacement from any antiderivative f as the numerical difference $f(b) - f(a)$ without knowing the exact values of C and s.

31. If $T(t)$ is the temperature of the thermometer at time t, then $T(0) = -19°$ C and $T(14) = 100°$ C. From the Mean Value Theorem there exists a $0 < t_0 < 14$ such that $\frac{T(14) - T(0)}{14 - 0} = 8.5°$ C/sec $= T'(t_0)$, the rate at which the temperature was changing at $t = t_0$ as measured by the rising mercury on the thermometer.

32. Because the trucker's average speed was 79.5 mph, and by the Mean Value Theorem, the trucker must have been going that speed at least once during the trip.

33. Because its average speed was approximately 7.667 knots, and by the Mean Value Theorem, it must have been going that speed at least once during the trip.

34. The runner's average speed for the marathon was approximately 11.909 mph. Therefore, by the Mean Value Theorem, the runner must have been going that speed at least once during the marathon. Since the initial speed and final speed are both 0 mph and the runner's speed is continuous, by the Intermediate Value Theorem, the runner's speed must have been 11 mph at least twice.

35. The conclusion of the Mean Value Theorem yields $\frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{c^2} \Rightarrow c^2\left(\frac{a-b}{ab}\right) = a - b \Rightarrow c = \sqrt{ab}.$

36. The conclusion of the Mean Value Theorem yields $\frac{b^2 - a^2}{b-a} = 2c \Rightarrow c = \frac{a+b}{2}.$

37. $f'(x) = [\cos x \sin(x+2) + \sin x \cos(x+2)] - 2\sin(x+1)\cos(x+1) = \sin(x+x+2) - \sin 2(x+1)$

$= \sin(2x+2) - \sin(2x+2) = 0$. Therefore, the function has the constant value $f(0) = -\sin^2 1 \approx -0.7081$ which explains why the graph is a horizontal line.

38. Example CAS commands:

<u>Maple</u>:

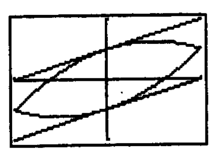
```
(x+2)*(x+1)*x*(x-1)*(x-2);
expand(%);
f:=unapply(%,x);
plot({f(x),diff(f(x),x)},x=-2..2);
```

<u>Mathematica</u>:

```
(x+2) (x+1)x(x-1) (x-2)
Expand[%]
f[x_] = %
Plot[ {f[x],f'[x]}, {x,-2,2} ]
```

39. f(x) must be zero at least once between a and b by the Intermediate Value Theorem. Now suppose that f(x) is zero twice between a and b. Then by the Mean Value Theorem, f′(x) would have to be zero at least once between the two zeros of f(x), but this can't be true since we are given that $f'(x) \neq 0$ on this interval. Therefore, f(x) is zero once and only once between a and b.

40. Consider the function $k(x) = f(x) - g(x)$. k(x) is continuous and differentiable on [a, b], and since $k(a) = f(a) - g(a) = 0$ and $k(b) = f(b) - g(b) = 0$, by the Mean Value Theorem, there must be a point c in (a, b) where $k'(c) = 0$. But since $k'(c) = f'(c) - g'(c)$, this means that $f'(c) = g'(c)$, and c is a point where the graphs of f and g have parallel or identical tangent lines.



[−1, 1] by [−2, 2]

41. Yes. By Corollary 2 we have $f(x) = g(x) + C$ since $f'(x) = g'(x)$. If the graphs start at the same point x = a, then $f(a) = g(a) \Rightarrow C = 0 \Rightarrow f(x) = g(x)$.

42. Let $f(x) = \sin x$ for $a \le x \le b$. From the Mean Value Theorem there exists a c between a and b such that $\frac{\sin b - \sin a}{b - a} = \cos c \Rightarrow -1 \le \frac{\sin b - \sin a}{b - a} \le 1 \Rightarrow \left|\frac{\sin b - \sin a}{b - a}\right| \le 1 \Rightarrow |\sin b - \sin a| \le |b - a|$.

43. By the Mean Value Theorem, $\frac{f(b) - f(a)}{b - a} = f'(c)$ for some point c between a and b. Since $b - a > 0$ and $f(b) < f(a)$, we have $f(b) - f(a) < 0 \Rightarrow f'(c) < 0$.

44. The condition is that f' should be continuous over $[a, b]$. The Mean Value Theorem then guarantees the existence of a point c in (a, b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$. If f' is continuous, then it has a minimum and maximum value on $[a, b]$, and $\min f' \le f'(c) \le \max f'$, as required.

45. $f'(x) = \left(1 + x^4 \cos x\right)^{-1} \Rightarrow f''(x) = -\left(1 + x^4 \cos x\right)^{-2}\left(4x^3 \cos x - x^4 \sin x\right)$
$= -x^3\left(1 + x^4 \cos x\right)^{-2}(4 \cos x - x \sin x) < 0$ for $0 \le x \le 0.1 \Rightarrow f'(x)$ is decreasing when $0 \le x \le 0.1$
$\Rightarrow \min f' \approx 0.9999$ and $\max f' = 1$. Now we have $0.9999 \le \frac{f(0.1) - 1}{0.1} \le 1 \Rightarrow 0.09999 \le f(0.1) - 1 \le 0.1$
$\Rightarrow 1.09999 \le f(0.1) \le 1.1$.

46. $f'(x) = \left(1 - x^4\right)^{-1} \Rightarrow f''(x) = -\left(1 - x^4\right)^{-2}\left(-4x^3\right) = \frac{4x^3}{\left(1 - x^4\right)^2} > 0$ for $0 < x \le 0.1 \Rightarrow f'(x)$ is increasing when $0 \le x \le 0.1 \Rightarrow \min f' = 1$ and $\max f' = 1.0001$. Now we have $1 \le \frac{f(0.1) - 2}{0.1} \le 1.0001$
$\Rightarrow 0.1 \le f(0.1) - 2 \le 0.10001 \Rightarrow 2.1 \le f(0.1) \le 2.10001$.

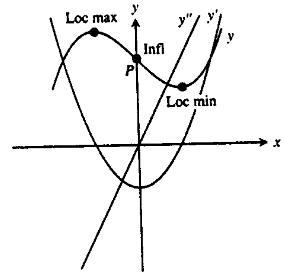
47-50. Example CAS commands

Maple:
```
with(plots): with(DEtools):
a:=0;b:=1;
eq:= D(y) (x)=x*sqrt(1-x);
sol:= dsolve({eq},y(x));
tograph:={seq(subs(_C1=i,sol),i={-2,-1,-,1,2})};
plot1:= implicitplot(tograph,x=a..b,y=-6..6):
display({plot1});
partsol:=dsolve({eq,y(1/2)=1},y(x));
implicitplot(partsol,x=a..b,y=-6..6,scaling=CONSTRAINED);
```

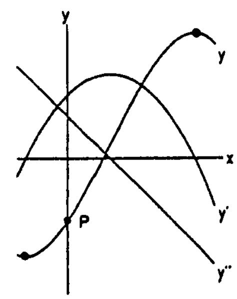
Mathematica:
```
a=0;b=1;
eq=D[y[x],x] = x*Sqrt[1-x]
sol=Flatten[DSolve[eq,y[x],x]]
cvals={-2,-1,1,2};
tograph=Table[y[x] /. (sol /. C[1] → cvals[[i]]), {i,1,4}]
Plot[Evaluate[tograph],{x,a,b}];
partsol = DSolve[{eq,y[1/2]=1},y[x],x]//Flatten
Plot[y[x] /. partsol,{x,a,b}]
```

3.3 THE SHAPE OF A GRAPH

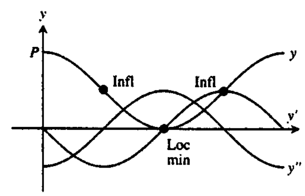
1. The graph of $y = f''(x) \Rightarrow$ the graph of $y = f(x)$ is concave up on $(0, \infty)$, concave down on $(-\infty, 0) \Rightarrow$ a point of inflection at $x = 0$; the graph of $y = f'(x) \Rightarrow y' = +++ \mid --- \mid +++ \Rightarrow$ the graph $y = f(x)$ has both a local maximum and a local minimum



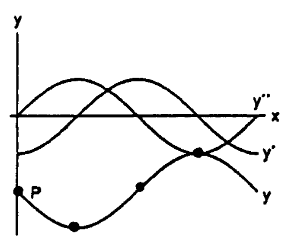
2. The graph of $y = f''(x) \Rightarrow y'' = +++ \mid --- \Rightarrow$ the graph of $y = f(x)$ has a point of inflection, the graph of $y = f'(x)$ $\Rightarrow y' = --- \mid +++ \mid --- \Rightarrow$ the graph of $y = f(x)$ has both a local maximum and a local minimum

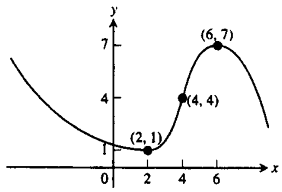


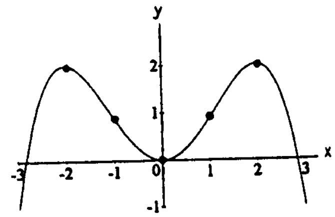
3. The graph of $y = f''(x) \Rightarrow y'' = --- \mid +++ \mid --- \Rightarrow$ the graph of $y = f(x)$ has two points of inflection, the graph of $y = f'(x) \Rightarrow y' = --- \mid +++ \Rightarrow$ the graph of $y = f(x)$ has a local minimum



4. The graph of $y = f''(x) \Rightarrow y'' = +++ \mid --- \Rightarrow$ the graph of $y = f(x)$ has a point of inflection; the graph of $y = f'(x)$ $\Rightarrow y' = --- \mid +++ \mid --- \Rightarrow$ the graph of $y = f(x)$ has both a local maximum and a local minimum



5. 

6. 

7. (a) Zero: $x = \pm 1$;
positive: $(-\infty, -1)$ and $(1, \infty)$;
negative: $(-1, 1)$
(b) Zero: $x = 0$;
positive: $(0, \infty)$;
negative: $(-\infty, 0)$

8. (a) Zero: $x \approx 0,\ \pm 1.25$;
positive: $(-1.25, 0)$ and $(1.25, \infty)$;
negative: $(-\infty, -1.25)$ and $(0, 1.25)$
(b) Zero: $x \approx \pm 0.7$;
positive: $(-\infty, -0.7)$ and $(0.7, \infty)$;
negative: $(-0.7, 0.7)$

9. (a) $(-\infty, -2]$ and $[0, 2]$
(b) $[-2, 0]$ and $[2, \infty)$
(c) Local maxima: $x = -2$ and $x = 2$;
local minimum: $x = 0$

10. (a) $[-2, 2]$
(b) $(-\infty, -2]$ and $[2, \infty)$
(c) Local maximum: $x = 2$;
local minimum: $x = -2$

11. (a) $[0, 1]$, $[3, 4]$, and $[5.5, 6]$
(b) $[1, 3]$ and $[4, 5.5]$
(c) Local maxima: $x = 1$, $x = 4$ (if f is continuous at $x = 4$), and $x = 6$;
local minima: $x = 0$, $x = 3$, and $x = 5.5$

12. If f is continuous on the interval $[0, 3]$;
(a) $[0, 3]$
(b) Nowhere
(c) Local maximum: $x = 3$;
local minimum: $x = 0$

13. (a) $f'(x) = (x-1)(x+2) \Rightarrow$ critical points at -2 and 1
(b) $f' = +++\underset{-2}{|}---\underset{1}{|}+++ \Rightarrow$ increasing on $(-\infty, -2]$ and $[1, \infty)$, decreasing on $[-2, 1]$
(c) Local maximum at $x = -2$ and a local minimum at $x = 1$

14. (a) $f'(x) = (x-1)^2(x+2) \Rightarrow$ critical points at -2 and 1
(b) $f' = ---\underset{-2}{|}+++\underset{1}{|}+++ \Rightarrow$ increasing on $[-2, 1]$ and $[1, \infty)$, decreasing on $(-\infty, -2]$
(c) No local maximum and a local minimum at $x = -2$

15. (a) $f'(x) = (x-1)e^{-x} \Rightarrow$ critical point at $x = 1$
(b) $f' = ----\underset{1}{|}++++ \Rightarrow$ decreasing on $(-\infty, 1]$, increasing on $[1, \infty)$
(c) Local (and absolute) minimum at $x = 1$

16. (a) $f'(x) = x^{-1/3}(x+2) \Rightarrow$ critical points at -2 and 0
(b) $f' = +++\underset{-2}{|}---\underset{0}{)(}+++ \Rightarrow$ increasing on $(-\infty, -2]$ and $[0, \infty)$, decreasing on $[-2, 0]$

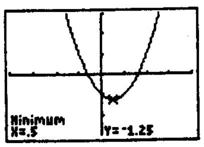
(c) Local maximum at $x = -2$, local minimum at $x = 0$

17. $y' = 2x - 1$

Intervals	$x < \frac{1}{2}$	$x > \frac{1}{2}$
Sign of y′	−	+
Behavior of y	Decreasing	Increasing

$y'' = 2$ (always positive: concave up)

Graphical support:



[−4, 4] by [−3, 3]

(a) $\left[\frac{1}{2}, \infty\right)$

(b) $\left(-\infty, \frac{1}{2}\right]$

(c) $(-\infty, \infty)$

(d) Nowhere

(e) Local (and absolute) minimum at $\left(\frac{1}{2}, -\frac{5}{4}\right)$

(f) None

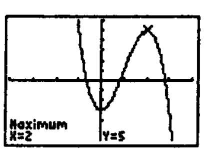
18. $y' = -6x^2 + 12x = -6x(x-2)$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y′	−	+	−
Behavior of y	Decreasing	Increasing	Decreasing

$y'' = -12x + 12 = -12(x-1)$

Intervals	$x < 1$	$x > 1$
Sign of y″	+	−
Behavior of y	Concave up	Concave down

Graphical support:



[−4, 4] by [−6, 6]

(a) $[0, 2]$

(b) $(-\infty, 0]$ and $[2, \infty)$

(c) $(-\infty, 1)$

(d) $(1, \infty)$

(e) Local maximum: $(2,5)$;
local minimum: $(0,-3)$

(f) At $(1,1)$

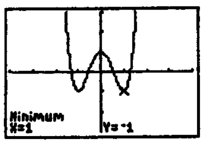
19. $y' = 8x^3 - 8x = 8x(x-1)(x+1)$

Intervals	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	$-$	$+$	$-$	$+$
Behavior of y	Decreasing	Increasing	Decreasing	Increasing

$y'' = 24x^2 - 8 = 8(\sqrt{3}x - 1)(\sqrt{3}x + 1)$

Intervals	$x < -\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}} < x$
Sign of y''	$+$	$-$	$+$
Behavior of y	Concave up	Concave down	Concave up

Graphical support:



$[-4, 4]$ by $[-3, 3]$

(a) $[-1,0]$ and $[1,\infty)$

(b) $(-\infty,1]$ and $[0,1]$

(c) $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$

(d) $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

(e) Local maximum: $(0,1)$;
local (and absolute) minima: $(-1,-1)$ and $(1,-1)$

(f) $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{9}\right)$

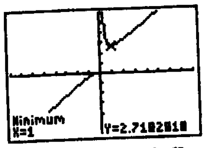
20. $y' = xe^{1/x}(-x^{-2}) + e^{1/x} = e^{1/x}\left(1 - \frac{1}{x}\right)$

Intervals	$x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	$+$	$-$	$+$
Behavior of y	Increasing	Decreasing	Increasing

$y'' = e^{1/x}(x^{-2}) + \left(1 - \frac{1}{x}\right)e^{1/x}(-x^{-2}) = \frac{e^{1/x}}{x^3}$

Intervals	$x < 0$	$x > 0$
Sign of y''	$-$	$+$
Behavior of y	Concave down	Concave down

Graphical support:



[−8, 8] by [−6, 6]

(a) $(-\infty, 0)$ and $[1, \infty)$
(b) $(0, 1]$
(c) $(0, \infty)$
(d) $(-\infty, 0)$
(e) Local minimum: $(1, e)$
(f) None

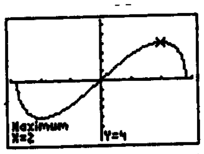
21. $y' = x \dfrac{1}{2\sqrt{8-x^2}}(-2x) + \left(\sqrt{8-x^2}\right)(1) = \dfrac{8-2x^2}{\sqrt{8-x^2}}$

Intervals	$-\sqrt{8} < x < -2$	$-2 < x < 2$	$2 < x < \sqrt{8}$
Sign of y′	−	+	−
Behavior of y	Decreasing	Increasing	Decreasing

$$y'' = \frac{\left(\sqrt{8-x^2}\right)(-4x) - (8-2x^2)\dfrac{1}{2\sqrt{8-x^2}}(-2x)}{\left(\sqrt{8-x^2}\right)^2} = \frac{2x^3-24x}{(8-x^2)^{3/2}} = \frac{2x(x^2-12)}{(8-x^2)^{3/2}}$$

Intervals	$-\sqrt{8} < x < 0$	$0 < x < \sqrt{8}$
Sign of y′	+	−
Behavior of y	Concave up	Concave down

Graphical support:



[−3.02, 3.02] by [−6.5, 6.5]

(a) $[-2, 2]$
(b) $[-\sqrt{8}, -2]$ and $[2, \sqrt{8}]$
(c) $(-\sqrt{8}, 0)$
(d) $(0, \sqrt{8})$
(e) Local maxima: $(-\sqrt{8}, 0)$ and $(2, 4)$;
local minima: $(-2, -4)$ and $(\sqrt{8}, 0)$
Note that the local extrema at $x = \pm 2$ are also absolute extrema
(f) $(0, 0)$

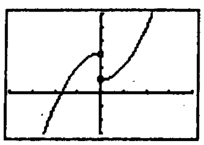
22. $y' = \begin{cases} -2x, & x < 0 \\ 2x, & x > 0 \end{cases}$

Intervals	$x < 0$	$x > 0$
Sign of y'	+	+
Behavior of y	Increasing	Increasing

$y'' = \begin{cases} -2, & x < 0 \\ 2, & x > 0 \end{cases}$

Intervals	$x < 0$	$x > 0$
Sign of y''	−	+
Behavior of y	Concave down	Concave up

Graphical support:



[−4, 4] by [−3, 6]

(a) $(-\infty, \infty)$
(b) None
(c) $(0, \infty)$
(d) $(-\infty, 0)$
(e) Local minimum: $(0, 1)$
(f) Note that $(0, 1)$ is not an inflection point because the graph has no tangent line at this point. There are no inflection points.

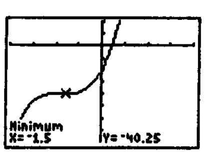
23. $y' = 12x^2 + 42x + 36 = 6(x+2)(2x+3)$

Intervals	$x < -2$	$-2 < x < -\frac{3}{2}$	$-\frac{3}{2} < x$
Sign of y'	+	−	+
Behavior of y	Increasing	Decreasing	Increasing

$y'' = 24x + 42 = 6(4x + 7)$

Intervals	$x < -\frac{7}{4}$	$-\frac{7}{4} < x$
Sign of y''	−	+
Behavior of y	Concave down	Concave up

Graphical support:



[−4, 4] by [−80, 20]

(a) $(-\infty, -2]$ and $\left[-\frac{3}{2}, \infty\right)$

(b) $\left[-2, -\frac{3}{2}\right]$

(c) $\left(-\frac{7}{4}, \infty\right)$

(d) $\left(-\infty, -\frac{7}{4}\right)$

(e) Local maximum: $(-2, -40)$; local minimum: $\left(-\frac{3}{2}, -\frac{161}{4}\right)$

(f) $\left(-\frac{7}{4}, -\frac{321}{8}\right)$

24. $y' = -4x^3 + 12x^2 - 4$

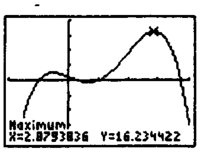
Using grapher techniques, the zeros of y' are $x \approx -0.53$, $x \approx 0.65$, and $x \approx 2.88$.

Intervals	$x < -0.53$	$-0.53 < x < 0.65$	$0.65 < x < 2.88$	$2.88 < x$
Sign of y'	+	−	+	−
Behavior of y	Increasing	Decreasing	Increasing	Decreasing

$y'' = -12x^2 + 24x = -12x(x-2)$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y''	−	+	−
Behavior of y	Concave down	Concave up	Concave down

Graphical support:



[−2, 4] by [−20, 20]

(a) $(-\infty, -0.53]$ and $[0.65, 2.88]$

(b) $[-0.53, 0.65]$ and $[2.88, \infty)$

(c) $(0, 2)$

(d) $(-\infty, 0)$ and $(2, \infty)$

(e) Local maxima: $(-0.53, 2.45)$ and $(2.88, 16.23)$; local minimum: $(0.65, -0.68)$
Note that the local maximum at $x \approx 2.88$ is also an absolute maximum.

(f) $(0, 1)$ and $(2, 9)$

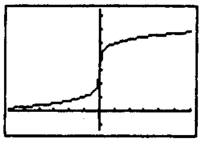
25. $y' = \frac{2}{5}x^{-4/5}$

Intervals	$x < 0$	$0 < x$
Sign of y'	+	+
Behavior of y	Increasing	Increasing

$y'' = -\frac{8}{25}x^{-9/5}$

Intervals	$x < 0$	$0 < x$
Sign of y''	+	−
Behavior of y	Concave up	Concave down

Graphical support:



[−6, 6] by [−1.5, 7.5]

(a) $(-\infty, \infty)$

(b) None

(c) $(-\infty, 0)$

(d) $(0, \infty)$

(e) None

(f) $(0, 3)$

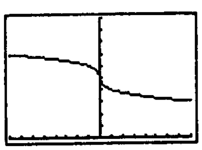
26. $y' = -\frac{1}{3}x^{-2/3}$

Intervals	$x < 0$	$0 < x$
Sign of y'	−	−
Behavior of y	Decreasing	Decreasing

$y'' = \frac{2}{9}x^{-5/3}$

Intervals	$x < 0$	$0 < x$
Sign of y''	−	+
Behavior of y	Concave down	Concave up

Graphical support:



[−8, 8] by [0, 10]

(a) None

(b) $(-\infty, \infty)$

(c) $(0, \infty)$

(d) $(-\infty, 0)$

(e) None

(f) $(0, 5)$

27. $y = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$

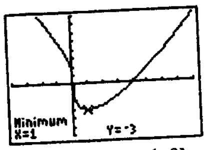
$y' = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4x-4}{3x^{2/3}}$

Intervals	$x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	−	−	+
Behavior of y	Decreasing	Decreasing	Increasing

$$y'' = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3} = \frac{4x+8}{9x^{5/3}}$$

Intervals	$x < -2$	$-2 < x < 0$	$0 < x$
Sign of y''	+	−	+
Behavior of y	Concave up	Concave down	Concave up

Graphical support:



[−4, 8] by [−6, 8]

(a) $[1,\infty)$

(b) $(-\infty,1]$

(c) $(-\infty,-2)$ and $(0,\infty)$

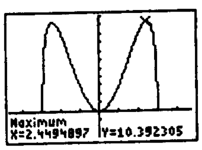
(d) $(-2,0)$

(e) Local minimum: $(1,-3)$

(f) $(-2,6\sqrt[3]{2}) \approx (-2,7.56)$ and $(0,0)$

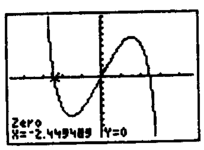
28. This problem can be solved using either graphical or analytic methods. The following is a graphical solution.

y



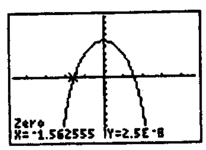
[−4.7, 4.7] by [−3, 11]

y′



[−4.7, 4.7] by [−10, 10]

y″



[−4.7, 4.7] by [−10, 10]

An analytic solution follows.

$$y' = x^2\frac{1}{2\sqrt{9-x^2}}(-2x) + \sqrt{9-x^2}\,(2x) = \frac{-3x^3+18x}{\sqrt{9-x^2}} = \frac{-3x(x^2-6)}{\sqrt{9-x^2}}$$

Intervals	$-3 < x < -\sqrt{6}$	$-\sqrt{6} < x < 0$	$0 < x < \sqrt{6}$	$\sqrt{6} < x < 3$
Sign of y'	+	−	+	−
Behavior of y	Increasing	Decreasing	Increasing	Decreasing

$$y'' = \frac{(\sqrt{9-x^2})(-9x^2+18) - (-3x^3+18x)\frac{1}{2\sqrt{9-x^2}}(-2x)}{(\sqrt{9-x^2})^2} = \frac{(9-x^2)(-9x^2+18)+(-3x^3+18x)(x)}{(9-x^2)^{3/2}}$$

$$= \frac{6x^4-81x^2+162}{(9-x^2)^{3/2}}$$

Find the zeros of y'': $\dfrac{3(2x^4-27x^2+54)}{(9-x^2)^{3/2}} = 0 \Rightarrow 2x^4-27x^2+54=0 \Rightarrow x^2 = \dfrac{27 \pm \sqrt{27^2-4(2)(54)}}{2(2)} = \dfrac{27 \pm 3\sqrt{33}}{4}$

$\Rightarrow x = \pm\sqrt{\frac{27-3\sqrt{33}}{4}} \approx \pm 1.56$

Note that we do not use $x = \pm\sqrt{\frac{27+3\sqrt{33}}{4}} \approx \pm 3.33$, because these values are outside of the domain.

Intervals	$-3 < x < -1.56$	$-1.56 < x < 1.56$	$1.56 < x < 3$
Sign of y″	−	+	−
Behavior of y	Concave down	Concave up	Concave down

(a) $[-3, -\sqrt{6}]$ and $[0, \sqrt{6}]$ or, $\approx [-3, -2.45]$ and $[0, 2.45]$

(b) $[-\sqrt{6}, 0]$ and $[\sqrt{6}, 3]$ or, $\approx [-2.45, 0]$ and $[2.45, 3]$

(c) Approximately $(-1.56, 1.56)$

(d) Approximately $(-3, -1.56)$ and $(1.56, 3)$

(e) Local maxima: $(\pm\sqrt{6}, 6\sqrt{3}) \approx (\pm 2.45, 10.39)$;
local minima: $(0,0)$ and $(\pm 3, 0)$

(f) $\approx (\pm 1.56, 6.25)$

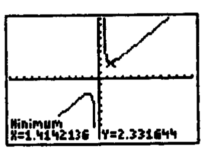
29. $y' = xe^{1/x^2}(-2x^{-3}) + (e^{1/x^2})(1) = e^{1/x^2}(1 - 2x^{-2}) = e^{1/x^2}\left(\frac{x^2-2}{x^2}\right)$

Intervals	$x < -\sqrt{2}$	$-\sqrt{2} < x < 0$	$0 < x < \sqrt{2}$	$\sqrt{2} < x$
Sign of y′	+	−	−	+
Behavior of y	Increasing	Decreasing	Decreasing	Increasing

$y'' = (e^{1/x^2})(4x^{-3}) + (1 - 2x^{-2})(e^{1/x^2})(-2x^{-3}) = (e^{1/x^2})(2x^{-3} + 4x^{-5}) = 2e^{1/x^2}\left(\frac{x^2+2}{x^5}\right)$

Intervals	$x < 0$	$0 < x$
Sign of y″	−	+
Behavior of y	Concave down	Concave up

Graphical support:

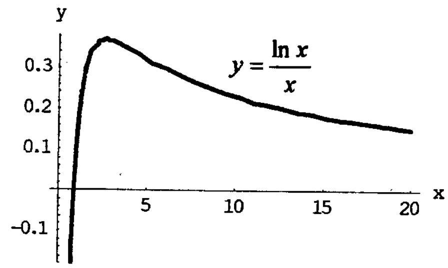


[−12, 12] by [−9, 9]

(a) $(-\infty, -\sqrt{2}]$ and $[\sqrt{2}, \infty)$
(b) $[-\sqrt{2}, 0)$ and $(0, \sqrt{2}]$
(c) $(0, \infty)$
(d) $(-\infty, 0)$
(e) Local maximum: $(-\sqrt{2}, -\sqrt{2e}) \approx (-1.41, -2.33)$; local minimum: $(\sqrt{2}, \sqrt{2e}) \approx (1.41, 2.33)$
(f) None

30. $y' = \frac{1 - \ln x}{x^2} \Rightarrow$ critical point at $x = e \Rightarrow y' = +++\underset{e}{|}---$

$y'' = \frac{2\ln x - 3}{x^3} \Rightarrow$ inflection point at $x = e^{3/2} \approx 4.48169 \Rightarrow y'' = \underset{e^{3/2}}{---|+++}$



The "maximum" function on the TI-89 calculator gives a maximum at $x = 2.71828 \approx e$, and the "inflection" function gives an inflection point at $x = 4.48169 \approx e^{3/2}$.

(a) $(0, e)$

(b) (e, ∞)

(c) $(0, e^{3/2})$

(d) $(e^{3/2}, \infty)$

(e) Local maximum at $\left(e, \frac{1}{e}\right)$; There is no local minima.

(f) $\left(e^{3/2}, \frac{3}{2e^{3/2}}\right) \approx (4.48, 0.33)$

31. $y = x^{1/4}(x+3) = x^{5/4} + 3x^{1/4}$

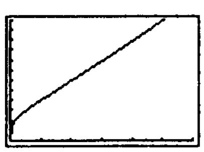
$y' = \frac{5}{4}x^{1/4} + \frac{3}{4}x^{-3/4} = \frac{5x+3}{4x^{3/4}}$

Since $y' > 0$ for all $x > 0$, y is always increasing on its domain $x \geq 0$.

$y'' = \frac{5}{16}x^{-3/4} - \frac{9}{16}x^{-7/4} = \frac{5x-9}{16x^{7/4}}$

Intervals	$0 < x < \frac{9}{5}$	$\frac{9}{5} < x$
Sign of y''	$-$	$+$
Behavior of y	Concave down	Concave up

Graphical support:



[0, 6] by [0, 12]

(a) $[0, \infty)$

(b) None

(c) $\left(\frac{9}{5}, \infty\right)$

(d) $\left(0, \frac{9}{5}\right)$

(e) Local (and absolute) minimum: $(0, 0)$

(f) $\left(\frac{9}{5}, \frac{24}{5}\cdot\sqrt[4]{\frac{9}{5}}\right) \approx (1.8, 5.56)$

32. $y' = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{-x^2+1}{(x^2+1)^2}$

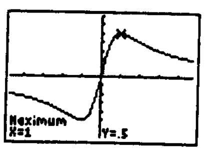
Intervals	$x < -1$	$-1 < x < 1$	$1 < x$
Sign of y'	$-$	$+$	$-$
Behavior of y	Decreasing	Increasing	Decreasing

$$y'' = \frac{(x^2+1)^2(-2x) - (-x^2+1)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{(x^2+1)(-2x) - 4x(-x^2+1)}{(x^3+1)^3}$$

$$= \frac{2x^3 - 6x}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}$$

Intervals	$x < -\sqrt{3}$	$-\sqrt{3} < x < 0$	$0 < x < \sqrt{3}$	$\sqrt{3} < x$
Sign of y''	$-$	$+$	$-$	$+$
Behavior of y	Concave down	Concave up	Concave down	Concave up

Graphical support:



[−4.7, 4.7] by [−0.7, 0.7]

(a) $[-1, 1]$

(b) $(-\infty, -1]$ and $[1, \infty)$

(c) $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$

(d) $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$

(e) Local maximum: $\left(1, \frac{1}{2}\right)$;

local minimum: $\left(-1, -\frac{1}{2}\right)$

(f) $(0, 0)$, $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$, and $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$

33. $y' = (x-1)^2(x-2)$

Intervals	$x < 1$	$1 < x < 2$	$2 < x$
Sign of y'	$-$	$-$	$+$
Behavior of y	Decreasing	Decreasing	Increasing

$y'' = (x-1)^2(1) + (x-2)(2)(x-1) = (x-1)[(x-1) + 2(x-2)] = (x-1)(3x-5)$

Intervals	$x < 1$	$1 < x < \frac{5}{3}$	$\frac{5}{3} < x$
Sign of y''	$+$	$-$	$+$
Behavior of y	Concave up	Concave down	Concave up

(a) There are no local maxima.

(b) There is a local (and absolute) minimum at $x = 2$.

(c) There are points of inflection at $x = 1$ and at $x = \frac{5}{3}$.

34. $y' = (x-1)^2(x-2)(x-4)$

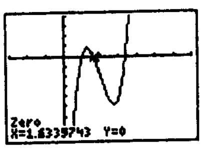
Intervals	$x < 1$	$1 < x < 2$	$2 < x < 4$	$4 < x$
Sign of y'	+	+	−	+
Behavior of y	Increasing	Increasing	Decreasing	Increasing

$$y'' = \frac{d}{dx}[(x-1)^2(x^2-6x+8)] = (x-1)^2(2x-6) + (x^2-6x+8)(2)(x-1)$$
$$= (x-1)[(x-1)(2x-6) + 2(x^2-6x+8)] = (x-1)(4x^2-20x+22)$$
$$= 2(x-1)(2x^2-10x+11)$$

Note that the zeros of y'' are $x = 1$ and

$$x = \frac{10 \pm \sqrt{10^2 - 4(2)(11)}}{4} = \frac{10 \pm \sqrt{12}}{4} = \frac{5 \pm \sqrt{3}}{2} \approx 1.63 \text{ or } 3.37.$$

The zeros of y'' can also be found graphically, as shown.



[−3, 7] by [−8, 4]

Intervals	$x < 1$	$1 < x < 1.63$	$1.63 < x < 3.37$	$3.37 < x$
Sign of y''	−	+	−	+
Behavior of y	Concave down	Concave up	Concave down	Concave up

(a) Local maximum at $x = 2$

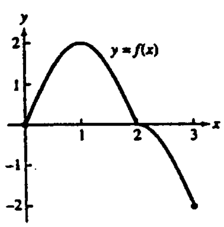
(b) Local minimum at $x = 4$

(c) Points of inflection at $x = 1$, at $x \approx 1.63$, and at $x \approx 3.37$.

35. (a) Absolute maximum at $(1, 2)$; absolute minimum at $(3, -2)$

(b) None

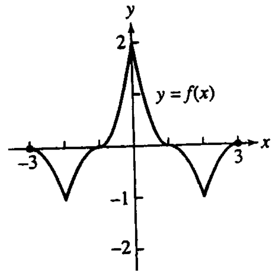
(c) One possible answer



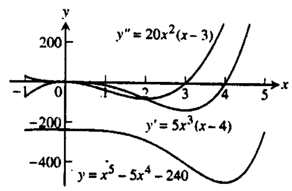
36. (a) Absolute maximum at $(0, 2)$; absolute minimum at $(2, -1)$ and $(-2, -1)$

(b) At $(1, 0)$ and $(-1, 0)$

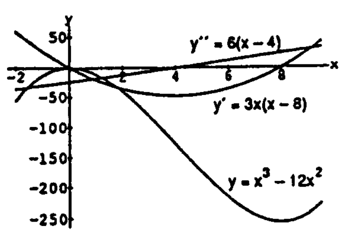
(c) One possible answer:



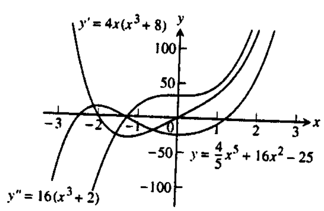
37. If $y = x^5 - 5x^4 - 240$, then $y' = 5x^3(x-4)$ and $y'' = 20x^2(x-3)$. The zeros of y' are extrema of y. The right-hand zero of y'' is a point of inflection for y. Inflection point at $x = 3$, local maximum at $x = 0$, local minimum at $x = 4$.



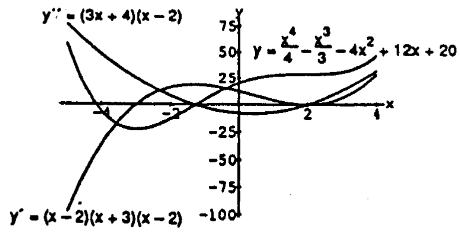
38. If $y = x^3 - 12x^2$, then $y' = 3x(x-8)$ and $y'' = 6(x-4)$. The zeros of y' and y'' are extrema and points of inflection, respectively.



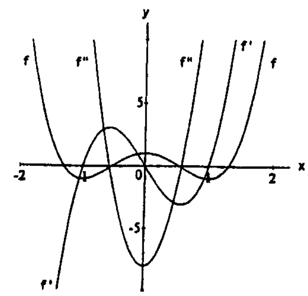
39. If $y = \frac{4}{5}x^5 + 16x^2 - 25$, then $y' = 4x\left(x^3 + 8\right)$ and $y'' = 16\left(x^3 + 2\right)$. The zeros of y' and y'' are extrema and points of inflection, respectively.



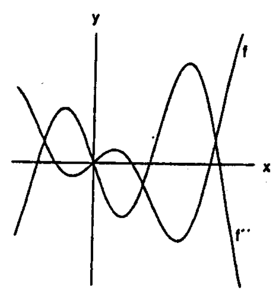
40. If $y = \frac{x^4}{4} - \frac{x^3}{3} - 4x^2 + 12x + 20$, then $y' = (x-2)^2(x+3)$ and $y'' = (3x+4)(x-2)$. The zeros of y' and y'' are extrema and points of inflection, respectively.



41. The graph of f falls where $f' < 0$, rises where $f' > 0$, and has horizontal tangents where $f' = 0$. It has local minima at points where f' changes from negative to positive and local maxima where f' changes from positive to negative. The graph of f is concave down where $f'' < 0$ and concave up where $f'' > 0$. It has points of inflection at values of x where f'' changes sign and a tangent line exists.



42. The graph f is concave down where $f'' < 0$, and concave up where $f'' > 0$. It has an inflection point each time f'' changes sign, provided a tangent line exists there.



43. (a) $v(t) = s'(t) = 2t - 4$ (b) $a(t) = v'(t) = 2$
(c) It begins at position 3 moving in a negative direction. It moves to position -1 when $t = 2$, and then changes direction, moving in a positive direction thereafter.

44. (a) $v(t) = s'(t) = -2 - 2t$ (b) $a(t) = v'(t) = -2$
(c) It begins at position 6 moving in the negative direction thereafter.

45. (a) $v(t) = s'(t) = 3t^2 - 3$ (b) $a(t) = v'(t) = 6t$
(c) It begins at position 3 moving in a negative direction. It moves to position 1 when $t = 1$, and then changes direction, moving in a positive direction thereafter.

46. (a) $v(t) = s'(t) = 6t - 6t^2$ (b) $a(t) = v'(t) = 6 - 12t$
(c) It begins at position 0. It starts moving in the positive direction under it reaches position 1 when $t = 1$, and then it changes direction. It moves in the negative direction thereafter.

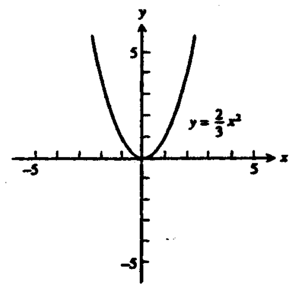
47. (a) The velocity is zero when the tangent line is horizontal, at approximately $t = 2.2$, $t = 6$, and $t = 9.8$.
(b) The acceleration is zero at the inflection points, approximately $t = 4$, $t = 8$, and $t = 12$.

48. (a) The velocity is zero when the tangent line is horizontal, at approximately $t = -0.2$, $t = 4$, and $t = 12$.
(b) The acceleration is zero at the inflection points, approximately $t = 1.5$, $t = 5.2$, $t = 8$, $t = 11$, and $t = 13$.

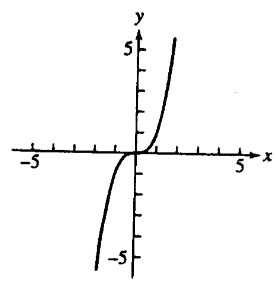
49. No. f must have a horizontal tangent at that point, but f could be increasing (or decreasing), and there would be no local extremum. For example, if $f(x) = x^3$, $f'(0) = 0$ but there is no local extremum at $x = 0$.

50. No. $f''(x)$ could still be positive (or negative) on both sides of $x = c$, in which case the concavity of the function would not change at $x = c$. For example, if $f(x) = x^4$, then $f''(0) = 0$, but f has no inflection point at $x = 0$.

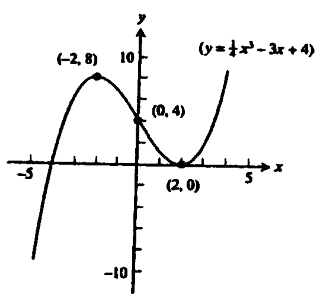
51. One possible answer:



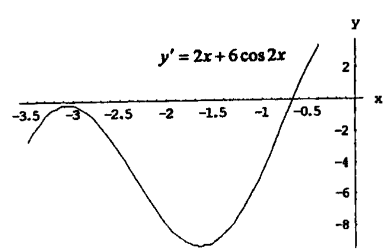
52. One possible answer:



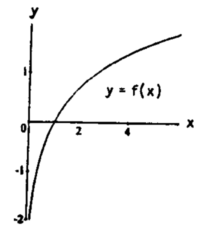
53. One possible answer:



54. No: $y = x^2 + 3\sin(2x) \Rightarrow y' = 2x + 6\cos(2x)$. The graph of y' does not touch the x-axis near $x = -3$ indicating that there is no horizontal tangent near $x = -3$.



55. The graph must be concave down for $x > 0$ because $f''(x) = -\frac{1}{x^2} < 0$.



56. The second derivative, being continuous and never zero, cannot change sign. Therefore the graph will always be concave up or concave down so it will have no inflection points.

57. A quadratic curve never has an inflection point. If $y = ax^2 + bx + c$ where $a \neq 0$, then $y' = 2ax + b$ and $y'' = 2a$. Since 2a is a constant, it is not possible for y'' to change signs.

58. A cubic curve always has exactly one inflection point. If $y = ax^3 + bx^2 + cx + d$ where $a \neq 0$, then $y' = 3ax^2 + 2bx + c$ and $y'' = 6ax + 2b$. Since $\frac{-b}{3a}$ is a solution of $y'' = 0$, we have that y'' changes its sign at $x = -\frac{b}{3a}$ and y' exists everywhere (so there is a tangent at $x = -\frac{b}{3a}$). Thus the curve has an inflection point at $x = -\frac{b}{3a}$. There are no other inflection points because y'' changes sign only at this zero.

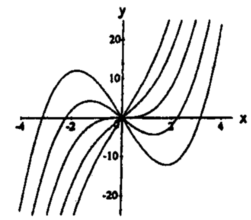
59. With $f(-2) = 11 > 0$ and $f(-1) = -1 < 0$ we conclude from the Intermediate Value Theorem that $f(x) = x^4 + 3x + 1$ has at least one zero between -2 and -1. Then $-2 < x < -1 \Rightarrow -8 < x^3 < -1$ $\Rightarrow -32 < 4x^3 < -4 \Rightarrow -29 < 4x^3 + 3 < -1 \Rightarrow f'(x) < 0$ for $-2 < x < -1 \Rightarrow f(x)$ is decreasing on $[-2, -1]$ $\Rightarrow f(x) = 0$ has exactly one solution in the interval $(-2, -1)$.

60. $g(t) = \sqrt{t} + \sqrt{t+1} - 4 \Rightarrow g'(t) = \frac{1}{2\sqrt{t}} + \frac{1}{2\sqrt{t+1}} > 0 \Rightarrow g(t)$ is increasing for t in $(0, \infty)$; $g(3) = \sqrt{3} - 2 < 0$ and $g(15) = \sqrt{15} > 0 \Rightarrow g(t)$ has exactly one zero in $(0, \infty)$.

61. $r(\theta) = \theta + \sin^2\left(\frac{\theta}{3}\right) - 8 \Rightarrow r'(\theta) = 1 + \frac{2}{3}\sin\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right) = 1 + \frac{1}{3}\sin\left(\frac{2\theta}{3}\right) > 0$ on $(-\infty, \infty) \Rightarrow r(\theta)$ is increasing on $(-\infty, \infty)$; $r(0) = -8$ and $r(8) = \sin^2\left(\frac{8}{3}\right) > 0 \Rightarrow r(\theta)$ has exactly one zero in $(-\infty, \infty)$.

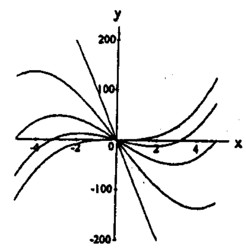
62. $r(\theta) = \tan\theta - \cot\theta - \theta \Rightarrow r'(\theta) = \sec^2\theta + \csc^2\theta - 1 = \sec^2\theta + \cot^2\theta > 0$ on $\left(0, \frac{\pi}{2}\right) \Rightarrow r(\theta)$ is increasing on $\left(0, \frac{\pi}{2}\right)$; $r\left(\frac{\pi}{4}\right) = -\frac{\pi}{4} < 0$ and $r(1.57) \approx 1254.2 \Rightarrow r(\theta)$ has exactly one zero in $\left(0, \frac{\pi}{2}\right)$.

63. (a) It appears to control the number and magnitude of the local extrema. If $k < 0$, there is a local maximum to the left of the origin and a local minimum to the right. The larger the magnitude of k $(k < 0)$, the greater the magnitude of the extrema. If $k > 0$, the graph has only positive slopes and lies entirely in the first and third quadrants with no local extrema. The graph becomes increasingly steep and straight as $k \to \infty$.

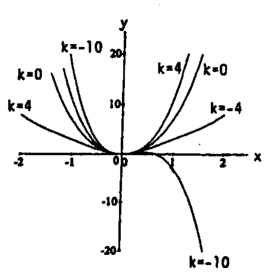


(b) $f'(x) = 3x^2 + k \Rightarrow$ the discriminant $0^2 - 4(3)(k) = -12k$ is positive for $k < 0$, zero for $k = 0$, and negative for $k > 0$; f' has two zeros $x = \pm\sqrt{-\frac{k}{3}}$ when $k < 0$, one zero $x = 0$ when $k = 0$ and no real zeros when $k > 0$; the sign of k controls the number of local extrema.

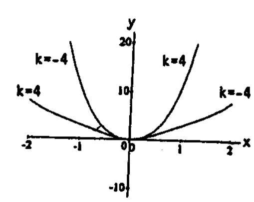
(c) As $k \to \infty$, $f'(x) \to \infty$ and the graph becomes increasingly steep and straight. As $k \to -\infty$, the crest of the graph (local maximum) in the second quadrant becomes increasingly high and the trough (local minimum) in the fourth quadrant becomes increasingly deep.



64. (a) It appears to control the concavity and the number of local extrema.



(b) $f(x) = x^4 + kx^3 + 6x^2 \Rightarrow f'(x) = 4x^3 + 3kx^2 + 12x$ $\Rightarrow f''(x) = 12x^2 + 6kx + 12 \Rightarrow$ the discriminant is $36k^2 - 4(12)(12) = 36(k+4)(k-4)$, so the sign line of the discriminant is $+++\underset{-4}{|}---\underset{4}{|}+++ \Rightarrow$ the discriminant is positive when $|k| > 4$, zero when $k = \pm 4$, and negative when $|k| < 4$; $f''(x) = 0$ has two zeros when $|k| > 4$, one zero when $k = \pm 4$, and



no real zeros for $|k| < 4$; the value of k controls the number of possible points of inflection.

65. (a) $f'(x) = \frac{(1+ae^{-bx})(0)-(c)(-abe^{-bx})}{(1+ae^{-bx})^2} = \frac{abce^{-bx}}{(1+ae^{-bx})^2} = \frac{abce^{bx}}{(e^{bx}+a)^2}$

so the sign of $f'(x)$ is the same as the sign of abc.

(b) $f''(x) = \frac{(e^{bx}+a)^2(ab^2ce^{bx})-(abce^{bx})2(e^{bx}+a)(be^{bx})}{(e^{bx}+a)^4} = \frac{(e^{bx}+a)(ab^2ce^{bx})-(abce^{bx})(2be^{bx})}{(e^{bx}+a)^3 2}$

$= -\frac{ab^2ce^{bx}(e^{bx}-a)}{(e^{bx}+a)^3}$

Since $a > 0$, this changes sign when $x = \frac{\ln a}{b}$ due to the $e^{bx} - a$ factor in the numerator, and f(x) has a point of inflection at that location.

66. (a) $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$f''(x) = 12ax^2 + 6bx + 2c$

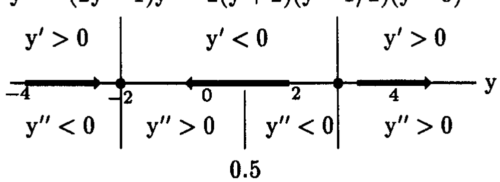
Since $f''(x)$ is quadratic, it must have 0, 1, or 2 zeros. If $f''(x)$ has 0 or 1 zeros, it will not change sign and the concavity of f(x) will not change, so there is no point of inflection. If $f''(x)$ has 2 zeros, it will change sign twice, and f(x) will have 2 points of inflection.

(b) If f has no points of inflection, then $f''(x)$ has 0 or 1 zeros, so the discriminant of $f''(x)$ is ≤ 0. This gives $(6b)^2 - 4(12a)(2c) \leq 0$, or $3b^2 \leq 8ac$.

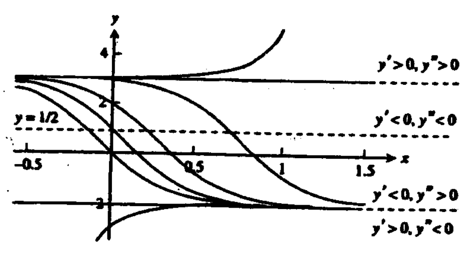
If f has 2 points of inflection, then $f''(x)$ has 2 zeros and the inequality is reversed, so $3b^2 > 8ac$. In summary, f has 2 points of inflection if and only if $3b^2 > 8ac$.

3.4 GRAPHICAL SOLUTIONS TO DIFFERENTIAL EQUATIONS

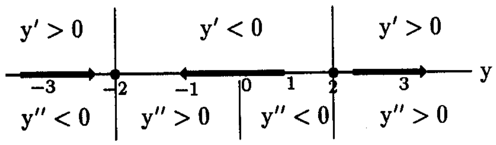
1. $y' = (y+2)(y-3)$
   (a) $y = -2$ is a stable equilibrium value and $y = 3$ is an unstable equilibrium.
   (b) $y'' = (2y-1)y' = 2(y+2)(y-1/2)(y-3)$



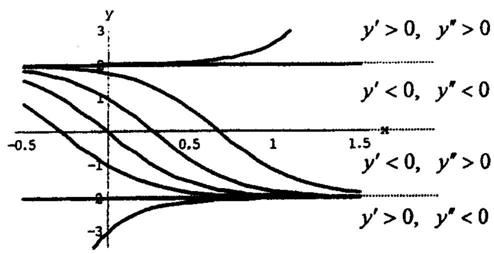
(c)



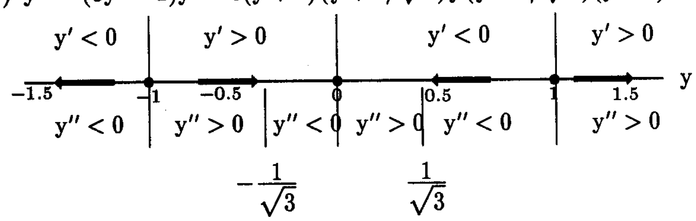
2. $y' = (y+2)(y-2)$
   (a) $y = -2$ is a stable equilibrium value and $y = 2$ is an unstable equilibrium value.
   (b) $y'' = 2yy' = 2(y+2)y(y-2)$



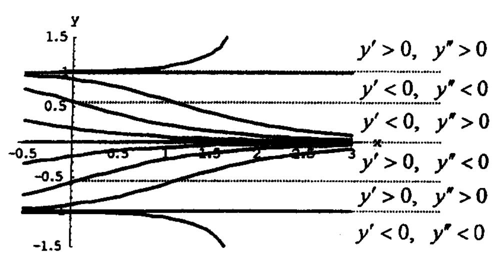
   (c)



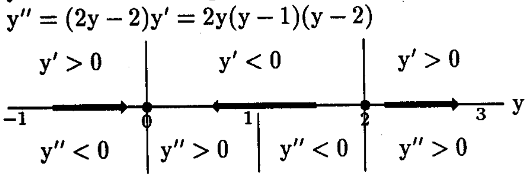
3. $y' = y^3 - y = (y+1)y(y-1)$
   (a) $y = -1$ and $y = 1$ are unstable equilibria and $y = 0$ is a stable equilibrium.
   (b) $y'' = (3y^2 - 1)y' = 3(y+1)\left(y + 1/\sqrt{3}\right)y\left(y - 1/\sqrt{3}\right)(y-1)$



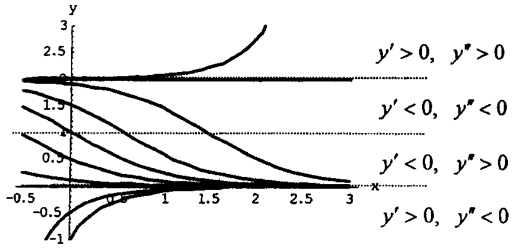
   (c)



4. $y' = y(y-2)$
   (a) $y = 0$ is a stable equilibrium and $y = 2$ is an unstable equilibrium.
   (b) $y'' = (2y-2)y' = 2y(y-1)(y-2)$



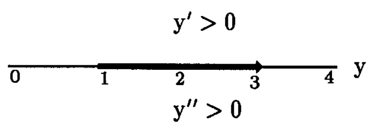
(c)



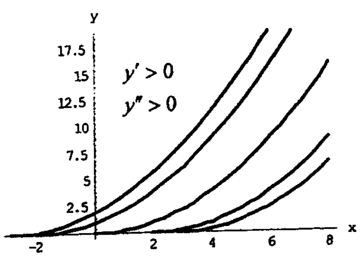
5. $y' = \sqrt{y},\ y > 0$

(a) There are no equilibrium values.

(b) $y'' = \frac{1}{2\sqrt{y}} y' = \frac{1}{2\sqrt{y}} \cdot \sqrt{y} = \frac{1}{2}$



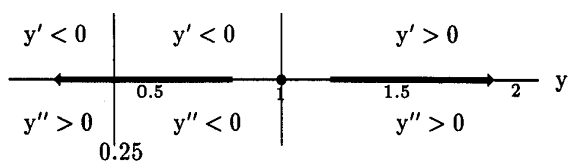
(c)



6. $y' = y - \sqrt{y},\ y > 0$

(a) $y = 1$ is an unstable equilibrium value.

(b) $y'' = \left(1 - \frac{1}{2\sqrt{y}}\right) y' = \left(1 - \frac{1}{2\sqrt{y}}\right)(y - \sqrt{y}) = \left(\sqrt{y} - \frac{1}{2}\right)(\sqrt{y} - 1)$



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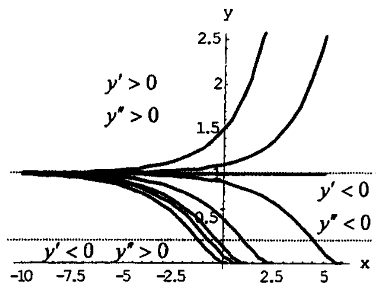
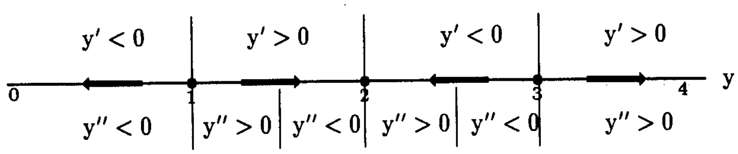
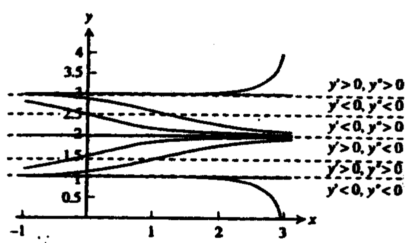


7. $y' = (y-1)(y-2)(y-3)$
   (a) $y = 1$ and $y = 3$ are unstable equilibria and $y = 2$ is a stable equilibrium.
   (b) $y'' = (3y^2 - 12y + 11)(y-1)(y-2)(y-3) = 3(y-1)\left(y - \frac{6-\sqrt{3}}{3}\right)(y-2)\left(y - \frac{6+\sqrt{3}}{3}\right)(y-3)$

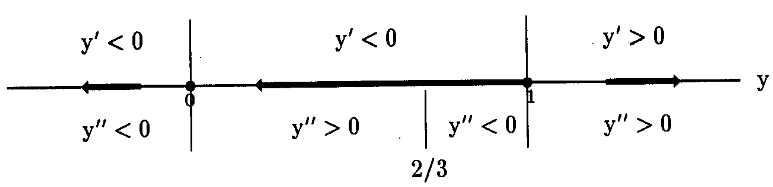


$\frac{6-\sqrt{3}}{3} \approx 1.42 \quad \frac{6+\sqrt{3}}{3} \approx 2.58$

(c)



8. $y' = y^3 - y^2 = y^2(y-1)$
   (a) $y = 0$ and $y = 1$ are unstable equilibria.
   (b) $y'' = (3y^2 - 2y)(y^3 - y^2) = y^3(3y-2)(y-1)$



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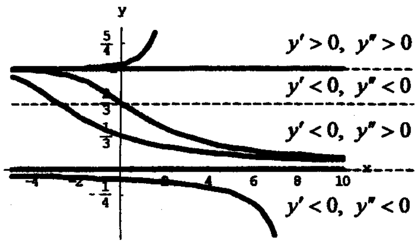
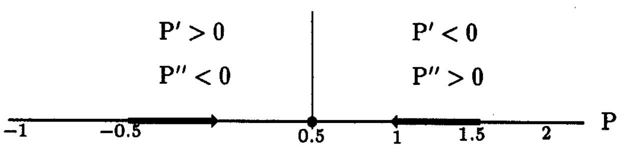
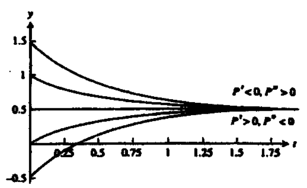


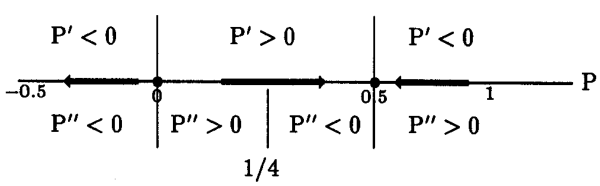
9. $\frac{dP}{dt} = 1 - 2P$ has a stable equilibrium at $P = \frac{1}{2}$. $\frac{d^2P}{dt^2} = -2\,\frac{dP}{dt} = -2(1 - 2P)$

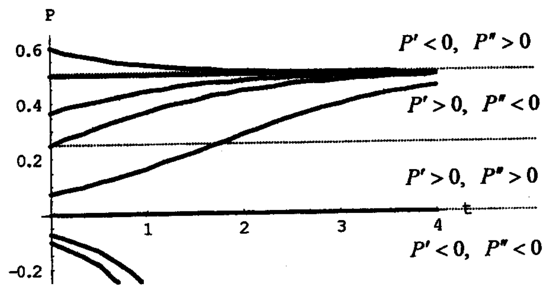




10. $\frac{dP}{dt} = P(1 - 2P)$ has an unstable equilibrium at $P = 0$ and a stable equilibrium at $P = \frac{1}{2}$.

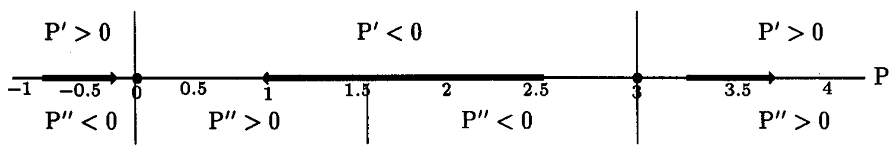
$\frac{d^2P}{dt^2} = (1 - 4P)\,\frac{dP}{dt} = P(1 - 4P)(1 - 2P)$

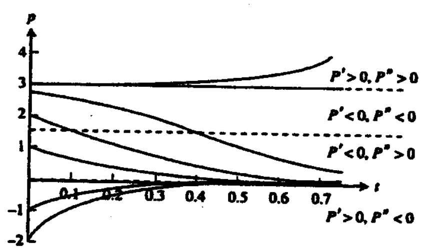




11. $\frac{dP}{dt} = 2P(P-3)$ has a stable equilibrium at $P = 0$ and an unstable equilibrium at $P = 3$.

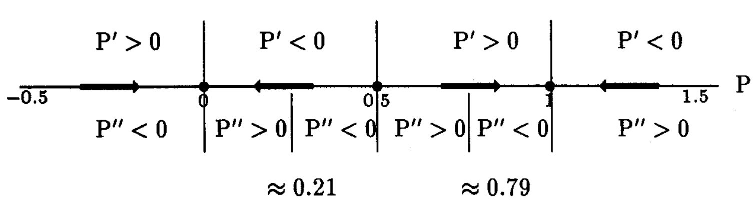
$\frac{d^2P}{dt^2} = 2(2P-3)\,\frac{dP}{dt} = 4P(2P-3)(P-3)$

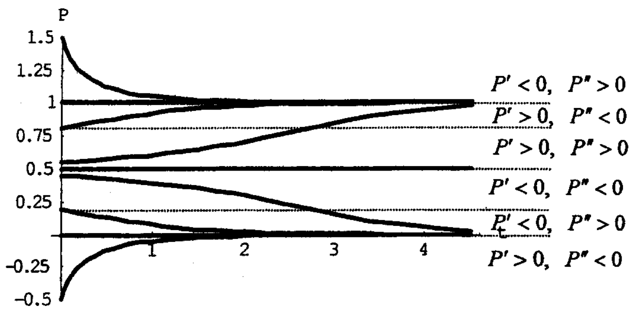




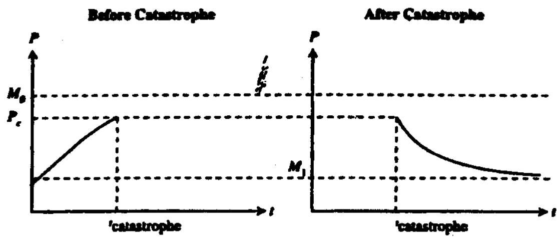
12. $\frac{dP}{dt} = 3P(1-P)\left(P-\frac{1}{2}\right)$ has stable equilibria at $P = 0$ and $P = 1$ and an unstable equilibrium at $P = \frac{1}{2}$.

$\frac{d^2P}{dt^2} = -\frac{3}{2}(6P^2-6P+1)\,\frac{dP}{dt} = \frac{3}{2}P\left(P-\frac{3-\sqrt{3}}{6}\right)\left(P-\frac{1}{2}\right)\left(P-\frac{3+\sqrt{3}}{6}\right)(P-1)$



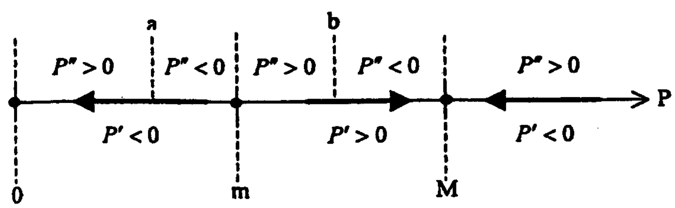


13.



Before the catastrophe, the population exhibits logistic growth and $P(t) \to M_0$, the stable equilibrium. After the catastrophe, the population declines logistically and $P(t) \to M_1$, the new stable equilibrium.

14. $\frac{dP}{dt} = rP(M - P)(P - m)$, r, M, $m > 0$



The model has 3 equilibrium points. The rest points $P = 0$, $P = M$ are asymptotically stable while $P = m$ is unstable. For initial populations greater than m, the model predicts P approaches M for large t. For initial populations less than m, the model predicts extinction. Points of inflection occur at $P = a$ and $P = b$ where $a = \frac{1}{3}\left[M + m - \sqrt{M^2 - mM + m^2}\right]$ and $b = \frac{1}{3}\left[M + m + \sqrt{M^2 - mM + m^2}\right]$.

(a) The model is reasonable in the sense that if $P < m$, then $P \to 0$ as $t \to \infty$; if $m < P < M$, then $P \to M$ as $t \to \infty$; if $P > M$, then $P \to M$ as $t \to \infty$.

(b) It is different if the population falls below m, for then $P \to 0$ as $t \to \infty$ (extinction). It is probably a more realistic model for that reason because we know some populations have become extinct after the population level became too low.

(c) For $P > M$ we see that $\frac{dP}{dt} = rP(M-P)(P-m)$ is negative. Thus the curve is everywhere decreasing. Moreover, $P \equiv M$ is a solution to the differential equation. Since the equation satisfies the existence and uniqueness conditions, solution trajectories cannot cross. Thus, $P \to M$ as $t \to \infty$.

(d) See the initial discussion above.

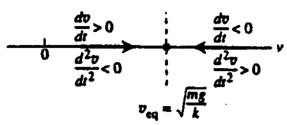
(e) See the initial discussion above.

15. $\frac{dv}{dt} = g - \frac{k}{m}v^2$, g, k, $m > 0$ and $v(t) \geq 0$

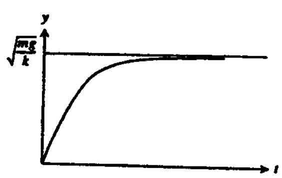
Equilibrium: $\frac{dv}{dt} = g - \frac{k}{m}v^2 = 0 \Rightarrow v = \sqrt{\frac{mg}{k}}$

Concavity: $\frac{d^2v}{dt^2} = -2\left(\frac{k}{m}v\right)\frac{dv}{dt} = -2\left(\frac{k}{m}v\right)\left(g - \frac{k}{m}v^2\right)$

(a)



(b)



(c) $v_{terminal} = \sqrt{\frac{160}{0.005}} = 178.9\ \frac{ft}{s} = 122$ mph

16. $F = F_p - F_r$

$ma = mg - k\sqrt{v}$

$\frac{dv}{dt} = g - \frac{k}{m}\sqrt{v}$, $v(0) = v_0$

Thus, $\frac{dv}{dt} = 0$ implies $v = \left(\frac{mg}{k}\right)^2$, the terminal velocity. If $v_0 < \left(\frac{mg}{k}\right)^2$, the object will fall faster and faster, approaching the terminal velocity; if $v_0 > \left(\frac{mg}{k}\right)^2$, the object will slow down to the terminal velocity.

17. $F = F_p - F_r$

$ma = 50 - 5|v|$

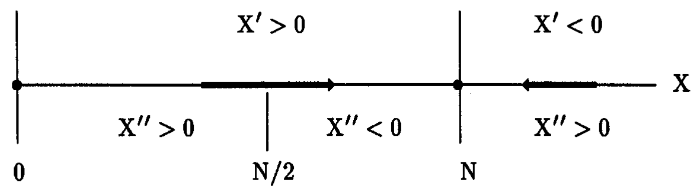
$\frac{dv}{dt} = \frac{1}{m}(50 - 5|v|)$

The maximum velocity occurs when $\frac{dv}{dt} = 0$ or $v = 10$ ft/sec.

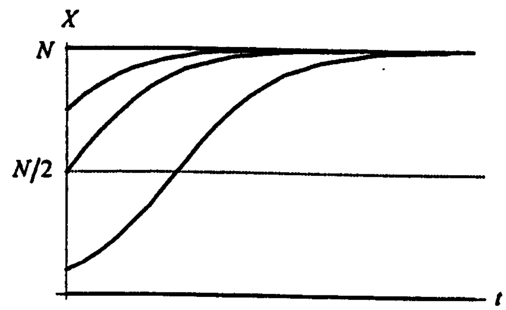
18. (a) The model seems reasonable because the rate of spread of a piece of information, an innovation, or a cultural fad is proportional to the product of the number of individuals who have it (X) and those who do not (N − X). When X is small, there are only a few individuals to spread the item so the rate of spread is slow. On the other hand, when (N − X) is small the rate of spread will be slow because there are only a few individuals who can receive it during an interval of time. The rate of spread will be fastest when both X and (N − X) are large because then there are a lot of individuals to spread the item and a lot of individuals to receive it.

(b) There is a stable equilibrium at $X = N$ and an unstable equilibrium at $X = 0$.

$\frac{d^2X}{dt^2} = k\frac{dX}{dt}(N - X) - kX\frac{dX}{dt} = k^2X(N - X)(N - 2X) \Rightarrow$ inflection points at $X = 0$, $X = \frac{N}{2}$, and $X = N$.



(c)



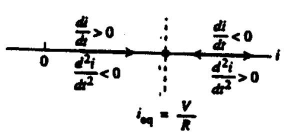
(d) The spread rate is most rapid when $X = \frac{N}{2}$. Eventually all of the people will receive the item.

19. $L\frac{di}{dt} + Ri = V \Rightarrow \frac{di}{dt} = \frac{V}{L} - \frac{R}{L}i = \frac{R}{L}\left(\frac{V}{R} - i\right)$, $V, L, R > 0$

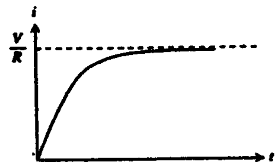
Equilibrium: $\frac{di}{dt} = \frac{R}{L}\left(\frac{V}{R} - i\right) = 0 \Rightarrow i = \frac{V}{R}$,

Concavity: $\frac{d^2i}{dt^2} = -\left(\frac{R}{L}\right)\frac{di}{dt} = -\left(\frac{R}{L}\right)^2\left(\frac{V}{R} - i\right)$

Phase Line:

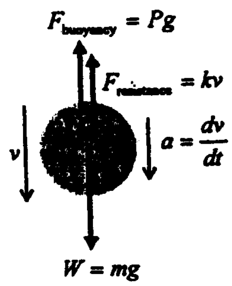


If the switch is closed at $t = 0$, then $i(0) = 0$, and the graph of the solution looks like this:



As $t \to \infty$, $i(t) \to i_{\text{steady state}} = \frac{V}{R}$. (In the steady state condition, the self-inductance acts like a simple wire connector and, as a result, the current through the resistor can be calculated using the familiar version of Ohm's Law.)

20. (a) Free body diagram of the pearl:

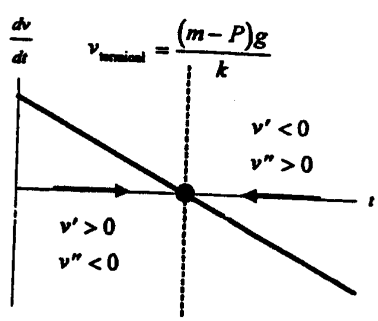


(b) Use Newton's Second Law, summing forces in the direction of the acceleration:

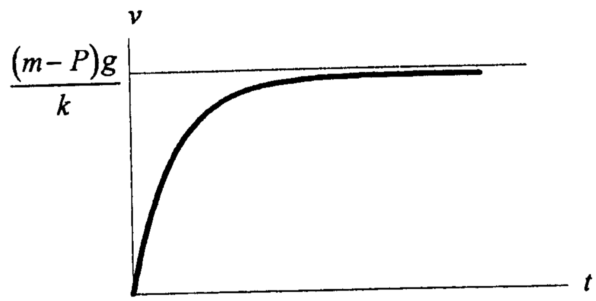
$$mg - Pg - kv = ma \Rightarrow \frac{dv}{dt} = \left(\frac{m - P}{m}\right)g - \frac{k}{m}v.$$

(c) Equilibrium: $\frac{dv}{dt} = \frac{k}{m}\left(\frac{(m - P)g}{k} - v\right) = 0 \Rightarrow v_{\text{terminal}} = \frac{(m - P)g}{k}$

Concavity: $\frac{d^2v}{dt^2} = -\left(\frac{k}{m}\right)\frac{dv}{dt} = -\left(\frac{k}{m}\right)^2\left(\frac{(m - P)g}{k} - v\right)$



(d)



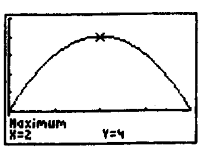
(e) The terminal velocity of the pearl is $\frac{(m-P)g}{k}$.

3.5 MODELING AND OPTIMIZATION

1. Let ℓ and w represent the length and width of the rectangle, respectively. With an area of 16 in.^{2}, we have that $(\ell)(w) = 16 \Rightarrow w = 16\ell^{-1} \Rightarrow$ the perimeter is $P = 2\ell + 2w = 2\ell + 32\ell^{-1}$ and $P'(\ell) = 2 - \frac{32}{\ell^2} = \frac{2(\ell^2 - 16)}{\ell^2}$. Solving $P'(\ell) = 0 \Rightarrow \frac{2(\ell+4)(\ell-4)}{\ell^2} = 0 \Rightarrow \ell = -4, 4$. Since $\ell > 0$ for the length of a rectangle, ℓ must be 4 and $w = 4 \Rightarrow$ the perimeter is 16 in., a minimum since $P''(\ell) = \frac{64}{\ell^3} > 0$.

2. Let x represent the length of the rectangle in meters $(0 < x < 4)$. Then the width is $4 - x$ and the area is $A(x) = x(4 - x) = 4x - x^2$. Since $A'(x) = 4 - 2x$, the critical point occurs at $x = 2$. Since $A'(x) > 0$ for $0 < x < 2$ and $A'(x) < 0$ for $2 < x < 4$, this critical point corresponds to the maximum area. The rectangle with the largest area measures 2 m by $4 - 2 = 2$m, so it is a square.

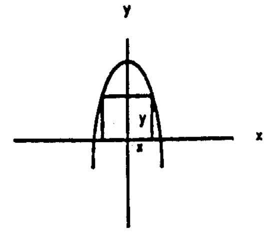
   Graphical support:

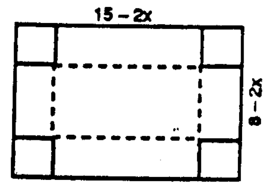
   [0, 4] by [−1.5, 5]

3. (a) The line containing point P also contains the points $(0, 1)$ and $(1, 0) \Rightarrow$ the line containing P is $y = 1 - x$ $\Rightarrow$ a general point on that line is $(x, 1 - x)$.
   (b) The area $A(x) = 2x(1 - x)$, where $0 \le x \le 1$.
   (c) When $A(x) = 2x - 2x^2$, then $A'(x) = 0 \Rightarrow 2 - 4x = 0 \Rightarrow x = \frac{1}{2}$. Since $A(0) = 0$ and $A(1) = 0$, we conclude that $A\left(\frac{1}{2}\right) = \frac{1}{2}$ sq units is the largest area. The dimensions are 1 unit by $\frac{1}{2}$ unit.

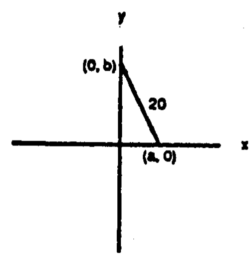
4. The area of the rectangle is $A = 2xy = 2x\left(12 - x^2\right)$, where $0 \le x \le \sqrt{12}$. Solving $A'(x) = 0 \Rightarrow 24 - 6x^2 = 0$ $\Rightarrow x = -2$ or 2. Now -2 is not in the domain, and since $A(0) = 0$ and $A\left(\sqrt{12}\right) = 0$, we conclude that $A(2) = 32$ sq units is the maximum area. The dimensions are 4 units by 8 units.



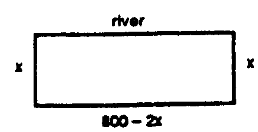
5. The volume of the box is $V(x) = x(15 - 2x)(8 - 2x)$ $= 120x - 46x^2 + 4x^3$, where $0 \le x \le 4$. Solving $V'(x) = 0$ $\Rightarrow 120 - 92x + 12x^2 = 4(6 - x)(5 - 3x) = 0 \Rightarrow x = \frac{5}{3}$ or 6, but 6 is not in the domain. Since $V(0) = V(4) = 0$, $V\left(\frac{5}{3}\right)$ $= \frac{2450}{27} \approx 91$ sq units must be the maximum volume of the box with dimensions $\frac{14}{3} \times \frac{35}{3} \times \frac{5}{3}$ inches.



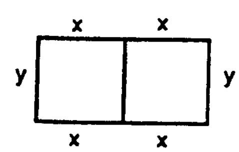
6. The area of the triangle is $A = \frac{1}{2}ba = \frac{b}{2}\sqrt{400 - b^2}$, where $0 \le b \le 20$. Then $\frac{dA}{db} = \frac{1}{2}\sqrt{400 - b^2} - \frac{b^2}{2\sqrt{400 - b^2}} = \frac{200 - b^2}{\sqrt{400 - b^2}}$ $= 0 \Rightarrow$ the interior critical point is $b = 10\sqrt{2}$. When $b = 0$ or 20, the area is zero $\Rightarrow A\left(10\sqrt{2}\right)$ is the maximum area. When $a^2 + b^2 = 400$ and $b = 10\sqrt{2}$, the value of a is also $10\sqrt{2} \Rightarrow$ the maximum area occurs when $a = b$.



7. The area is $A(x) = x(800 - 2x)$, where $0 \le x \le 400$. Solving $A'(x) = 800 - 4x = 0 \Rightarrow x = 200$. With $A(0) = A(400) = 0$, the maximum area is $A(200) = 80{,}000\ m^2$. The dimensions are 200 m by 400 m.



8. The area is $2xy = 216 \Rightarrow y = \frac{108}{x}$. The perimeter is $P = 4x + 3y$ $= 4x + 324x^{-1}$, where $0 < x$; $\frac{dP}{dx} = 4 - \frac{324}{x^2} = 0 \Rightarrow x^2 - 81 = 0$ $\Rightarrow$ the critical points are 0 and ± 9, but 0 and -9 are not in the domain. Then $P''(9) > 0 \Rightarrow$ at $x = 9$ there is a minimum $\Rightarrow$ the dimensions of the outer rectangle are 18 m by 12 m $\Rightarrow$ 72 meters of fence will be needed.



9. (a) We minimize the weight $= tS$ where S is the surface area, and t is the thickness of the steel walls of the tank. The surface area is $S = x^2 + 4xy$ where x is the length of a side of the square base of the tank, and y is its depth. The volume of the tank must be 500 $ft^3 \Rightarrow y = \frac{500}{x^2}$. Therefore, the weight of the tank is $w(x) = t\left(x^2 + \frac{2000}{x}\right)$. Treating the thickness as a constant gives $w'(x) = t\left(2x - \frac{2000}{x^2}\right)$ for $x > 0$. The

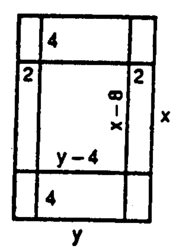
critical value is at x = 10. Since $w''(10) = t\left(2 + \frac{4000}{10^3}\right) > 0$, there is a minimum at x = 10. Therefore, the optimum dimensions of the tank are 10 ft on the base edges and 5 ft deep.

(b) Minimizing the surface area of the tank minimizes its weight for a given wall thickness. The thickness of the steel walls would likely be determined by other considerations such as structural requirements.

10. (a) With the volume of the tank being 1125 ft^3, we have that $yx^2 = 1125 \Rightarrow y = \frac{1125}{x^2}$. The cost of building the tank is $c(x) = 5x^2 + 30x\left(\frac{1125}{x^2}\right)$, where $0 < x$. Then $c'(x) = 10x - \frac{33{,}750}{x^2} = 0 \Rightarrow$ the critical points are 0 and 15, but 0 is not in the domain. Thus $c''(15) > 0 \Rightarrow$ at x = 15 we have a minimum. The values of x = 15 ft and y = 5 ft will minimize the cost.

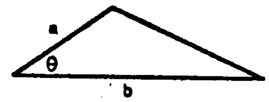
(b) The cost function, $c = 5(x^2 + 4xy) + 10xy$, can be separated into two items: (1) the cost of materials and labor to fabricate the tank, and (2) the cost for the excavation. Since the area of the sides and bottom of the tank is $(x^2 + 4xy)$, it can be deduced that the unit cost to fabricate the tank is $\$5/ft^2$. Normally, excavation costs are per unit volume of excavated material. Consequently, the total excavation cost can be taken as $10xy = \left(\frac{10}{x}\right)(x^2y)$. This suggests that the unit cost of excavation is $\frac{\$10/ft^2}{x}$ where x is the length of a side of the square base of the tank in feet. For the least expensive tank, the unit cost for the excavation is $\frac{\$10/ft^2}{15\ ft} = \frac{\$0.67}{ft^3} = \frac{\$18}{yd^3}$. The total cost of the least expensive tank is \$3375, which is the sum of \$2625 for fabrication and \$750 for the excavation.

11. The area of the printing is $(y-4)(x-8) = 50$. Consequently, $y = \left(\frac{50}{x-8}\right) + 4$. The area of the paper is $A(x) = x\left(\frac{50}{x-8} + 4\right)$, where $8 < x$. Then $A'(x) = \left(\frac{50}{x-8} + 4\right) - x\left(\frac{50}{(x-8)^2}\right) = \frac{4(x-8)^2 - 400}{(x-8)^2} = 0 \Rightarrow$ the critical points are -2 and 18, but -2 is not in the domain. Thus $A''(18) > 0 \Rightarrow$ at x = 18 we have a minimum. Therefore the dimensions 18 by 9 inches minimize the amount of paper.

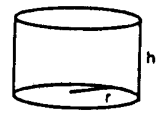


12. The volume of the cone is $V = \frac{1}{3}\pi r^2 h$, where $r = x = \sqrt{9 - y^2}$ and $h = y + 3$ (from the figure in the text). Thus, $V(y) = \frac{\pi}{3}(9 - y^2)(y + 3) = \frac{\pi}{3}(27 + 9y - 3y^2 - y^3) \Rightarrow V'(y) = \frac{\pi}{3}(9 - 6y - 3y^2) = \pi(1 - y)(3 + y)$. The critical points are -3 and 1, but -3 is not in the domain. Thus $V''(1) = \frac{\pi}{3}(-6 - 6(1)) < 0 \Rightarrow$ at y = 1 we have a maximum volume of $V(1) = \frac{\pi}{3}(8)(4) = \frac{32\pi}{3}$ cubic units.

13. The area of the triangle is $A(\theta) = \frac{ab \sin \theta}{2}$, where $0 < \theta < \pi$. Solving $A'(\theta) = 0 \Rightarrow \frac{ab \cos \theta}{2} = 0 \Rightarrow \theta = \frac{\pi}{2}$. Since $A''(\theta) = -\frac{ab \sin \theta}{2} \Rightarrow A''\left(\frac{\pi}{2}\right) < 0$, there is a maximum at $\theta = \frac{\pi}{2}$.



14. A volume $V = \pi r^2 h = 1000 \Rightarrow h = \frac{1000}{\pi r^2}$. The amount of material is the surface area given by the sides and bottom of the can $\Rightarrow S = 2\pi rh + \pi r^2 = \frac{2000}{r} + \pi r^2$, $0 < r$. Then $\frac{dS}{dr} = -\frac{2000}{r^2} + 2\pi r$



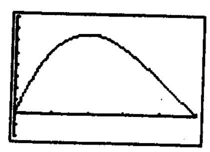
$= 0 \Rightarrow \frac{\pi r^3 - 1000}{r^2} = 0$. The critical points are 0 and $\frac{10}{\sqrt[3]{\pi}}$, but 0 is not in the domain. Since $\frac{d^2S}{dr^2} = \frac{4000}{r^3} + 2\pi$ > 0, we have a minimum surface area when $r = \frac{10}{\sqrt[3]{\pi}}$ cm and $h = \frac{1000}{\pi r^2} = \frac{10}{\sqrt[3]{\pi}}$ cm.

15. With a volume of 1000 cm and $V = \pi r^2 h$, then $h = \frac{1000}{\pi r^2}$. The amount of aluminum used per can is $A = 8r^2 + 2\pi rh = 8r^2 + \frac{2000}{r}$. Then $A'(r) = 16r - \frac{2000}{r^2} = 0 \Rightarrow \frac{8r^3 - 1000}{r^2} = 0 \Rightarrow$ the critical points are 0 and 5, but $r = 0$ results in no can. Since $A''(r) = 16 + \frac{4000}{r^3} > 0$ we have a minimum at $r = 5 \Rightarrow h = \frac{40}{\pi}$ and $h{:}r = 8{:}\pi$.

16. (a) The base measures $10 - 2x$ in. by $\frac{15 - 2x}{2}$ in., so the volume formula is

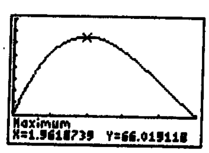
$$V(x) = \frac{x(10 - 2x)(15 - 2x)}{2} = 2x^3 - 25x^2 + 75x.$$

(b) We require $x > 0$, $2x < 10$, and $2x < 15$. Combining these requirements, the domain is the interval $(0, 5)$.



[0, 5] by [−20, 80]

(c)



[0, 5] by [−20, 80]

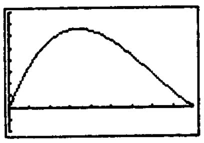
The maximum volume is approximately 66.02 in.^{3} when $x \approx 1.96$ in.

(d) $V'(x) = 6x^2 - 50x + 75$

The critical point occurs when $V'(x) = 0$, at $x = \frac{50 \pm \sqrt{(-50)^2 - 4(6)(75)}}{2(6)} = \frac{50 \pm \sqrt{700}}{12}$ $= \frac{25 \pm 5\sqrt{7}}{6}$, that is, $x \approx 1.96$ or $x \approx 6.37$. We discard the larger value because it is not in the domain. Since $V''(x) = 12x - 50$, which is negative when $x \approx 1.96$, the critical point corresponds to the maximum volume. The maximum volume occurs when $x = \dfrac{25 - 5\sqrt{7}}{6} \approx 1.96$, which confirms the result in (c).

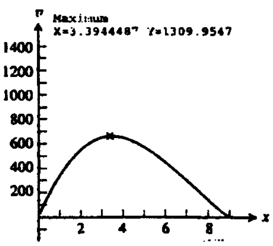
17. (a) The "sides" of the suitcase will measure $24 - 2x$ in. by $18 - 2x$ in. and will be $2x$ in. apart, so the volume formula is $V(x) = 2x(24 - 2x)(18 - 2x) = 8x^3 - 168x^2 + 864x$.

(b) We require $x > 0$, $2x < 18$, and $2x < 24$. Combining these requirements, the domain is the interval $(0, 9)$.



[0, 9] by [−400, 1600]

(c)



This maximum volume is approximately 1309.95 in.^{3} when $x \approx 3.39$ in.

(d) $V'(x) = 24x^2 - 336x + 864 = 24(x^2 - 14x + 36)$
The critical point is at

$$x = \frac{14 \pm \sqrt{(-14)^2 - 4(1)(36)}}{2(1)} = \frac{14 \pm \sqrt{52}}{2} = 7 \pm \sqrt{13},$$

that is, $x \approx 3.39$ or $x \approx 10.61$. We discard the larger value because it is not in the domain. Since $V''(x) = 24(2x - 14)$, which is negative when $x \approx 3.39$, the critical point corresponds to the maximum volume. The maximum value occurs at $x = 7 - \sqrt{13} \approx 3.39$, which confirms the results in (c).

(e) $8x^3 - 168x^2 + 864x = 1120$
$8(x^3 - 21x^2 + 108x - 140) = 0$
$8(x - 2)(x - 5)(x - 14) = 0$
Since 14 is not in the domain, the possible values of x are $x = 2$ in. or $x = 5$ in.

(f) The dimensions of the resulting box are $2x$ in., $(24 - 2x)$ in., and $(18 - 2x)$ in. Each of these measurements must be positive, so that gives the domain of $(0, 9)$.

18. If the upper right vertex of the rectangle is located at $(x, 4 \cos 0.5x)$ for $0 < x < \pi$, then the rectangle has width $2x$ and height $4 \cos 0.5x$, so the area is $A(x) = 8x \cos 0.5x$. Then $A'(x) = 8x(-0.5 \sin 0.5x) + 8(\cos 0.5x)(1) = -4x \sin 0.5x + 8 \cos 0.5x$. Solving $A'(x) = 0$ graphically for $0 < x < \pi$, we find that $x \approx 1.72$. Evaluating $2x$ and $4 \cos 0.5x$ for $x \approx 1.72$, the dimensions of the rectangle are approximately 3.44 (width) by 2.61 (height), and the maximum area is approximately 8.98.

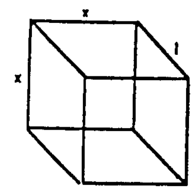
19. Let the radius of the cylinder be r cm, $0 < r < 10$. Then the height is $2\sqrt{100 - r^2}$ and the volume is

$V(r) = 2\pi r^2\sqrt{100-r^2}$ cm^3. Then

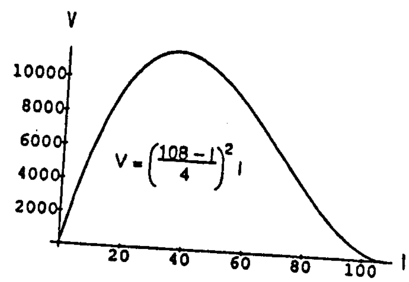
$$V'(r) = 2\pi r^2\left(\frac{1}{2\sqrt{100-r^2}}\right)(-2r) + \left(2\pi\sqrt{100-r^2}\right)(2r) = \frac{-2\pi r^3 + 4\pi r(100-r^2)}{\sqrt{100-r^2}} = \frac{2\pi r(200-3r^2)}{\sqrt{100-r^2}}$$

The critical point for $0 < r < 10$ occurs at $r = \sqrt{\frac{200}{3}} = 10\sqrt{\frac{2}{3}}$. Since $V'(r) > 0$ for $0 < r < 10\sqrt{\frac{2}{3}}$ and $V'(r) < 0$ for $10\sqrt{\frac{2}{3}} < r < 10$, the critical point corresponds to the maximum volume. The dimensions are $r = 10\sqrt{\frac{2}{3}} \approx 8.16$ cm and $h = \frac{20}{\sqrt{3}} \approx 11.55$ cm, and the volume is $\frac{4000\pi}{3\sqrt{3}} \approx 2418.40$ cm^3.

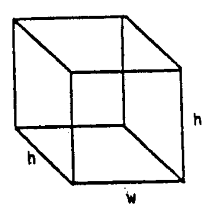
20. (a) From the diagram we have $4x + \ell = 108$ and $V = x^2\ell$. The volume of the box is $V(x) = x^2(108-4x)$, where $0 \le x < 27$. Then $V'(x) = 216x - 12x^2 = 12x(18-x) = 0 \Rightarrow$ the critical points are 0 and 18, but $x = 0$ results in no box. Since $V''(x) = 216 - 24x < 0$ at $x = 18$ we have a maximum. The dimensions of the box are $18 \times 18 \times 36$ in.



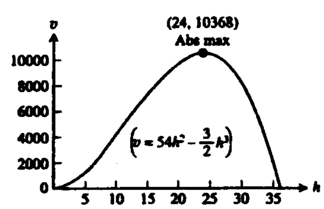
(b) In terms of length, $V(\ell) = x^2\ell = \left(\frac{108-\ell}{4}\right)^2 \ell$. The graph indicates that the maximum volume occurs near $\ell = 36$, which is consistent with the result of part (a).



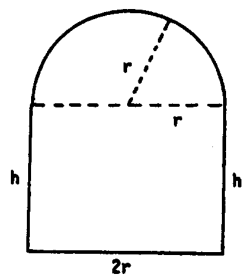
21. (a) From the diagram we have $3h + 2w = 108$ and $V = h^2w$ $\Rightarrow V(h) = h^2\left(54 - \frac{3}{2}h\right) = 54h^2 - \frac{3}{2}h^3$. Then $V'(h) = 108h - \frac{9}{2}h^2 = \frac{9}{2}h(24-h) = 0 \Rightarrow h = 0$ or $h = 24$, but $h = 0$ results in no box. Since $V''(h) = 108 - 9h < 0$ at $h = 24$, we have a maximum volume at $h = 24$ in. and $w = 54 - \frac{3}{2}h = 18$ in.



(b)



22. From the diagram the perimeter is $P = 2r + 2h + \pi r$, where r is the radius of the semicircle and h is the height of the rectangle. The amount of light transmitted is $A = 2rh + \frac{1}{4}\pi r^2 = r(P - 2r - \pi r) + \frac{1}{4}\pi r^2 = rP - 2r^2 - \frac{3}{4}\pi r^2$. Then $\frac{dA}{dr} = P - 4r - \frac{3}{2}\pi r = 0 \Rightarrow r = \frac{2P}{8+3\pi} \Rightarrow$ $2h = P - \frac{4P}{8+3\pi} - \frac{2\pi P}{8+3\pi} = \frac{(4+\pi)P}{8+3\pi}$. Therefore, $\frac{2r}{h} = \frac{8}{4+\pi}$ gives the proportions that admit the <u>most</u> light since $\frac{d^2A}{dr^2} = -4 - \frac{3}{2}\pi < 0$.



23. The fixed volume is $V = \pi r^2 h + \frac{2}{3}\pi r^3 \Rightarrow h = \frac{V}{\pi r^2} - \frac{2r}{3}$, where h is the height of the cylinder and r is the radius of the hemisphere. To minimize the cost we must minimize surface area of the cylinder added to twice the surface area of the hemisphere. Thus, we minimize $C = 2\pi rh + 4\pi r^2 = 2\pi r\left(\frac{V}{\pi r^2} - \frac{2r}{3}\right) + 4\pi r^2 = \frac{2V}{r} + \frac{8}{3}\pi r^2$. Then $\frac{dC}{dr} = -\frac{2V}{r^2} + \frac{16}{3}\pi r = 0 \Rightarrow V = \frac{8}{3}\pi r^3 \Rightarrow r = \left(\frac{3V}{8\pi}\right)^{1/3}$. From the volume equation, $h = \frac{V}{\pi r^2} - \frac{2r}{3}$ $= \frac{4V^{1/3}}{\pi^{1/3}\cdot 3^{2/3}} - \frac{2\cdot 3^{1/3}\cdot V^{1/3}}{3\cdot 2\cdot \pi^{1/3}} = \frac{3^{1/3}\cdot 2\cdot 4\cdot V^{1/3} - 2\cdot 3^{1/3}\cdot V^{1/3}}{3\cdot 2\cdot \pi^{1/3}} = \left(\frac{3V}{\pi}\right)^{1/3}$. Since $\frac{d^2C}{dr^2} = \frac{4V}{r^3} + \frac{16}{3}\pi > 0$, these dimensions do minimize the cost.

24. The volume of the trough is maximized when the area of the cross section is maximized. From the diagram the area of the cross section is $A(\theta) = \cos\theta + \sin\theta\cos\theta$, $0 < \theta < \frac{\pi}{2}$. Then $A'(\theta) = -\sin\theta + \cos^2\theta - \sin^2\theta$ $= -(2\sin^2\theta + \sin\theta - 1) = -(2\sin\theta - 1)(\sin\theta + 1)$ so $A'(\theta) = 0 \Rightarrow \sin\theta = \frac{1}{2}$ or $\sin\theta = -1 \Rightarrow \theta = \frac{\pi}{6}$ because $\sin\theta \neq -1$ when $0 < \theta < \frac{\pi}{2}$. Also, $A'(\theta) > 0$ for $0 < \theta < \frac{\pi}{6}$ and $A'(\theta) < 0$ for $\frac{\pi}{6} < \theta < \frac{\pi}{2}$. Therefore, at $\theta = \frac{\pi}{6}$ there is a maximum.

25. (a) From the diagram we have: $\overline{AP} = x$, $\overline{RA} = \sqrt{L^2 - x^2}$,

$\overline{PB} = 8.5 - x$, $\overline{CH} = \overline{DR} = 11 - \overline{RA} = 11 - \sqrt{L^2 - x^2}$,

$\overline{QB} = \sqrt{x^2 - (8.5 - x)^2}$, $\overline{HQ} = 11 - \overline{CH} - \overline{QB}$

$= 11 - \left[11 - \sqrt{L^2 - x^2} + \sqrt{x^2 - (8.5 - x)^2}\right]$

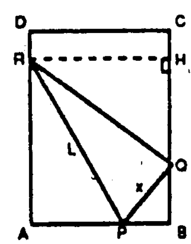
$= \sqrt{L^2 - x^2} - \sqrt{x^2 - (8.5 - x)^2}$,

$\overline{RQ}^2 = \overline{RH}^2 + \overline{HQ}^2 = (8.5)^2 + \left(\sqrt{L^2 - x^2} - \sqrt{x^2 - (8.5 - x)^2}\right)^2$. It follows that

$\overline{RP}^2 = \overline{PQ}^2 + \overline{RQ}^2 \Rightarrow L^2 = x^2 + \left(\sqrt{L^2 - x^2} - \sqrt{x^2 - (x - 8.5)^2}\right)^2 + (8.5)^2$

$\Rightarrow L^2 = x^2 + L^2 - x^2 - 2\sqrt{L^2 - x^2}\,\sqrt{17x - (8.5)^2} + 17x - (8.5)^2 + (8.5)^2$

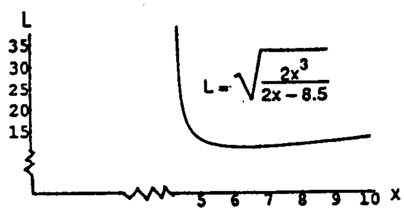
$\Rightarrow 17^2 x^2 = 4(L^2 - x^2)(17x - (8.5)^2) \Rightarrow L^2 = x^2 + \frac{17^2 x^2}{4[17x - (8.5)^2]} = \frac{17x^3}{17x - (8.5)^2}$



$$= \frac{17x^3}{17x - \left(\frac{17}{2}\right)^2} = \frac{4x^3}{4x - 17} = \frac{2x^3}{(2x - 8.5)}.$$

(b) If $f(x) = \frac{4x^3}{4x - 17}$ is minimized, then L^2 is minimized. Now $f'(x) = \frac{4x^2(8x - 51)}{(4x - 17)^2} \Rightarrow f'(x) < 0$ when $x < \frac{51}{8}$ and $f'(x) > 0$ when $x > \frac{51}{8}$. Thus L^2 is minimized when $x = \frac{51}{8}$.

(c) When $x = \frac{51}{8}$, then $L \approx 11.0$ in.



26. (a) From the figure in the text we have $P = 2x + 2y \Rightarrow y = \frac{P}{2} - x$. If $P = 36$, then $y = 18 - x$. When the cylinder is formed, $x = 2\pi r \Rightarrow r = \frac{x}{2\pi}$ and $h = y \Rightarrow h = 18 - x$. The volume of the cylinder is $V = \pi r^2 h$ $\Rightarrow V(x) = \frac{18x^2 - x^3}{4\pi}$. Solving $V'(x) = \frac{3x(12 - x)}{4\pi} = 0 \Rightarrow x = 0$ or 12; but when $x = 0$, there is no cylinder. Then $V''(x) = \frac{3}{\pi}\left(3 - \frac{x}{2}\right) \Rightarrow V''(12) < 0 \Rightarrow$ there is a maximum at $x = 12$. The values of $x = 12$ cm and $y = 6$ cm give the largest volume.

(b) In this case $V(x) = \pi x^2(18 - x)$. Solving $V'(x) = 3\pi x(12 - x) = 0 \Rightarrow x = 0$ or 12; but $x = 0$ would result in no cylinder. Then $V''(x) = 6\pi(6 - x) \Rightarrow V''(12) < 0 \Rightarrow$ there is a maximum at $x = 12$. The values of $x = 12$ cm and $y = 6$ cm give the largest volume.

27. Note that $h^2 + r^2 = 3$ and so $r = \sqrt{3 - h^2}$. Then the volume is given by $V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}(3 - h^2)h = \pi h - \frac{\pi}{3} h^3$ for $0 < h < \sqrt{3}$, and so $\frac{dV}{dh} = \pi - \pi h^2 = \pi(1 - h^2)$. The critical point (for $h > 0$) occurs at $h = 1$. Since $\frac{dV}{dh} > 0$ for $0 < h < 1$ and $\frac{dV}{dh} < 0$ for $1 < h < \sqrt{3}$, the critical point corresponds to the maximum volume. The cone of greatest volume has radius $\sqrt{2}$ m, height 1 m, and volume $\frac{2\pi}{3}$ m^3.

28. (a) $f(x) = x^2 + \frac{a}{x} \Rightarrow f'(x) = x^{-2}(2x^3 - a)$, so that $f'(x) = 0$ when $x = 2$ implies $a = 16$

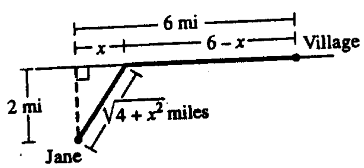
(b) $f(x) = x^2 + \frac{a}{x} \Rightarrow f''(x) = 2x^{-3}(x^3 + a)$, so that $f''(x) = 0$ when $x = 1$ implies $a = -1$

29. $A = xy = xe^{-x^2} \Rightarrow \frac{dA}{dx} = e^{-x^2} + (x)(-2x)e^{-x^2} = e^{-x^2}(1 - 2x^2)$. Solving $\frac{dA}{dx} = 0 \Rightarrow 1 - 2x^2 = 0$ $\Rightarrow x = \frac{1}{\sqrt{2}}$; $\frac{dA}{dx} < 0$ for $x > \frac{1}{\sqrt{2}}$ and $\frac{dA}{dx} > 0$ for $0 < x < \frac{1}{\sqrt{2}} \Rightarrow$ absolute maximum of $\frac{1}{\sqrt{2}}e^{-1/2} = \frac{1}{\sqrt{2e}}$ at $x = \frac{1}{\sqrt{2}}$ units long by $y = e^{-1/2} = \frac{1}{\sqrt{e}}$ units high.

30. $A = xy = x\left(\frac{\ln x}{x^2}\right) = \frac{\ln x}{x} \Rightarrow \frac{dA}{dx} = \frac{1}{x^2} - \frac{\ln x}{x^2} = \frac{1 - \ln x}{x^2}$. Solving $\frac{dA}{dx} = 0 \Rightarrow 1 - \ln x = 0 \Rightarrow x = e$; $\frac{dA}{dx} < 0$ for $x > e$ and $\frac{dA}{dx} > 0$ for $x < e \Rightarrow$ absolute maximum of $\frac{\ln e}{e} = \frac{1}{e}$ at $x = e$ units long and $y = \frac{1}{e^2}$ units high.

31. (a) $s(t) = -16t^2 + 96t + 112$
$v(t) = s'(t) = -32t + 96$
At $t = 0$, the velocity is $v(0) = 96$ ft/sec.
(b) The maximum height occurs when $v(t) = 0$, when $t = 3$. The maximum height is $s(3) = 256$ ft and it occurs at $t = 3$ sec.
(c) Note that $s(t) = -16t^2 + 96t + 112 = -16(t+1)(t-7)$, so $s = 0$ at $t = -1$ or $t = 7$. Choosing the positive value of t, the velocity when $s = 0$ is $v(7) = -128$ ft/sec.

32.

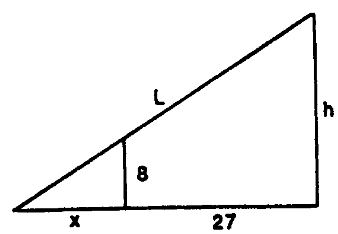


Let x be the distance from the point on the shoreline nearest Jane's boat to the point where she lands her boat. Then she needs to row $\sqrt{4+x^2}$ mi at 2 mph and walk $6 - x$ mi at 5 mph. The total amount of time to reach the village is $f(x) = \frac{\sqrt{4+x^2}}{2} + \frac{6-x}{5}$ hours $(0 \le x \le 6)$. Then $f'(x) = \frac{1}{2}\frac{1}{2\sqrt{4+x^2}}(2x) - \frac{1}{5} = \frac{x}{2\sqrt{4+x^2}} - \frac{1}{5}$.

Solving $f'(x) = 0$, we have: $\frac{x}{2\sqrt{4+x^2}} = \frac{1}{5} \Rightarrow 5x = 2\sqrt{4+x^2} \Rightarrow 25x^2 = 4(4+x^2) \Rightarrow 21x^2 = 16 \Rightarrow x = \pm\frac{4}{\sqrt{21}}$

We discard the negative value of x because it is not in the domain. Checking the endpoints and critical point, we have $f(0) = 2.2$, $f\left(\frac{4}{\sqrt{21}}\right) \approx 2.12$, and $f(6) \approx 3.16$. Jane should land her boat $\frac{4}{\sqrt{21}} \approx 0.87$ miles down the shoreline from the point nearest her boat.

33. $\frac{8}{x} = \frac{h}{x+27} \Rightarrow h = 8 + \frac{216}{x}$ and $L(x) = \sqrt{h^2 + (x+27)^2}$
$= \sqrt{\left(8 + \frac{216}{x}\right)^2 + (x+27)^2}$ when $x \ge 0$. Note that $L(x)$ is minimized when $f(x) = \left(8 + \frac{216}{x}\right)^2 + (x+27)^2$ is minimized.

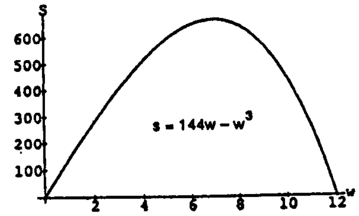


If $f'(x) = 0$, then $2\left(8 + \frac{216}{x}\right)\left(-\frac{216}{x^2}\right) + 2(x+27) = 0$
$\Rightarrow (x+27)\left(1 - \frac{1728}{x^3}\right) = 0 \Rightarrow x = -27$ (not acceptable since distance is never negative) or $x = 12$. Then $L(12) = \sqrt{2197} \approx 46.87$ ft.

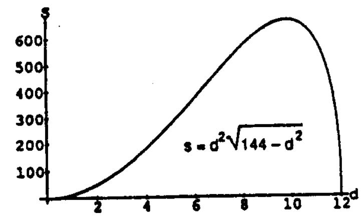
34. (a) From the diagram we have $d^2 = 144 - w^2$. The strength of the beam is $S = kwd^2 = kw(144 - w^2)$
$\Rightarrow S = 144kw - kw^3 \Rightarrow S'(w) = 144k - 3kw^2 = 3k(48 - w^2)$ so $S'(w) = 0 \Rightarrow w = \pm 4\sqrt{3}$;

$S''(4\sqrt{3}) < 0$ and $-4\sqrt{3}$ is not acceptable. Therefore $S(4\sqrt{3})$ is the maximum strength. The dimensions of the strongest beam are $4\sqrt{3}$ by $4\sqrt{6}$ inches.

(b)



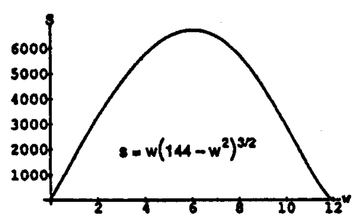
(c)



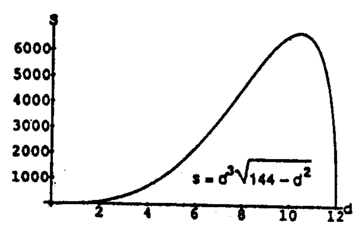
Both graphs indicate the same maximum value and are consistent with each other. Changing k does not change the dimensions that give the strongest beam (i.e., does not change the values of w and d that produce the strongest beam).

35. (a) From the situation we have $w^2 = 144 - d^2$. The stiffness of the beam is $S = kwd^3 = kd^3(144 - d^2)^{1/2}$, where $0 \le d \le 12$. Also, $S'(d) = \frac{4kd^2(108 - d^2)}{\sqrt{144 - d^2}} \Rightarrow$ critical points at 0, 12, and $6\sqrt{3}$. Both $d = 0$ and $d = 12$ cause $S = 0$. The maximum occurs at $d = 6\sqrt{3}$. The dimensions are 6 by $6\sqrt{3}$ inches.

(b)



(c)



Both graphs indicate the same maximum value and are consistent with each other. The changing of k has no effect.

36. $\theta = \pi - \cot^{-1}\left(\frac{x}{60}\right) - \cot^{-1}\left(\frac{5}{3} - \frac{x}{30}\right),\ 0 < x < 50 \Rightarrow \frac{d\theta}{dx} = \frac{\left(\frac{1}{60}\right)}{1 + \left(\frac{x}{60}\right)^2} + \frac{\left(-\frac{1}{30}\right)}{1 + \left(\frac{50 - x}{30}\right)^2}$

$= 30\left[\frac{2}{60^2 + x^2} - \frac{1}{30^2 + (50 - x)^2}\right]$; solving $\frac{d\theta}{dx} = 0 \Rightarrow x^2 - 200x + 3200 = 0 \Rightarrow x = 100 \pm 20\sqrt{17}$, but $100 + 20\sqrt{17}$ is not in the domain; $\frac{d\theta}{dx} > 0$ for $x < 20(5 - \sqrt{17})$ and $\frac{d\theta}{dx} < 0$ for $20(5 - \sqrt{17}) < x < 50$ $\Rightarrow x = 20(5 - \sqrt{17}) \approx 17.54$ m maximizes θ

37. (a) $s_1 = s_2 \Rightarrow \sin t = \sin\left(t + \frac{\pi}{3}\right) \Rightarrow \sin t = \sin t \cos \frac{\pi}{3} + \sin \frac{\pi}{3} \cos t \Rightarrow \sin t = \frac{1}{2} \sin t + \frac{\sqrt{3}}{2} \cos t \Rightarrow \tan t = \sqrt{3}$ $\Rightarrow t = \frac{\pi}{3}$ or $\frac{4\pi}{3}$

(b) The distance between the particles is $s(t) = |s_1 - s_2| = \left|\sin t - \sin\left(t + \frac{\pi}{3}\right)\right| = \frac{1}{2}\left|\sin t - \sqrt{3} \cos t\right|$

$\Rightarrow s'(t) = \frac{(\sin t - \sqrt{3} \cos t)(\cos t + \sqrt{3} \sin t)}{2\left|\sin t - \sqrt{3} \cos t\right|}$ since $\frac{d}{dx}|x| = \frac{x}{|x|} \Rightarrow$ critical times and endpoints are $0, \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}, 2\pi$; then $s(0) = \frac{\sqrt{3}}{2}$, $s\left(\frac{\pi}{3}\right) = 0$, $s\left(\frac{5\pi}{6}\right) = 1$, $s\left(\frac{4\pi}{3}\right) = 0$, $s\left(\frac{11\pi}{6}\right) = 1$, $s(2\pi) = \frac{\sqrt{3}}{2} \Rightarrow$ the greatest distance between the particles is 1

(c) Since $s'(t) = \frac{(\sin t - \sqrt{3}\cos t)(\cos t + \sqrt{3}\sin t)}{2\left|\sin t - \sqrt{3}\cos t\right|}$ we can conclude that at $t = \frac{\pi}{3}$ and $\frac{4\pi}{3}$, $s'(t)$ has cusps and the distance between the particles is changing the fastest near these points

38. $\frac{di}{dt} = -2\sin t + 2\cos t$, solving $\frac{di}{dt} = 0 \Rightarrow \tan t = 1 \Rightarrow t = \frac{\pi}{4} + n\pi$ where n is a nonnegative integer (in this Exercise t is never negative) $\Rightarrow$ the peak current is $2\sqrt{2}$ amps

39. (a) $s = 10\cos(\pi t) \Rightarrow v = -10\pi\sin(\pi t) \Rightarrow$ speed $= \left|10\pi\sin(\pi t)\right| = 10\pi\left|\sin(\pi t)\right| \Rightarrow$ the maximum speed is $10\pi \approx 31.42$ cm/sec since the maximum value of $\left|\sin(\pi t)\right|$ is 1; the cart is moving the fastest at t = 0.5 sec, 1.5 sec, 2.5 sec and 3.5 sec when $\left|\sin(\pi t)\right|$ is 1. At these times the distance is $s = 10\cos\left(\frac{\pi}{2}\right) = 0$ cm and $a = -10\pi^2\cos(\pi t) \Rightarrow |a| = 10\pi^2\left|\cos(\pi t)\right| \Rightarrow |a| = 0$ cm/sec^2

(b) $|a| = 10\pi^2\left|\cos(\pi t)\right|$ is greatest at t = 0.0 sec, 1.0 sec, 2.0 sec, 3.0 sec and 4.0 sec, and at these times the magnitude of the cart's position is $|s| = 10$ cm from the rest position and the speed is 0 cm/sec.

40. (a) $2\sin t = \sin 2t \Rightarrow 2\sin t - 2\sin t\cos t = 0 \Rightarrow (2\sin t)(1 - \cos t) = 0 \Rightarrow t = k\pi$ where k is a positive integer

(b) The vertical distance between the masses is $s(t) = |s_1 - s_2| = \left((s_1 - s_2)^2\right)^{1/2} = \left((\sin 2t - 2\sin t)^2\right)^{1/2}$

$\Rightarrow s'(t) = \left(\frac{1}{2}\right)\left((\sin 2t - 2\sin t)^2\right)^{-1/2}(2)(\sin 2t - 2\sin t)(2\cos 2t - 2\cos t)$

$= \frac{2(\cos 2t - \cos t)(\sin 2t - 2\sin t)}{|\sin 2t - 2\sin t|} = \frac{4(2\cos t + 1)(\cos t - 1)(\sin t)(\cos t - 1)}{|\sin 2t - 2\sin t|} \Rightarrow$ critical times at $0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$; then $s(0) = 0$, $s\left(\frac{2\pi}{3}\right) = \left|\sin\left(\frac{4\pi}{3}\right) - 2\sin\left(\frac{2\pi}{3}\right)\right| = \frac{3\sqrt{3}}{2}$, $s(\pi) = 0$, $s\left(\frac{4\pi}{3}\right)$

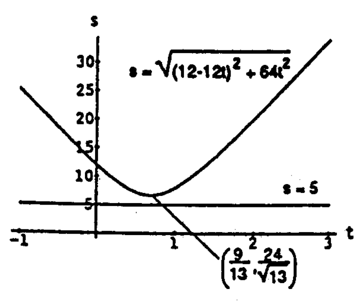
$= \left|\sin\left(\frac{8\pi}{3}\right) - 2\sin\left(\frac{4\pi}{3}\right)\right| = \frac{3\sqrt{3}}{2}$, $s(2\pi) = 0 \Rightarrow$ the greatest distance is $\frac{3\sqrt{3}}{2}$ at $t = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$

41. (a) $s = \sqrt{(12 - 12t)^2 + (8t)^2} = \left((12 - 12t)^2 + 64t^2\right)^{1/2}$

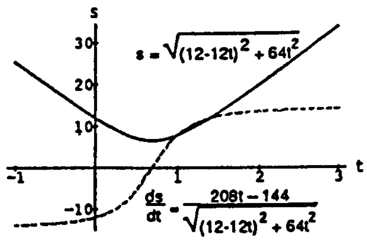
(b) $\frac{ds}{dt} = \frac{1}{2}\left((12 - 12t)^2 + 64t^2\right)^{-1/2}[2(12 - 12t)(-12) + 128t] = \frac{208t - 144}{\sqrt{(12 - 12t)^2 + 64t^2}}$

$\Rightarrow \left.\frac{ds}{dt}\right|_{t=0} = -12$ knots and $\left.\frac{ds}{dt}\right|_{t=1} = 8$ knots

(c) The graph indicates that the ships did not see each other because $s(t) > 5$ for all values of t.



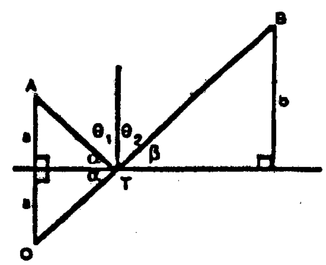
(d) The graph supports the conclusions in parts (b) and (c).



(e) $\lim_{t\to\infty} \frac{ds}{dt} = \sqrt{\lim_{t\to\infty} \frac{(208t-144)^2}{144(1-t)^2+64t^2}} = \sqrt{\lim_{t\to\infty} \frac{\left(208-\frac{144}{t}\right)^2}{144\left(\frac{1}{t}-1\right)^2+64}} = \sqrt{\frac{208^2}{144+64}} = \sqrt{208} = 4\sqrt{13}$

which equals the square root of the sums of the squares of the individual speeds.

42. The distance $\overline{OT} + \overline{TB}$ is minimized when $\overline{OB}$ is a straight line. Hence $\angle\alpha = \angle\beta \Rightarrow \theta_1 = \theta_2$.



43. If $v = kax - kx^2$, then $v' = ka - 2kx$ and $v'' = -2k$, so $v' = 0 \Rightarrow x = \frac{a}{2}$. At $x = \frac{a}{2}$ there is a maximum since $v''\left(\frac{a}{2}\right) = -2k < 0$. The maximum value of v is $\frac{ka^2}{4}$.

44. (a) According to the graph, $y'(0) = 0$.

(b) According to the graph, $y'(-L) = 0$.

(c) $y(0) = 0$, so $d = 0$.

Now $y'(x) = 3ax^2 + 2bx + c$, so $y'(0) = 0$ implies that $c = 0$. Therefore, $y(x) = ax^3 + bx^2$ and $y'(x) = 3ax^2 + 2bx$. Then $y(-L) = -aL^3 + bL^2 = H$ and $y'(-L) = 3aL^2 - 2bL = 0$, so we have two linear equations in the two unknowns a and b. The second equation gives $b = \frac{3aL}{2}$. Substituting into the first equation, we have $-aL^3 + \frac{3aL^3}{2} = H$, or $\frac{aL^3}{2} = H$, so $a = 2\frac{H}{L^3}$. Therefore, $b = 3\frac{H}{L^2}$ and the equation for y is $y(x) = 2\frac{H}{L^3}x^3 + 3\frac{H}{L^2}x^2$, or $y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right]$.

45. The profit is $p = nx - nc = n(x-c) = \left[a(x-c)^{-1} + b(100-x)\right](x-c) = a + b(100-x)(x-c)$ $= a + (bc + 100b)x - 100bc - bx^2$. Then $p'(x) = bc + 100b - 2bx$ and $p''(x) = -2b$. Solving $p'(x) = 0 \Rightarrow$ $x = \frac{c}{2} + 50$. At $x = \frac{c}{2} + 50$ there is a maximum profit since $p''(x) = -2b < 0$ for all x.

46. Let x represent the number of people over 50. The profit is $p(x) = (50+x)(200-2x) - 32(50+x) - 6000$ $= -2x^2 + 68x + 2400$. Then $p'(x) = -4x + 68$ and $p'' = -4$. Solving $p'(x) = 0 \Rightarrow x = 17$. At $x = 17$ there is a maximum since $p''(17) < 0$. It would take 67 people to maximize the profit.

47. (a) $A(q) = kmq^{-1} + cm + \frac{h}{2}q$, where $q > 0 \Rightarrow A'(q) = -kmq^{-2} + \frac{h}{2} = \frac{hq^2 - 2km}{2q^2}$ and $A''(q) = 2kmq^{-3}$. The critical points are $-\sqrt{\frac{2km}{h}}$, 0, and $\sqrt{\frac{2km}{h}}$, but only $\sqrt{\frac{2km}{h}}$ is in the domain. Then $A''\left(\sqrt{\frac{2km}{h}}\right) > 0 \Rightarrow$ at $q = \sqrt{\frac{2km}{h}}$ there is a minimum average weekly cost.

(b) $A(q) = \frac{(k+bq)m}{q} + cm + \frac{h}{2}q = kmq^{-1} + bm + cm + \frac{h}{2}q$, where $q > 0 \Rightarrow A'(q) = 0$ at $q = \sqrt{\frac{2km}{h}}$ as in (a). Also $A''(q) = 2kmq^{-3} > 0$ so the most economical quantity to order is still $q = \sqrt{\frac{2km}{h}}$ which minimizes the average weekly cost.

48. We start with $c(x)$ = the cost of producing x items, $x > 0$, and $\frac{c(x)}{x}$ = the average cost of producing x items, assumed to be differentiable. If the average cost can be minimized, it will be at a production level at which $\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 0 \Rightarrow \frac{xc'(x) - c(x)}{x^2} = 0$ (by the quotient rule) $\Rightarrow xc'(x) - c(x) = 0$ (multiply both sides by x^2) $\Rightarrow c'(x) = \frac{c(x)}{x}$ where $c'(x)$ is the marginal cost. This concludes the proof. (Note: The theorem does not assure a production level that will give a minimum average cost, but rather, it indicates where to look to see if there is one. Find the production levels where the average cost equals the marginal cost, then check to see if any of them give a minimum.)

49. The profit $p(x) = r(x) - c(x) = 6x - (x^3 - 6x^2 + 15x) = -x^3 + 6x^2 - 9x$, where $x \geq 0$. Then $p'(x) = -3x^2 + 12x - 9 = -3(x-3)(x-1)$ and $p''(x) = -6x + 12$. The critical points are 1 and 3. Thus $p''(1) = 6 > 0 \Rightarrow$ at $x = 1$ there is a local minimum, and $p''(3) = -6 < 0 \Rightarrow$ at $x = 3$ there is a local maximum. But $p(3) = 0 \Rightarrow$ the best you can do is break even.

50. The cost is $c(x) = x^3 - 20x^2 + 20{,}000x$, where $x > 0$. Since the production level at which average cost is smallest is a level at which the average cost equals the marginal cost, the solution of $\frac{c(x)}{x} = c'(x)$ will minimize the average cost of making x items. Now $\frac{x^3 - 20x^2 + 20{,}000x}{x} = 3x^2 - 40x + 20{,}000 \Rightarrow x^2 - 20x + 20{,}000$ $= 3x^2 - 40x + 20{,}000 \Rightarrow 2x^2 - 20x = 0 \Rightarrow 2x(x-10) = 0 \Rightarrow x = 0$ or $x = 10 \Rightarrow x = 10$ since $x > 0$.

51. (a) The artisan should order px units of material in order to have enough until the next delivery.

(b) The average number of units in storage until the next delivery is $\frac{px}{2}$ and so the cost of storing them is $s\left(\frac{px}{2}\right)$ per day, and the total cost for x days is $\left(\frac{px}{2}\right)sx$. When added to the delivery cost, the total cost for delivery and storage for each cycle is: cost per cycle $= d + \frac{px}{2}sx$.

(c) The average cost per day for storage and delivery of materials is:

average cost per day $= \frac{\left(d + \frac{ps}{2}x^2\right)}{x} = \frac{d}{x} + \frac{ps}{2}x$. To minimize the average cost per day, set the derivative equal to zero. $\frac{d}{dx}\left(d(x)^{-1} + \frac{ps}{2}x\right) = -d(x)^{-2} + \frac{ps}{2} = 0 \Rightarrow x = \pm\sqrt{\frac{2d}{ps}}$. Only the positive root makes sense in this context so that $x^* = \sqrt{\frac{2d}{ps}}$. To verify that x^* gives a minimum, check the second derivative

$$\left[\frac{d}{dx}\left(-dx^{-2} + \frac{ps}{2}\right)\right]\Big|_{\sqrt{\frac{2d}{ps}}} = \frac{2d}{x^3}\Big|_{\sqrt{\frac{2d}{ps}}} = \frac{2d}{\left(\sqrt{\frac{2d}{ps}}\right)^3} > 0 \Rightarrow \text{a minimum.}$$

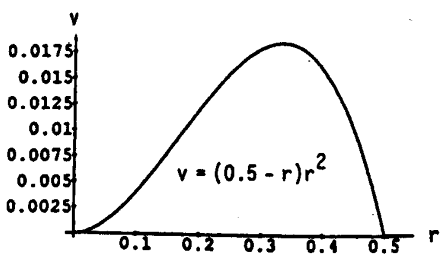
(d) The line and hyperbola intersect when $\frac{d}{x} = \frac{ps}{2}x$. Solving for x gives $x_{intersection} = \pm\sqrt{\frac{2d}{ps}}$. For $x > 0$, $x_{intersection} = \sqrt{\frac{2d}{ps}} = x^*$. From this result, the average cost per day is minimized when the average daily cost of delivery is equal to the average daily cost of storage.

52. Average Cost: $\frac{c(x)}{x} = \frac{2000}{x} + 96 + 4x^{1/2} \Rightarrow \frac{d}{dx}\left(\frac{c(x)}{x}\right) = -\frac{2000}{x^2} + 2x^{-1/2} = 0 \Rightarrow x = 100$. Check for a minimum: $\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right)\Big|_{x=100} = \frac{4000}{100^3} - 100^{-3/2} = 0.003 > 0 \Rightarrow$ a minimum at $x = 100$. At a production level of 100,000 units, the average cost will be minimized at \$156 per unit.

53. We have $\frac{dR}{dM} = CM - M^2$. Solving $\frac{d^2R}{dM^2} = C - 2M = 0 \Rightarrow M = \frac{C}{2}$. Also, $\frac{d^3R}{dM^3} = -2 < 0 \Rightarrow$ at $M = \frac{C}{2}$ there is a maximum.

54. (a) If $v = cr_0r^2 - cr^3$, then $v' = 2cr_0r - 3cr^2 = cr(2r_0 - 3r)$ and $v'' = 2cr_0 - 6cr = 2c(r_0 - 3r)$. The solution of $v' = 0$ is $r = 0$ or $\frac{2r_0}{3}$, but 0 is not in the domain. Also, $v' > 0$ for $r < \frac{2r_0}{3}$ and $v' < 0$ for $r > \frac{2r_0}{3} \Rightarrow$ at $r = \frac{2r_0}{3}$ there is a maximum.

(b)



55. If $x > 0$, then $(x-1)^2 \geq 0 \Rightarrow x^2 + 1 \geq 2x \Rightarrow \frac{x^2+1}{x} \geq 2$. In particular if a, b, c and d are positive integers, then $\left(\frac{a^2+1}{a}\right)\left(\frac{b^2+1}{b}\right)\left(\frac{c^2+1}{c}\right)\left(\frac{d^2+1}{d}\right) \geq 16$.

56. (a) $f(x) = \frac{x}{\sqrt{a^2+x^2}} \Rightarrow f'(x) = \frac{(a^2+x^2)^{1/2} - x^2(a^2+x^2)^{-1/2}}{(a^2+x^2)} = \frac{a^2+x^2-x^2}{(a^2+x^2)^{3/2}} = \frac{a^2}{(a^2+x^2)^{3/2}} > 0$

$\Rightarrow$ f(x) is an increasing function of x

(b) $g(x) = \frac{d-x}{\sqrt{b^2+(d-x)^2}} \Rightarrow g'(x) = \frac{-\left(b^2+(d-x)^2\right)^{1/2} + (d-x)^2\left(b^2+(d-x)^2\right)^{-1/2}}{b^2+(d-x)^2}$

$= \frac{-\left(b^2+(d-x)^2\right)+(d-x)^2}{\left(b^2+(d-x)^2\right)^{3/2}} = \frac{-b^2}{\left(b^2+(d-x)^2\right)^{3/2}} < 0 \Rightarrow g(x)$ is a decreasing function of x

(c) Since $c_1, c_2 > 0$, the derivative $\frac{dt}{dx}$ is an increasing function of x (from part (a)) minus a decreasing function of x (from part (b)): $\frac{dx}{dt} = \frac{1}{c_1}f(x) - \frac{1}{c_2}g(x) \Rightarrow \frac{d^2x}{dt^2} = \frac{1}{c_1}f'(x) - \frac{1}{c_2}g'(x) > 0$ since $f'(x) > 0$ and $g'(x) < 0 \Rightarrow \frac{dx}{dt}$ is an increasing function of x.

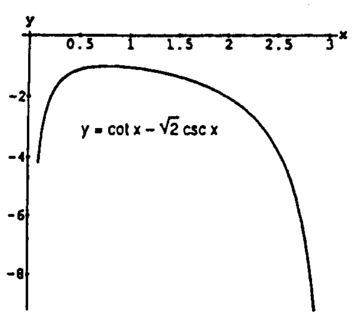
57. At $x = c$, the tangents to the curves are parallel. Justification: The vertical distance between the curves is $D(x) = f(x) - g(x)$, so $D'(x) = f'(x) - g'(x)$. The maximum value of D will occur at a point c where $D' = 0$. At such a point, $f'(c) = g'(c) = 0$, or $f'(c) = g'(c)$.

58. (a) $f(x) = 3 + 4\cos x + \cos 2x$ is a periodic function with period 2π

(b) No, $f(x) = 3 + 4\cos x + \cos 2x = 3 + 4\cos x + \left(2\cos^2 x - 1\right) = 2\left(1 + 2\cos x + \cos^2 x\right) = 2(1+\cos x)^2 \geq 0$ $\Rightarrow f(x)$ is never negative

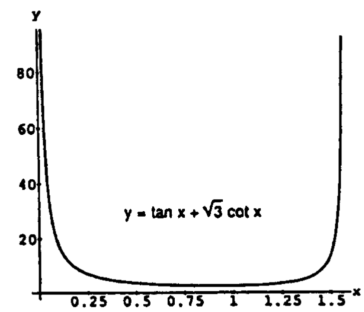
59. (a) If $y = \cot x - \sqrt{2}\csc x$ where $0 < x < \pi$, then $y' = (\csc x)\left(\sqrt{2}\cot x - \csc x\right)$. Solving $y' = 0$ $\Rightarrow \cos x = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{\pi}{4}$. For $0 < x < \frac{\pi}{4}$ we have $y' > 0$, and $y' < 0$ when $\frac{\pi}{4} < x < \pi$. Therefore, at $x = \frac{\pi}{4}$ there is a maximum value of $y = -1$.

(b)



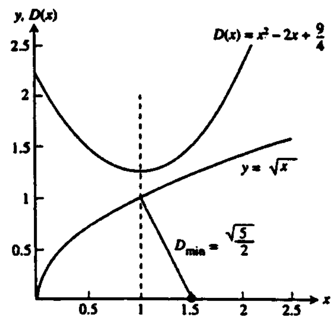
60. (a) If $y = \tan x + 3\cot x$ where $0 < x < \frac{\pi}{2}$, then $y' = \sec^2 x - 3\csc^2 x$. Solving $y' = 0 \Rightarrow \tan x = \pm\sqrt{3}$ $\Rightarrow x = \pm\frac{\pi}{3}$, but $-\frac{\pi}{3}$ is not in the domain. Also, $y'' = 2\sec^2 x \tan x + 6\csc^2 x \cot x > 0$ for all $0 < x < \frac{\pi}{2}$. Therefore at $x = \frac{\pi}{3}$ there is a minimum value of $y = 2\sqrt{3}$.

(b)



61. (a) The square of the distance is $D(x) = \left(x - \frac{3}{2}\right)^2 + \left(\sqrt{x} + 0\right)^2 = x^2 - 2x + \frac{9}{4}$, so $D'(x) = 2x - 2$ and the critical point occurs at $x = 1$. Since $D'(x) < 0$ for $x < 1$ and $D'(x) > 0$ for $x > 1$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(1)} = \frac{\sqrt{5}}{2}$.

(b)



The minimum distance is from the point $(3/2, 0)$ to the point $(1, 1)$ on the graph of $y = \sqrt{x}$, and this occurs at the value $x = 1$ where $D(x)$, the distance squared, has its minimum value.

62. (a) Calculus method:

The square of the distance from the point $(1, \sqrt{3})$ to $\left(x, \sqrt{16 - x^2}\right)$ is given by

$D(x) = (x-1)^2 + \left(\sqrt{16 - x^2} - \sqrt{3}\right)^2 = x^2 - 2x + 1 + 16 - x^2 - 2\sqrt{48 - 3x^2} + 3 = -2x + 20 - 2\sqrt{48 - 3x^2}$.

Then $D'(x) = -2 - \frac{2}{2\sqrt{48 - 3x^2}}(-6x) = -2 + \frac{6x}{\sqrt{48 - 3x^2}}$. Solving $D'(x) = 0$ we have:

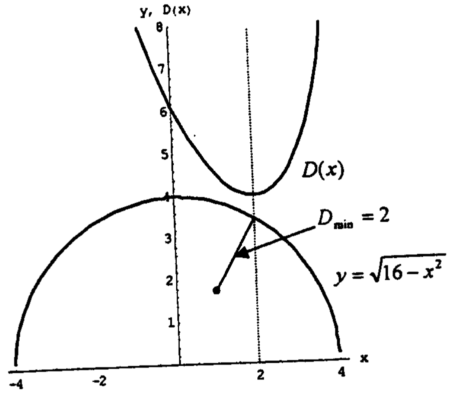
$6x = 2\sqrt{48 - 3x^2} \Rightarrow 36x^2 = 4(48 - 3x^2) \Rightarrow 9x^2 = 48 - 3x^2 \Rightarrow 12x^2 = 48 \Rightarrow x = \pm 2$

We discard $x = -2$ as an extraneous solution, leaving $x = 2$. Since $D'(x) < 0$ for $-4 < x < 2$ and $D'(x) > 0$ for $2 < x < 4$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(2)} = 2$.

Geometry method:

The semicircle is centered at the origin and has radius 4. The distance from the origin to $(1, \sqrt{3})$ is $\sqrt{1^2 + (\sqrt{3})^2} = 2$. The shortest distance from the point to the semicircle is the distance along the radius containing the point $(1, \sqrt{3})$. That distance is $4 - 2 = 2$.

(b)



The minimum distance is from the point $(1, \sqrt{3})$ to the point $(2, 2\sqrt{2})$ on the graph of $y = \sqrt{16 - x^2}$, and this occurs at the value $x = 2$ where D(x), the distance squared, has its minimum value.

63. (a) The base radius of the cone is $r = \frac{2\pi a - x}{2\pi}$ and so the height is $h = \sqrt{a^2 - r^2} = \sqrt{a^2 - \left(\frac{2\pi a - x}{2\pi}\right)^2}$.

Therefore, $V(x) = \frac{\pi}{3} r^2 h = \frac{\pi}{3}\left(\frac{2\pi a - x}{2\pi}\right)^2 \sqrt{a^2 - \left(\frac{2\pi a - x}{2\pi}\right)^2}$.

(b) To simplify the calculations, we shall consider the volume as a function of r:

volume $= f(r) = \frac{\pi}{3} r^2 \sqrt{a^2 - r^2}$, where $0 < r < a$.

$$f'(r) = \frac{\pi}{3}\frac{d}{dr}\sqrt{r^2(a^2 - r^2)} = \frac{\pi}{3}\left[r^2 \cdot \frac{1}{2\sqrt{a^2 - r^2}} \cdot (-2r) + \left(\sqrt{a^2 - r^2}\right)(2r)\right] = \frac{\pi}{3}\left[\frac{-r^3 + 2r\left(a^2 - r^2\right)}{\sqrt{a^2 - r^2}}\right]$$

$$= \frac{\pi}{3}\left[\frac{2a^2 r - 3r^3}{\sqrt{a^2 - r^2}}\right] = \frac{\pi r\left(2a^2 - 3r^2\right)}{3\sqrt{a^2 - r^2}}$$

The critical point occurs when $r^2 = \frac{2a^2}{3}$, which gives $r = a\sqrt{\frac{2}{3}} = \frac{a\sqrt{6}}{3}$. Then $h = \sqrt{a^2 - r^2} = \sqrt{a^2 - \frac{2a^2}{3}} = \sqrt{\frac{a^2}{3}} = \frac{a\sqrt{3}}{3}$. Using $r = \frac{a\sqrt{6}}{3}$ and $h = \frac{a\sqrt{3}}{3}$, we may now find the values of r and h for the given values of a.

When $a = 4$: $r = \frac{4\sqrt{6}}{3}$, $h = \frac{4\sqrt{3}}{3}$; when $a = 5$: $r = \frac{5\sqrt{6}}{3}$, $h = \frac{5\sqrt{3}}{3}$; when $a = 6$: $r = 2\sqrt{6}$, $h = 2\sqrt{3}$;

when $a = 8$: $r = \frac{8\sqrt{6}}{3}$, $h = \frac{8\sqrt{3}}{3}$

(c) Since $r = \frac{a\sqrt{6}}{3}$ and $h = \frac{a\sqrt{3}}{3}$, the relationship is $\frac{r}{h} = \sqrt{2}$.

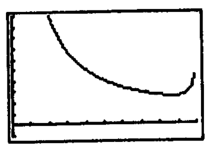
64. (a) Let x_0 represent the fixed value of x at point P, so that P has coordinates (x_0, a), and let $m = f'(x_0)$ be the slope of line RT. Then the equation of line RT is $y = m(x - x_0) + a$. The y-intercept of this line is $m(0 - x_0) + a = a - mx_0$, and the x-intercept is the solution of $m(x - x_0) + a = 0$, or $x = \frac{mx_0 - a}{m}$. Let O designate the origin. Then

(Area of triangle RST) = 2(Area of triangle ORT) = $2 \cdot \frac{1}{2}$(x-intercept of line RT)(y-intercept of line RT)

$$= 2 \cdot \frac{1}{2}\left(\frac{mx_0 - a}{m}\right)(a - mx_0) = -m\left(\frac{mx_0 - a}{m}\right)\left(\frac{mx_0 - a}{m}\right) = -m\left(\frac{mx_0 - a}{m}\right)^2 = -m\left(x_0 - \frac{a}{m}\right)^2$$

Substituting x for x_0, $f'(x)$ for m, and $f(x)$ for a, we have $A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2$.

(b) The domain is the open interval (0, 10). To graph, let $y_1 = f(x) = 5 + 5\sqrt{1 - \frac{x^2}{100}}$, $y_2 = f'(x) = \text{NDER}(y_1)$, and $y_3 = A(x) = -y_2\left(x - \frac{y_1}{y_2}\right)^2$. The graph of the area function $y_3 = A(x)$ is shown below.



[0, 10] by [−100, 1000]

The vertical asymptotes at x = 0 and x = 10 correspond to horizontal or vertical tangent lines, which do not form triangles.

(c) Using our expression for the y-intercept of the tangent line, the height of the triangle is

$$a - mx = f(x) - f'(x) \cdot x = 5 + \frac{1}{2}\sqrt{100 - x^2} - \frac{-x}{2\sqrt{100 - x^2}}x = 5 + \frac{1}{2}\sqrt{100 - x^2} + \frac{x^2}{2\sqrt{100 - x^2}}$$

We may use graphing methods or the analytic method in part (d) to find that the minimum value of A(x) occurs at $x \approx 8.66$. Substituting this value into the expression above, the height of the triangle is 15. This is 3 times the y-coordinate of the center of the ellipse.

(d) Part (a) remains unchanged. The domain is (0, C). To graph, note that

$f(x) = B + B\sqrt{1 - \frac{x^2}{C^2}} = B + \frac{B}{C}\sqrt{C^2 - x^2}$ and $f'(x) = \frac{B}{C}\frac{1}{2\sqrt{C^2 - x^2}}(-2x) = \frac{-Bx}{C\sqrt{C^2 - x^2}}$. Therefore we have

$$A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2 = \frac{Bx}{C\sqrt{C^2 - x^2}}\left[x - \frac{B + \frac{B}{C}\sqrt{C^2 - x^2}}{\frac{-Bx}{C\sqrt{C^2 - x^2}}}\right]^2$$

$$= \frac{Bx}{C\sqrt{C^2 - x^2}}\left[x - \frac{(BC + B\sqrt{C^2 - x^2})(\sqrt{C^2 - x^2})}{-Bx}\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2-x^2}}\left[Bx^2 + \left(BC + B\sqrt{C^2-x^2}\right)\left(\sqrt{C^2-x^2}\right)\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2-x^2}}\left[Bx^2 + BC\sqrt{C^2-x^2} + B\left(C^2-x^2\right)\right]^2 = \frac{1}{BCx\sqrt{C^2-x^2}}\left[BC\left(C+\sqrt{C^2-x^2}\right)\right]^2$$

$$= \frac{BC\left(C+\sqrt{C^2-x^2}\right)^2}{x\sqrt{C^2-x^2}}$$

$$A'(x) = BC \cdot \frac{\left(x\sqrt{C^2-x^2}\right)(2)\left(C+\sqrt{C^2-x^2}\right)\left(\frac{-x}{\sqrt{C^2-x^2}}\right) - \left(C+\sqrt{C^2-x^2}\right)^2\left(x\frac{-x}{\sqrt{C^2-x^2}} + \sqrt{C^2-x^2}(1)\right)}{x^2\left(C^2-x^2\right)}$$

$$= \frac{BC\left(C+\sqrt{C^2-x^2}\right)}{x^2\left(C^2-x^2\right)}\left[-2x^2 - \left(C+\sqrt{C^2-x^2}\right)\left(\frac{-x^2}{\sqrt{C^2-x^2}} + \sqrt{C^2-x^2}\right)\right]$$

$$= \frac{BC\left(C+\sqrt{C^2-x^2}\right)}{x^2\left(C^2-x^2\right)}\left[-2x^2 + \frac{Cx^2}{\sqrt{C^2-x^2}} - C\sqrt{C^2-x^2} + x^2 - \left(C^2-x^2\right)\right]$$

$$= \frac{BC\left(C+\sqrt{C^2-x^2}\right)}{x^2\left(C^2-x^2\right)}\left(\frac{Cx^2}{\sqrt{C^2-x^2}} - C\sqrt{C^2-x^2} - C^2\right)$$

$$= \frac{BC\left(C+\sqrt{C^2-x^2}\right)}{x^2\left(C^2-x^2\right)^{3/2}}\left[Cx^2 - C\left(C^2-x^2\right) - C^2\sqrt{C^2-x^2}\right]$$

$$= \frac{BC^2\left(C+\sqrt{C^2-x^2}\right)}{x^2\left(C^2-x^2\right)^{3/2}}\left(2x^2 - C^2 - C\sqrt{C^2-x^2}\right)$$

To find the critical points for $0 < x < C$, we solve:

$$2x^2 - C^2 = C\sqrt{C^2-x^2} \Rightarrow 4x^4 - 4C^2x^2 + C^4 = C^4 - C^2x^2 \Rightarrow 4x^4 - 3C^2x^2 = 0 \Rightarrow x^2(4x^2 - 3C^2) = 0$$

The minimum value of $A(x)$ for $0 < x < C$ occurs at the critical point $x = \frac{C\sqrt{3}}{2}$, or $x^2 = \frac{3C^2}{4}$. The corresponding triangle height is

$$a - mx = f(x) - f'(x)\cdot x = B + \frac{B}{C}\sqrt{C^2-x^2} + \frac{Bx^2}{C\sqrt{C^2-x^2}} = B + \frac{B}{C}\sqrt{C^2 - \frac{3C^2}{4}} + \frac{B\left(\frac{3C^2}{4}\right)}{C\sqrt{C^2-\frac{3C^2}{4}}}$$

$$= B + \frac{B}{C}\left(\frac{C}{2}\right) + \frac{\frac{3BC^2}{4}}{\frac{C^2}{2}} = B + \frac{B}{2} + \frac{3B}{2} = 3B$$

This shows that the triangle has minimum area when its height is 3B.

3.6 LINEARIZATION AND DIFFERENTIALS

1. $f(x) = x^3 - 2x + 3 \Rightarrow f'(x) = 3x^2 - 2 \Rightarrow L(x) = f'(2)(x-2) + f(2) = 10(x-2) + 7 \Rightarrow L(x) = 10x - 13$ at $x = 2$

2. $f(x) = \sqrt{x^2+9} = (x^2+9)^{1/2} \Rightarrow f'(x) = \left(\frac{1}{2}\right)(x^2+9)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+9}} \Rightarrow L(x) = f'(-4)(x+4) + f(-4)$
$= -\frac{4}{5}(x+4) + 5 \Rightarrow L(x) = -\frac{4}{5}x + \frac{9}{5}$ at $x = -4$

3. $f(x) = x + \frac{1}{x} \Rightarrow f'(x) = 1 - x^{-2} \Rightarrow L(x) = f(1) + f'(1)(x-1) = 2 + 0(x-1) = 2$

4. $f(x) = x^{1/3} \Rightarrow f'(x) = \frac{1}{3x^{2/3}} \Rightarrow L(x) = f'(-8)(x-(-8)) + f(-8) = \frac{1}{12}(x+8) - 2 \Rightarrow L(x) = \frac{x}{12} - \frac{4}{3}$

5. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow L(x) = f(\pi) + f'(\pi)(x-\pi) = 0 + 1(x-\pi) = x - \pi$

6. (a) $f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow L(x) = f'(0)(x-0) + f(0) = x \Rightarrow L(x) = x$
(b) $f(x) = \cos x \Rightarrow f'(x) = -\sin x \Rightarrow L(x) = f'(0)(x-0) + f(0) = 1 \Rightarrow L(x) = 1$
(c) $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow L(x) = f'(0)(x-0) + f(0) = x \Rightarrow L(x) = x$
(d) $f(x) = e^x \Rightarrow f'(x) = e^x \Rightarrow L(x) = f'(0)(x-0) + f(0) = x + 1 \Rightarrow L(x) = x + 1$
(e) $f(x) = \ln(1+x) \Rightarrow f'(x) = 1/(1+x) \Rightarrow L(x) = f'(0)(x-0) + f(0) = x \Rightarrow L(x) = x$

7. $f'(x) = k(1+x)^{k-1}$
We have $f(0) = 1$ and $f'(0) = k$. $L(x) = f(0) + f'(0)(x-0) = 1 + k(x-0) = 1 + kx$

8. (a) $f(x) = (1-x)^6 = [1+(-x)]^6 \approx 1 + 6(-x) = 1 - 6x$
(b) $f(x) = \frac{2}{1-x} = 2[1+(-x)]^{-1} \approx 2[1+(-1)(-x)] = 2 + 2x$
(c) $f(x) = (1+x)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)x = 1 - \frac{x}{2}$
(d) $f(x) = \sqrt{2+x^2} = \sqrt{2}\left(1+\frac{x^2}{2}\right)^{1/2} \approx \sqrt{2}\left(1+\frac{1}{2}\frac{x^2}{2}\right) = \sqrt{2}\left(1+\frac{x^2}{4}\right)$
(e) $f(x) = (4+3x)^{1/3} = 4^{1/3}\left(1+\frac{3x}{4}\right)^{1/3} \approx 4^{1/3}\left(1+\frac{1}{3}\frac{3x}{4}\right) = 4^{1/3}\left(1+\frac{x}{4}\right)$
(f) $f(x) = \left(1-\frac{1}{2+x}\right)^{2/3} = \left[1+\left(-\frac{1}{2+x}\right)\right]^{2/3} \approx 1 + \frac{2}{3}\left(-\frac{1}{2+x}\right) = 1 - \frac{2}{6+3x}$

9. Center $= -1$ and $f'(x) = 4x + 4$. We have $f(-1) = -5$ and $f'(-1) = 0$. $L(x) = f(-1) + f'(-1)(x-(-1))$
$= -5 + 0(x+1) = -5$

10. Center $= 8$ and $f'(x) = \frac{1}{3}x^{-2/3}$. We have $f(8) = 2$ and $f'(8) = \frac{1}{12}$. $L(x) = f(8) + f'(8)(x-8)$
$= 2 + \frac{1}{12}(x-8) = \frac{x}{12} + \frac{4}{3}$

11. Center $= 1$ and $f'(x) = \frac{(x+1)(1)-(x)(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$. We have $f(1) = \frac{1}{2}$ and $f'(1) = \frac{1}{4}$. $L(x) = f(1) + f'(1)(x-1)$

$= \frac{1}{2} + \frac{1}{4}(x-1) = \frac{1}{4}x + \frac{1}{4}$

Alternate solution: Using center $= \frac{3}{2}$, we have $f\left(\frac{3}{2}\right) = \frac{3}{5}$ and $f'\left(\frac{3}{2}\right) = \frac{4}{25}$. $L(x) = f\left(\frac{3}{2}\right) + f'\left(\frac{3}{2}\right)\left(x - \frac{3}{2}\right)$

$= \frac{3}{5} + \frac{4}{25}\left(x - \frac{3}{2}\right) = \frac{4}{25}x + \frac{9}{25}$

12. Center $= \frac{\pi}{2}$

$f'(x) = -\sin x$

We have $f\left(\frac{\pi}{2}\right) = 0$ and $f'\left(\frac{\pi}{2}\right) = -1$.

$L(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) = 0 - 1\left(x - \frac{\pi}{2}\right) = -x + \frac{\pi}{2}$

13. (a) $(1.0002)^{50} = (1 + 0.0002)^{50} \approx 1 + 50(0.0002) = 1 + .01 = 1.01$

(b) $\sqrt[3]{1.009} = (1 + 0.009)^{1/3} \approx 1 + \left(\frac{1}{3}\right)(0.009) = 1 + 0.003 = 1.003$

14. $f(x) = \sqrt{x+1} + \sin x = (x+1)^{1/2} + \sin x \Rightarrow f'(x) = \left(\frac{1}{2}\right)(x+1)^{-1/2} + \cos x \Rightarrow L_f(x) = f'(0)(x-0) + f(0)$ $= \frac{3}{2}(x-0) + 1 \Rightarrow L_f(x) = \frac{3}{2}x + 1$, the linearization of f(x); $g(x) = \sqrt{x+1} = (x+1)^{1/2} \Rightarrow g'(x)$ $= \left(\frac{1}{2}\right)(x+1)^{-1/2} \Rightarrow L_g(x) = g'(0)(x-0) + g(0) = \frac{1}{2}(x-0) + 1 \Rightarrow L_g(x) = \frac{1}{2}x + 1$, the linearization of g(x); $h(x) = \sin x \Rightarrow h'(x) = \cos x \Rightarrow L_h(x) = h'(0)(x-0) + h(0) = (1)(x-0) + 0 \Rightarrow L_h(x) = x$, the linearization of h(x). $L_f(x) = L_g(x) + L_h(x)$ implies that the linearization of a sum is equal to the sum of the linearizations.

15. $y = x^3 - 3\sqrt{x} = x^3 - 3x^{1/2} \Rightarrow dy = \left(3x^2 - \frac{3}{2}x^{-1/2}\right) dx \Rightarrow dy = \left(3x^2 - \frac{3}{2\sqrt{x}}\right) dx$

16. $y = x\sqrt{1-x^2} = x\left(1-x^2\right)^{1/2} \Rightarrow dy = \left[(1)\left(1-x^2\right)^{1/2} + (x)\left(\frac{1}{2}\right)\left(1-x^2\right)^{-1/2}(-2x)\right] dx$

$= \left(1-x^2\right)^{-1/2}\left[\left(1-x^2\right) - x^2\right] dx = \frac{\left(1-2x^2\right)}{\sqrt{1-x^2}} dx$

17. $y = x^2 \ln x \Rightarrow \frac{dy}{dx} = \left(x^2\right)\left(\frac{1}{x}\right) + (\ln x)(2x) = 2x \ln x + x$, $dy = (2x \ln x + x)\, dx$.

18. $y = \frac{2\sqrt{x}}{3\left(1+\sqrt{x}\right)} = \frac{2x^{1/2}}{3\left(1+x^{1/2}\right)} \Rightarrow dy = \left[\frac{x^{-1/2}\left(3\left(1+x^{1/2}\right)\right) - 2x^{1/2}\left(\frac{3}{2}x^{-1/2}\right)}{9\left(1+x^{1/2}\right)^2}\right] dx = \frac{3x^{-1/2} + 3 - 3}{9\left(1+x^{1/2}\right)^2} dx$

$\Rightarrow dy = \frac{1}{3\sqrt{x}\left(1+\sqrt{x}\right)^2} dx$

19. $2y^{3/2} + xy - x = 0 \Rightarrow 3y^{1/2}\, dy + y\, dx + x\, dy - dx = 0 \Rightarrow \left(3y^{1/2} + x\right) dy = (1-y)\, dx \Rightarrow dy = \frac{1-y}{3\sqrt{y} + x} dx$

20. $xy^2 - 4x^{3/2} - y = 0 \Rightarrow y^2\,dx + 2xy\,dy - 6x^{1/2}\,dx - dy = 0 \Rightarrow (2xy-1)\,dy = \left(6x^{1/2} - y^2\right)dx$
$\Rightarrow dy = \frac{6\sqrt{x} - y^2}{2xy - 1}\,dx$

21. $y = e^{\sin x} \Rightarrow \frac{dy}{dx} = e^{\sin x}\cos x \Rightarrow dy = (\cos x)e^{\sin x}\,dx.$

22. $y = \cos\left(x^2\right) \Rightarrow dy = \left[-\sin\left(x^2\right)\right](2x)\,dx = -2x\sin\left(x^2\right)dx$

23. $y = xe^x \Rightarrow \frac{dy}{dx} = (1+x)e^x \Rightarrow dy = \frac{dy}{dx}dx = (1+x)e^x\,dx$

24. $y = \sec\left(x^2-1\right) \Rightarrow dy = \left[\sec\left(x^2-1\right)\tan\left(x^2-1\right)\right](2x)\,dx = 2x\left[\sec\left(x^2-1\right)\tan\left(x^2-1\right)\right]dx$

25. (a) $\Delta f = f(0.1) - f(0) = 0.21 - 0 = 0.21$
(b) Since $f'(x) = 2x + 2$, $f'(0) = 2$. Therefore, $df = 2\,dx = 2(0.1) = 0.2$.
(c) $|\Delta f - df| = |0.21 - 0.2| = 0.01$

26. (a) $\Delta f = f(1.1) - f(1) = 0.231 - 0 = 0.231$
(b) Since $f'(x) = 3x^2 - 1$, $f'(1) = 2$. Therefore, $df = 2\,dx = 2(0.1) = 0.2$.
(c) $|\Delta f - df| = |0.231 - 0.2| = 0.031$

27. (a) $\Delta f = f(0.55) - f(0.5) = \frac{20}{11} - 2 = -\frac{2}{11}$
(b) Since $f'(x) = -x^{-2}$, $f'(0.5) = -4$. Therefore, $df = -4\,dx = -4(0.05) = -0.2 = -\frac{1}{5}$
(c) $|\Delta f - df| = \left|-\frac{2}{11} + \frac{1}{5}\right| = \frac{1}{55}$

28. (a) $\Delta f = f(1.01) - f(1) = 1.04060401 - 1 = 0.04060401$
(b) Since $f'(x) = 4x^3$, $f'(1) = 4$. Therefore, $df = 4\,dx = 4(0.01) = 0.04$.
(c) $|\Delta f - df| = |0.04060401 - 0.04| = 0.00060401$

29. Note that $\frac{dV}{dr} = 4\pi r^2$, so $dV = 4\pi r^2\,dr$. When r changes from a to a + dr the change in volume is approximately $4\pi a^2\,dr$.

30. Note that $\frac{dS}{dr} = 8\pi r$, so $dS = 8\pi r\,dr$. When r changes from a to a + dr, the change in surface area is approximately $8\pi a\,dr$.

31. Note that $\frac{dV}{dx} = 3x^2$, so $dV = 3x^2\,dx$. When x changes from a to a + dx, the change in volume is approximately $3a^2\,dx$.

32. Note that $\frac{dS}{dx} = 12x$, so $dS = 12x\,dx$. When x changes from a to a + dx, the change in surface area is approximately $12a\,dx$.

33. Given $r = 2$ m, $dr = .02$ m

(a) $A = \pi r^2 \Rightarrow dA = 2\pi r\ dr = 2\pi(2)(.02) = .08\pi\ m^2$

(b) $\left(\frac{.08\pi}{4\pi}\right)(100\%) = 2\%$

34. $C = 2\pi r$ and $dC = 2$ in. $\Rightarrow dC = 2\pi\ dr \Rightarrow dr = \frac{1}{\pi} \Rightarrow$ the diameter grew about $\frac{2}{\pi}$ in.; $A = \pi r^2 \Rightarrow dA = 2\pi r\ dr$ $= 2\pi(5)\left(\frac{1}{\pi}\right) = 10\ in.^2$

35. The volume of a cylinder is $V = \pi r^2 h$. When h is held fixed, we have $\frac{dV}{dr} = 2\pi rh$, and so $dV = 2\pi rh\ dr$. For $h = 30$ in., $r = 6$ in., and $dr = 0.5$ in., the thickness of the shell is approximately $dV = 2\pi rh\ dr = 2\pi(6)(30)(0.5)$ $= 180\pi \approx 565.5\ in.^3$

36. Let θ = angle of elevation and h = height of building. Then $h = 30 \tan\theta$, so $\frac{dh}{d\theta} = 30 \sec^2\theta\ d\theta$. We want $|dh| < 0.04h$, which gives:

$|30 \sec^2\theta\ d\theta| < 0.04\ (30 \tan\theta) \Rightarrow \frac{1}{\cos^2\theta}|d\theta| < \frac{0.04 \sin\theta}{\cos\theta} \Rightarrow |d\theta| < 0.04 \sin\theta \cos\theta$

$|d\theta| < 0.04 \sin\frac{5\pi}{12} \cos\frac{5\pi}{12} = 0.01$ radian. The angle should be measured with an error of less than 0.01 radian (or approximately 0.57 degrees), which is a percentage error of approximately 0.76%.

37. $V = \pi h^3 \Rightarrow dV = 3\pi h^2\ dh$; recall that $\Delta V \approx dV$. Then $|\Delta V| \le (1\%)(V) = \frac{(1)(\pi h^3)}{100} \Rightarrow |dV| \le \frac{(1)(\pi h^3)}{100}$ $\Rightarrow |3\pi h^2\ dh| \le \frac{(1)(\pi h^3)}{100} \Rightarrow |dh| \le \frac{1}{300}h = \left(\frac{1}{3}\%\right)h$. Therefore the greatest tolerated error in the measurement of h is $\frac{1}{3}\%$.

38. (a) Let D_i represent the inside diameter. Then $V = \pi r^2 h = \pi\left(\frac{D_i}{2}\right)^2 h = \frac{\pi D_i^2 h}{4}$ and $h = 10 \Rightarrow V = \frac{5\pi D_i^2}{2} \Rightarrow$ $dV = 5\pi D_i\ dD_i$. Recall that $\Delta V \approx dV$. We want $|\Delta V| \le (1\%)(V) \Rightarrow |dV| \le \left(\frac{1}{100}\right)\left(\frac{5\pi D_i^2}{2}\right) = \frac{\pi D_i^2}{40}$ $\Rightarrow |5\pi D_i\ dD_i| \le \frac{\pi D_i^2}{40} \Rightarrow |dD_i| \le \frac{D_i}{200} = \left(\frac{1}{2}\%\right) D_i \Rightarrow$ the measurement must have an error less than $\frac{1}{2}\%$.

(b) Let D_e represent the exterior diameter. Then $S = 2\pi rh = \frac{2\pi D_e h}{2} = \pi D_e h$, when $h = 10 \Rightarrow S = 10\pi D_e$ $\Rightarrow dS = 10\pi\ dD_e$. Recall that $\Delta S \approx dS$. We want $|\Delta S| \le (5\%)(S) \Rightarrow |dS| \le \left(\frac{5}{100}\right)(10\pi D_e) \Rightarrow |10\pi\ dD_e|$ $\le \frac{\pi D_e}{2} \Rightarrow |dD_e| \le \frac{D_e}{20} = (5\%)\ D_e \Rightarrow$ the measurement must have an error less than 5%.

39. $V = \pi r^2 h$, h is constant $\Rightarrow dV = 2\pi rh\ dr$; recall that $\Delta V \approx dV$. We want $|\Delta V| \le \frac{1}{1000}V \Rightarrow |dV| \le \frac{\pi r^2 h}{1000}$ $\Rightarrow |2\pi rh\ dr| \le \frac{\pi r^2 h}{1000} \Rightarrow |dr| \le \frac{r}{2000} = (.05\%)r \Rightarrow$ a 0.05% variation in the radius can be tolerated.

40. $\frac{\Delta P}{P} \times 100\% \approx \frac{dP}{P} \times 100\% = \frac{(200 - x/2)e^{-x/400}}{200xe^{-x/400}}\,dx$. As sales change from x = 145 to x = 150,

$\Delta x = dx = 5$ and $\frac{\Delta P}{P} \times 100\% \approx \frac{(200 - 145/2)e^{-145/400}}{200(145)e^{-145/400}}(5) \times 100\% = 2.2\%$

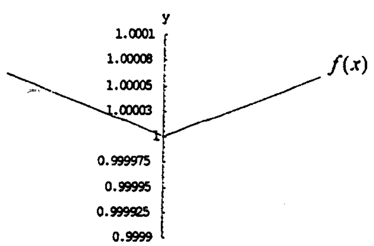
41. $W = a + \frac{b}{g} = a + bg^{-1} \Rightarrow dW = -bg^{-2}\,dg = -\frac{b\,dg}{g^2} \Rightarrow \frac{dW_{moon}}{dW_{earth}} = \frac{\left(-\frac{b\,dg}{(5.2)^2}\right)}{\left(-\frac{b\,dg}{(32)^2}\right)} = \left(\frac{32}{5.2}\right)^2 = 37.87$, so a change of gravity on the moon has about 38 times the effect that a change of the same magnitude has on Earth.

42. (a) $T = 2\pi\left(\frac{L}{g}\right)^{1/2} \Rightarrow dT = 2\pi\sqrt{L}\left(-\frac{1}{2}g^{-3/2}\right)dg = -\pi\sqrt{L}\,g^{-3/2}\,dg$

(b) If g increases, then $dg > 0 \Rightarrow dT < 0$. The period T decreases and the clock ticks more frequently. Both the pendulum speed and clock speed increase.

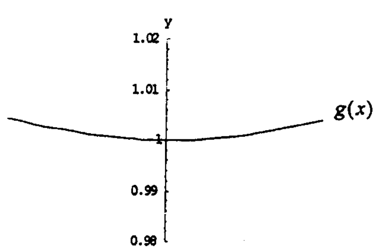
(c) $0.001 = -\pi\sqrt{100}\left(980^{-3/2}\right)dg \Rightarrow dg \approx -0.977$ cm/sec^2 $\Rightarrow$ the new $g \approx 979$ cm/sec^2

43. (a) Window: $-0.00006 \le x \le 0.00006$, $0.9999 \le y \le 1.0001$



After zooming in seven times, starting with the window $-1 \le x \le 1$ and $0 \le y \le 2$ on a TI-92 Plus calculator, the graph of f(x) shows no signs of straightening out.

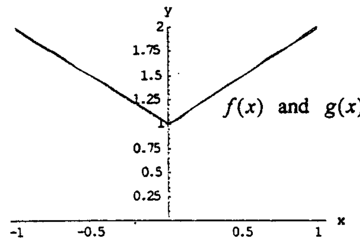
(b) Window: $-0.01 \le x \le 0.01$, $0.98 \le y \le 1.02$



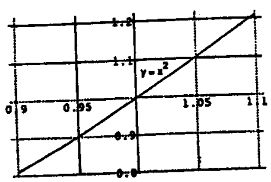
After zooming in only twice, starting with the window $-1 \le x \le 1$ and $0 \le y \le 2$ on a TI-92 Plus calculator, the graph of g(x) already appears to be smoothing toward a horizontal straight line.

(c) After seven zooms, starting with the window $-1 \le x \le 1$ and $0 \le y \le 2$ on a TI-92 Plus calculator, the graph of g(x) looks exactly like a horizontal straight line.

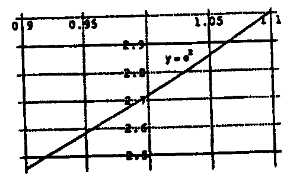
(d) Window: $-1 \le x \le 1$, $0 \le y \le 2$



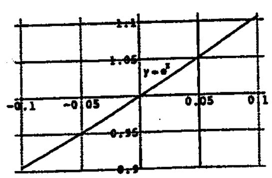
44. (a) $y = x^2$ at $x = 1$



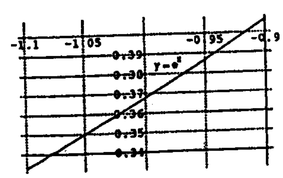
(b) $y = e^x$ at $x = 1$



(b) $y = e^x$ at $x = 0$



(b) $y = e^x$ at $x = -1$



45. $E(x) = f(x) - g(x) \Rightarrow E(x) = f(x) - m(x-a) - c$. Then $E(a) = 0 \Rightarrow f(a) - m(a-a) - c = 0 \Rightarrow c = f(a)$. Next we calculate m: $\lim\limits_{x \to a} \frac{E(x)}{x-a} = 0 \Rightarrow \lim\limits_{x \to a} \frac{f(x) - m(x-a) - c}{x-a} = 0 \Rightarrow \lim\limits_{x \to a} \left[\frac{f(x) - f(a)}{x-a} - m\right] = 0$ (since $c = f(a)$) $\Rightarrow f'(a) - m = 0 \Rightarrow m = f'(a)$. Therefore, $g(x) = m(x-a) + c = f'(a)(x-a) + f(a)$ is the linear approximation, as claimed.

46. (a) i. $Q(a) = f(a)$ implies that $b_0 = f(a)$.

ii. Since $Q'(x) = b_1 + 2b_2(x-a)$, $Q'(a) = f'(a)$ implies that $b_1 = f'(a)$.

iii. Since $Q''(x) = 2b_2$, $Q''(a) = f''(a)$ implies that $b_2 = \frac{f''(a)}{2}$.

In summary, $b_0 = f(a)$, $b_1 = f'(a)$, and $b_2 = \frac{f''(a)}{2}$.

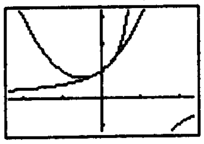
(b) $f(x) = (1-x)^{-1}$

$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$

$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$

Since $f(0) = 1$, $f'(0) = 1$, and $f''(0) = 2$, the coefficients are $b_0 = 1$, $b_1 = 1$, $b_2 = \frac{2}{2} = 1$. The quadratic approximation is $Q(x) = 1 + x + x^2$.

(c)

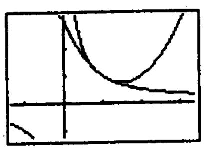


[−2.35, 2.35] by [−1.25, 3.25]

As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(d) $g(x) = x^{-1} \Rightarrow g'(x) = -x^{-2} \Rightarrow g''(x) = 2x^{-3}$

Since $g(1) = 1$, $g'(1) = -1$, and $g''(1) = 2$, the coefficients are $b_0 = 1$, $b_1 = -1$, and $b_2 = \frac{2}{2} = 1$. The quadratic approximation is $Q(x) = 1 - (x - 1) + (x - 1)^2$.

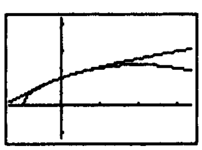


[−1.35, 3.35] by [−1.25, 3.25]

As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(e) $h(x) = (1 + x)^{1/2} \Rightarrow h'(x) = \frac{1}{2}(1 + x)^{-1/2} \Rightarrow h''(x) = -\frac{1}{4}(1 + x)^{-3/2}$

Since $h(0) = 1$, $h'(0) = \frac{1}{2}$, and $h''(0) = -\frac{1}{4}$, the coefficients are $b_0 = 1$, $b_1 = \frac{1}{2}$, and $b_2 = \frac{-\frac{1}{4}}{2} = -\frac{1}{8}$. The quadratic approximation is $Q(x) = 1 + \frac{x}{2} - \frac{x^2}{8}$.



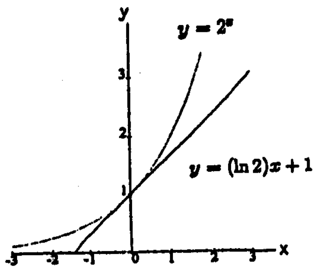
[−1.35, 3.35] by [−1.25, 3.25]

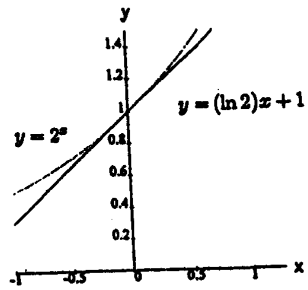
As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(f) The linearization of any differentiable function $u(x)$ at $x = a$ is $L(x) = u(a) + u'(a)(x - a) = b_0 + b_1(x - a)$, where b_0 and b_1 are the coefficients of the constant and linear terms of the quadratic approximation. Thus, the linearization for $f(x)$ at $x = 0$ is $1 + x$; the linearization for $g(x)$ at $x = 1$ is $1 - (x - 1)$ or $2 - x$; and the linearization for $h(x)$ at $x = 0$ is $1 + \frac{x}{2}$.

47. (a) $f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2$; $L(x) = \left(2^0 \ln 2\right)x + 2^0 = x \ln 2 + 1 \approx 0.69x + 1$

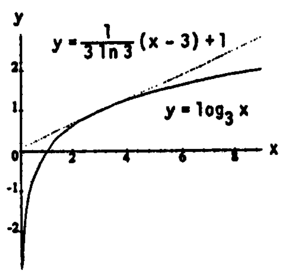
(b)

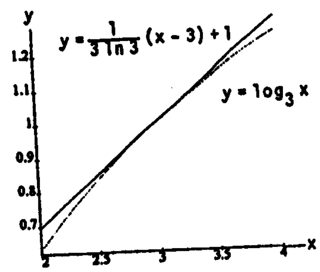




48. (a) $f(x) = \log_3 x \Rightarrow f'(x) = \frac{1}{x \ln 3}$, and $f(3) = \frac{\ln 3}{\ln 3} \Rightarrow L(x) = \frac{1}{3 \ln 3}(x-3) + \frac{\ln 3}{\ln 3} = \frac{x}{3 \ln 3} - \frac{1}{\ln 3} + 1$
$\approx 0.30x + 0.09$

(b)



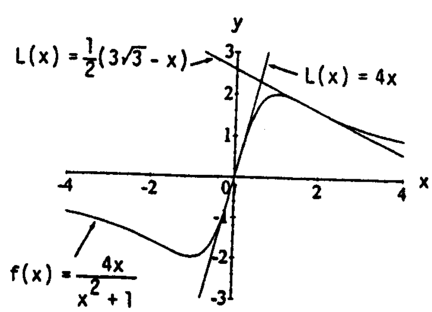


49. $f(x) = \frac{4x}{x^2+1} \Rightarrow f'(x) = \frac{4\left(1-x^2\right)}{\left(x^2+1\right)^2}$;

At $x = 0$: $L(x) = f'(0)(x-0) + f(0) = 4x$;

At $x = \sqrt{3}$: $L(x) = f'(\sqrt{3})(x - \sqrt{3}) + f(\sqrt{3})$

$= \left(-\frac{1}{2}\right)(x - \sqrt{3}) + \sqrt{3} \Rightarrow L(x) = \frac{1}{2}(3\sqrt{3} - x)$



50. (a) $\sqrt{1+x} \approx 1 + \frac{x}{2}$ gives the following: $\sqrt{1+1} \approx 1 + \frac{1}{2} \Rightarrow \sqrt{\sqrt{1+1}} \approx \sqrt{1+\frac{1}{2}}$

$\approx 1 + \frac{1}{4} \Rightarrow \sqrt{\sqrt{\sqrt{1+1}}} \approx \sqrt{1+\frac{1}{4}} \approx 1 + \frac{1}{8}$, and so forth. That is, $\underbrace{\sqrt{\cdots\sqrt{\sqrt{1+1}}}}_{\text{n square roots}} \approx 1 + \frac{1}{2n} \to 1$ as $n \to \infty$.

For successive tenth roots we obtain the approximation $1 + \frac{1}{10n} \to 1$ as $n \to \infty$.

(b) Yes, you can use any positive number in place of 2. Repeating the argument in (a) gives

$$\underbrace{\sqrt{\cdots\sqrt{\sqrt{1+x}}}}_{\text{n square roots}} \approx 1+\frac{x}{2n} \to 1+0 \text{ as } n \to \infty \text{ provided that the number } 1+x \text{ is positive.}$$

51-54. Example CAS commands:

Maple:

```
with(plots):
a:= 1: f:=x -> x ∧ 3 + x ∧ 2 − 2*x;
plot(f(x), x=−1..2);
diff(f(x),x);
fp := unapply (%,x);
L:=x -> f(a) + fp(a)*(x − a);
plot({f(x), L(x)}, x=−1..2);
err:=x -> abs(f(x) − L(x));
plot(err(x), x=−1..2, title = `absolute error function`);
err(−1);
```

Mathematica:

```
Clear[x]
{x1,x2} = {−1,2}; a = 1; f[x_] = x ∧ 3 + x ∧ 2 − 2x
Plot[ f[x], {x,x1,x2} ]
L[x_] = f[a] + f'[a] (x − a)
Plot[ {f[x], L[x]}, {x,x1,x2} ]
err[x_] := Abs[f[x] − L[x]]
Plot[ err[x], {x,x1,x2} ]
err[x1] // N

eps = 0.5; del = 0.3;
Plot[ {err[x],eps} {x,a−del,a+del} ]

eps = 0.1; del = 0.15;
Plot[ {err[x],eps}, {x,a−del,a+del} ]

eps = 0.01; del = 0.05;
Plot[ {err[x],eps}, {x,a−del,a+del} ]
```

3.7 NEWTON'S METHOD

1. $y = x^2 + x - 1 \Rightarrow y' = 2x + 1 \Rightarrow x_{n+1} = x_n - \frac{x_n^2 + x_n - 1}{2x_n + 1}$; $x_0 = 1 \Rightarrow x_1 = 1 - \frac{1+1-1}{2+1} = \frac{2}{3}$

$\Rightarrow x_2 = \frac{2}{3} - \frac{\frac{4}{9}+\frac{2}{3}-1}{\frac{4}{3}+1} \Rightarrow x_2 = \frac{2}{3} - \frac{4+6-9}{12+9} = \frac{2}{3} - \frac{1}{21} = \frac{13}{21} \approx .61905$; $x_0 = -1 \Rightarrow x_1 = 1 - \frac{1-1-1}{-2+1} = -2$

$\Rightarrow x_2 = -2 - \frac{4-2-1}{-4+1} = -\frac{5}{3} \approx -1.66667$

2. $y = x^3 + 3x + 1 \Rightarrow y' = 3x^2 + 3 \Rightarrow x_{n+1} = x_n - \frac{x_n^3 + 3x_n + 1}{3x_n^2 + 3}$; $x_0 = 0 \Rightarrow x_1 = 0 - \frac{1}{3} = -\frac{1}{3}$

$\Rightarrow x_2 = -\frac{1}{3} - \frac{-\frac{1}{27} - 1 + 1}{\frac{1}{3} + 3} = -\frac{1}{3} + \frac{1}{90} = -\frac{29}{90} \approx -0.32222$

3. $y = x^4 + x - 3 \Rightarrow y' = 4x^3 + 1 \Rightarrow x_{n+1} = x_n - \frac{x_n^4 + x_n - 3}{4x_n^3 + 1}$; $x_0 = 1 \Rightarrow x_1 = 1 - \frac{1+1-3}{4+1} = \frac{6}{5}$

$\Rightarrow x_2 = \frac{6}{5} - \frac{\frac{1296}{625} + \frac{6}{5} - 3}{\frac{864}{125} + 1} = \frac{6}{5} - \frac{1296 + 750 - 1875}{4320 + 625} = \frac{6}{5} - \frac{171}{4945} = \frac{5763}{4945} \approx 1.16542$; $x_0 = -1 \Rightarrow x_1 = -1 - \frac{1-1-3}{-4+1}$

$= -2 \Rightarrow x_2 = -2 - \frac{16-2-3}{-32+1} = -2 + \frac{11}{31} = -\frac{51}{31} \approx -1.64516$

4. $y = 2x - x^2 + 1 \Rightarrow y' = 2 - 2x \Rightarrow x_{n+1} = x_n - \frac{2x_n - x_n^2 + 1}{2 - 2x_n}$; $x_0 = 0 \Rightarrow x_1 = 0 - \frac{0-0+1}{2-0} = -\frac{1}{2}$

$\Rightarrow x_2 = -\frac{1}{2} - \frac{-1 - \frac{1}{4} + 1}{2+1} = -\frac{1}{2} + \frac{1}{12} = -\frac{5}{12} \approx -.41667$; $x_0 = 2 \Rightarrow x_1 = 2 - \frac{4-4+1}{2-4} = \frac{5}{2} \Rightarrow x_2 = \frac{5}{2} - \frac{5 - \frac{25}{4} + 1}{2-5}$

$= \frac{5}{2} - \frac{20 - 25 + 4}{-12} = \frac{5}{2} - \frac{1}{12} = \frac{29}{12} \approx 2.41667$

5. One obvious root is $x = 0$. Graphing e^{-x} and $2x + 1$ shows that $x = 0$ is the only root. Taking a naive approach we can use Newton's Method to estimate the root as follows: Let $f(x) = e^{-x} - 2x - 1$, $x_0 = 1$, and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n + \frac{e^{-x_n} - 2x_n - 1}{e^{-x_n} + 2}$. Performing iterations on a calculator, spreadsheet, or CAS gives $x_1 = -0.111594$, $x_2 = -0.00215192$, $x_3 = -0.000000773248$. You may get different results depending upon what you select for $f(x)$ and x_0, and what calculator or computer you use.

6. Graphing $\tan^{-1}(x)$ and $1 - 2x$ shows that there is only one root and it is between $x = 0.3$ and $x = 0.4$. Let $f(x) = \tan^{-1} x + 2x - 1$, $x_1 = 0.3$, and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\tan^{-1} x_n + 2x_n - 1}{\frac{1}{1+x_n^2} + 2}$. Performing iterations on a calculator, spreadsheet, or CAS gives $x_2 = 0.337205$, $x_3 = 0.337329$, $x_4 = 0.337329$. You may get different results depending upon what you select for $f(x)$ and x_1, and what calculator or computer you use.

7. $f(x_0) = 0$ and $f'(x_0) \neq 0 \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ gives $x_1 = x_0 \Rightarrow x_2 = x_0 \Rightarrow x_n = x_0$ for all $n \geq 0$. That is, all of the approximations in Newton's method will be the root of $f(x) = 0$ as well as x_0.

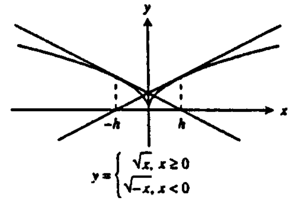
8. It does matter. If you start too far away from $x = \frac{\pi}{2}$, the calculated values may approach some other root. Starting with $x_0 = -0.5$, for instance, leads to $x = -\frac{\pi}{2}$ as the root, not $x = \frac{\pi}{2}$.

9. If $x_0 = h > 0 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = h - \frac{f(h)}{f'(h)}$

$= h - \frac{\sqrt{h}}{\left(\frac{1}{2\sqrt{h}}\right)} = h - (\sqrt{h})(2\sqrt{h}) = -h;$

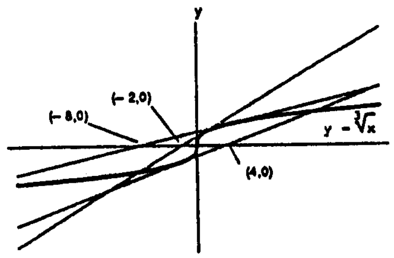
if $x_0 = -h < 0 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -h - \frac{f(-h)}{f'(-h)}$

$= -h - \frac{\sqrt{h}}{\left(\frac{-1}{2\sqrt{h}}\right)} = -h + (\sqrt{h})(2\sqrt{h}) = h.$



10. $f(x) = x^{1/3} \Rightarrow f'(x) = \left(\frac{1}{3}\right)x^{-2/3} \Rightarrow x_{n+1} = x_n - \frac{x_n^{1/3}}{\left(\frac{1}{3}\right)x_n^{-2/3}}$

$= -2x_n$; $x_0 = 1 \Rightarrow x_1 = -2$, $x_2 = 4$, $x_3 = -8$, and $x_4 = 16$ and so forth. Since $|x_n| = 2x_{n-1}$ we may conclude that $n \to \infty \Rightarrow |x_n| \to \infty$.



11. The points of intersection of $y = x^3$ and $y = 3x + 1$, or of $y = x^3 - 3x$ and $y = 1$, have the same x-values as the roots of $f(x) = x^3 - 3x - 1$ or the solutions of $g'(x) = 0$.

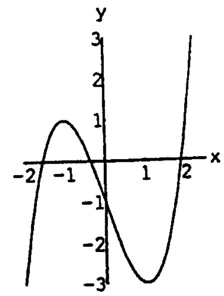
12. $f(x) = x - 1 - 0.5 \sin x \Rightarrow f'(x) = 1 - 0.5 \cos x \Rightarrow x_{n+1} = x_n - \frac{x_n - 1 - 0.5 \sin x_n}{1 - 0.5 \cos x_n}$; if $x_0 = 1.5$, then $x_1 = 1.49870$

13. The following commands are for the TI-92 Plus calculator. (Be sure your calculator is in approximate mode.) Go to the home screen and type the following:
 (a) Define f(x) = x^3+3*x+1 (enter)
  f(x) STO> y0 (enter)
  nDeriv(f(x),x) STO> yp (enter)
 (b) −0.3 STO> x (enter)
 (c) x − y0 ÷ yp STO> x (enter)(enter)(enter)
  After executing the last command two times the value, x = −0.322185, does not change in the sixth decimal place thereafter.
 (d) Now try x0 = 0 by typing the following commands:
  0 STO> x (enter)
  x − y0 ÷ yp STO> x (enter)(enter)(enter)(enter)
  After executing the last command three times the value, x = −0.322185, does not change in the sixth decimal place thereafter.
 (e) Try f(x) = sin x to estimate the zero at x = π by typing the following:
  Define f(x) = sin(x) (enter)
  3 STO> x (enter)
  x − y0 ÷ yp STO> x (enter)(enter)(enter)

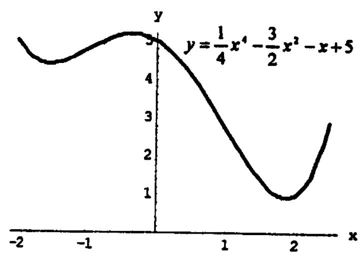
  After executing the last command two times the value, x = 3.14159, does not change in the fifth decimal place thereafter. The zeros (sin(x),x) command gives 3.14159 · @n1, which means any integer multiple of 3.14159. This matches the above result when @n1 = 1.

14. (a) $f(x) = x^3 - 3x - 1 \Rightarrow f'(x) = 3x^2 - 3 \Rightarrow x_{n+1} = x_n - \frac{x_n^3 - 3x_n - 1}{3x_n^2 - 3} \Rightarrow$ the two negative zeros are -1.53209 and -0.34730

(b) The estimated solutions of $x^3 - 3x - 1 = 0$ are $-1.53209, -0.34730, 1.87939$.



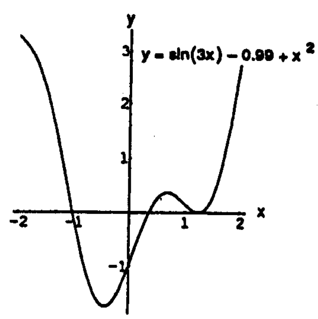
(c) The estimated x-values where $g(x) = 0.25x^4 - 1.5x^2 - x + 5$ has horizontal tangents are the roots of $g'(x) = x^3 - 3x - 1$, and these are $-1.53209, -0.34730, 1.87939$.



15. $f(x) = \tan x - 2x \Rightarrow f'(x) = \sec^2 x - 2 \Rightarrow x_{n+1} = x_n - \frac{\tan(x_n) - 2x_n}{\sec^2(x_n)}$; $x_0 = 1 \Rightarrow x_1 = 1.31047803$ $\Rightarrow x_2 = 1.223929097 \Rightarrow x_6 = x_7 = x_8 = 1.165561185$

16. $f(x) = x^4 - 2x^3 - x^2 - 2x + 2 \Rightarrow f'(x) = 4x^3 - 6x^2 - 2x - 2 \Rightarrow x_{n+1} = x_n - \frac{x_n^4 - 2x_n^3 - x_n^2 - 2x_n + 2}{4x_n^3 - 6x_n^2 - 2x_n - 2}$; if $x_0 = 0.5$, then $x_4 = 0.630115396$; if $x_0 = 2.5$, then $x_4 = 2.57327196$

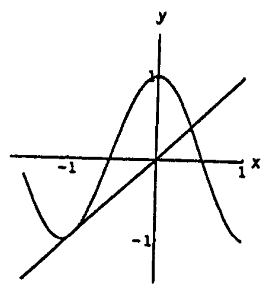
17. (a) The graph of $f(x) = \sin 3x - 0.99 + x^2$ in the window $-2 \le x \le 2$, $-2 \le y \le 3$ suggests three roots. However, when you zoom in on the x-axis near $x = 1.2$, you can see that the graph lies above the axis there. There are only two roots, one near $x = -1$, the other near $x = 0.4$.

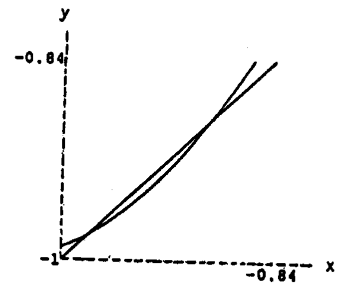


(b) $f(x) = \sin 3x - 0.99 + x^2 \Rightarrow f'(x) = 3 \cos 3x + 2x$ $\Rightarrow x_{n+1} = x_n - \frac{\sin(3x_n) - 0.99 + x_n^2}{3 \cos(3x_n) + 2x_n}$ and the solutions are approximately 0.35003501505249 and −1.0261731615301

18. (a) Yes, three times as indicted by the graphs

(b) $f(x) = \cos 3x - x \Rightarrow f'(x) = -3 \sin 3x - 1 \Rightarrow x_{n+1} = x_n - \frac{\cos(3x_n) - x_n}{-3 \sin(3x_n) - 1}$; at approximately -0.979367, -0.887726, and 0.39004 we have $\cos 3x = x$





19. $f(x) = 2x^4 - 4x^2 + 1 \Rightarrow f'(x) = 8x^3 - 8x \Rightarrow x_{n+1} = x_n - \frac{2x_n^4 - 4x_n^2 + 1}{8x_n^3 - 8x_n}$; if $x_0 = -2$, then $x_6 = -1.30656296$; if $x_0 = -0.5$, then $x_3 = -0.5411961$; the roots are approximately ± 0.5411961 and ± 1.30656296 because f(x) is an even function.

20. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow x_{n+1} = x_n - \frac{\tan(x_n)}{\sec^2(x_n)}$; $x_0 = 3 \Rightarrow x_1 = 3.13971 \Rightarrow x_2 = 3.14159$ and we approximate π to be 3.14159.

21. Graphing e^{-x^2} and $x^2 - x + 1$ shows that there are two places where the curves intersect, one at $x = 0$ and the other between $x = 0.5$ and $x = 0.6$. Let $f(x) = e^{-x^2} - x^2 + x - 1$, $x_0 = 0.5$, and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $= x_n - \frac{e^{-x_n^2} - x_n^2 + x_n - 1}{1 - 2x_n - 2x_n e^{-x_n^2}}$. Performing iterations on a calculator, spreadsheet, or CAS gives $x_1 = 0.536981$, $x_2 = 0.534856$, $x_3 = 0.53485$, $x_4 = 0.53485$. (You may get different results depending upon what you select for f(x) and x_1, and what calculator or computer you use.) Therefore, the two curves intersect at $x = 0$ and $x = 0.53485$.

22. Graphing $\ln(1 - x^2)$ and $x - 1$ shows that there are two places where the curves intersect, one between $x = -1$ and $x = -0.9$, the other between $x = 0.5$ and $x = 0.6$. Let $f(x) = \ln(1 - x^2) - x + 1$, and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $= x_n - \frac{\ln(1 - x_n^2) - x_n + 1}{-1 + \frac{2x_n}{x_n^2} - 1}$. Performing iterations on a calculator, spreadsheet, or CAS with $x_0 = 0.5$ gives $x_1 = 0.590992$, $x_2 = 0.583658$, $x_3 = 0.583597$, $x_4 = 0.583597$ and with $x_0 = -0.9$ gives $x_1 = -0.928237$, $x_2 = -0.924247$, $x_3 = -0.924119$, $x_4 = -0.924119$. (You may get different results depending upon what you select for f(x) and x_0, and what calculator or computer you use.) Therefore, the two curves intersect at $x = -0.924119$ and $x = 0.583597$.

23. If $f(x) = x^3 + 2x - 4$, then $f(1) = -1 < 0$ and $f(2) = 8 > 0 \Rightarrow$ by the Intermediate Value Theorem the equation $x^3 + 2x - 4 = 0$ has a solution between 1 and 2. Consequently, $f'(x) = 3x^2 + 2$ and $x_{n+1} = x_n - \frac{x_n^3 + 2x_n - 4}{3x_n^2 + 2}$. Then $x_0 = 1 \Rightarrow x_1 = 1.2 \Rightarrow x_2 = 1.17975 \Rightarrow x_3 = 1.179509 \Rightarrow x_4 = 1.1795090 \Rightarrow$ the root is approximately 1.17951.

24. We wish to solve $8x^4 - 14x^3 - 9x^2 + 11x - 1 = 0$. Let $f(x) = 8x^4 - 14x^3 - 9x^2 + 11x - 1$, then $f'(x) = 32x^3 - 42x^2 - 18x + 11 \Rightarrow x_{n+1} = x_n - \frac{8x_n^4 - 14x_n^3 - 9x_n^2 + 11x_n - 1}{32x_n^3 - 42x_n^2 - 18x_n + 11}$.

x_0	approximation of corresponding root
−1.0	−0.976823589
0.1	0.100363332
0.6	0.642746671
2.0	1.983713587

25. $f(x) = 4x^4 - 4x^2 \Rightarrow f'(x) = 16x^3 - 8x \Rightarrow x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^3 - x_i}{4x_i^2 - 2}$. Iterations are performed using the procedure in problem 13 in this section.

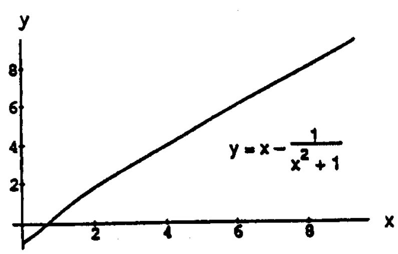
(a) For $x_0 = -2$ or $x_0 = -0.8$, $x_i \Rightarrow -1$ as i gets large.

(b) For $x_0 = -0.5$ or $x_0 = 0.25$, $x_i \Rightarrow 0$ as i gets large.

(c) For $x_0 = 0.8$ or $x_0 = 2$, $x_i \Rightarrow 1$ as i gets large.

(d) (If your calculator has a CAS, put it in exact mode, otherwise approximate the radicals with a decimal value.) For $x_0 = -\frac{\sqrt{21}}{7}$ or $x_0 = \frac{\sqrt{21}}{7}$, Newton's method does not converge. The values of x_i alternate between $-\frac{\sqrt{21}}{7}$ and $\frac{\sqrt{21}}{7}$ as i increases.

26. (a) The distance can be represented by $D(x) = \sqrt{(x-2)^2 + \left(x^2 + \frac{1}{2}\right)^2}$, where $x \geq 0$. The distance D(x) is minimized when $f(x) = (x-2)^2 + \left(x^2 + \frac{1}{2}\right)^2$ is minimized. If $f(x) = (x-2)^2 + \left(x^2 + \frac{1}{2}\right)^2$, then $f'(x) = 4\left(x^3 + x - 1\right)$ and $f''(x) = 4\left(3x^2 + 1\right) > 0$.



Now $f'(x) = 0 \Rightarrow x^3 + x - 1 = 0 \Rightarrow x\left(x^2 + 1\right) = 1 \Rightarrow x = \frac{1}{x^2 + 1}$.

(b) Let $g(x) = \frac{1}{x^2+1} - x = \left(x^2 + 1\right)^{-1} - x \Rightarrow g'(x) = -\left(x^2+1\right)^{-2}(2x) - 1 = \frac{-2x}{\left(x^2+1\right)^2} - 1$

$$\Rightarrow x_{n+1} = x_n - \frac{\left(\frac{1}{x_n^2+1} - x_n\right)}{\left(\frac{-2x_n}{\left(x_n^2+1\right)^2} - 1\right)};\ x_0 = 1 \Rightarrow x_4 = 0.68233 \text{ to five decimal places.}$$

27. $f(x) = (x-1)^{40} \Rightarrow f'(x) = 40(x-1)^{39} \Rightarrow x_{n+1} = x_n - \frac{(x_n-1)^{40}}{40(x_n-1)^{39}} = \frac{39x_n+1}{40}$. With $x_0 = 2$, our computer gave $x_{87} = x_{88} = x_{89} = \cdots = x_{200} = 1.11051$, coming within 0.11051 of the root $x = 1$.

28. $f(x) = 4x^4 - 4x^2 \Rightarrow f'(x) = 16x^3 - 8x = 8x(2x^2-1) \Rightarrow x_{n+1} = x_n - \frac{x_n(x_n^2-1)}{2(2x_n^2-1)}$; if $x_0 = .65$, then $x_{12} \approx -.000004$, if $x_0 = .7$, then $x_{12} = -1.000004$; if $x_0 = .8$, then $x_6 = 1.000000$. NOTE: $\frac{\sqrt{21}}{7} \approx .654654$

29. $f(x) = x^3 + 3.6x^2 - 36.4 \Rightarrow f'(x) = 3x^2 + 7.2x \Rightarrow x_{n+1} = x_n - \frac{x_n^3 + 3.6x_n^2 - 36.4}{3x_n^2 + 7.2x_n}$; $x_0 = 2 \Rightarrow x_1 = 2.53\overline{03}$ $\Rightarrow x_2 = 2.45418225 \Rightarrow x_3 = 2.45238021 \Rightarrow x_4 = 2.45237921$ which is 2.45 to two decimal places. Recall that $x = 10^4[H_3O^+] \Rightarrow [H_3O^+] = (x)(10^{-4}) = (2.45)(10^{-4}) = 0.000245$

30. Newton's method yields the following:

the initial value	2	i	$\sqrt{3}+i$
the approached value	1	$-5.55931i$	$-29.5815 - 17.0789i$

CHAPTER 3 PRACTICE EXERCISES

1. The global minimum value of $\frac{1}{2}$ occurs at $x = 2$.

2. (a) The values of y' and y'' are both negative where the graph is decreasing and concave down, at T.
   (b) The value of y' is negative and the value of y'' is positive where the graph is decreasing and concave up, at P.

3. (a) The function is increasing on the intervals $[-3,-2]$ and $[1,2]$.
   (b) The function is decreasing on the intervals $[-2,0)$ and $(0,2]$.
   (c) Local maximum values occur only at $x = -2$ and at $x = 2$; local minimum values occur at $x = -3$ and at $x = 1$ provided f is continuous at $x = 0$.

4. The 24th day

5. No, since $f(x) = x^3 + 2x + \tan x \Rightarrow f'(x) = 3x^2 + 2 + \sec^2 x > 0 \Rightarrow f(x)$ is always increasing on its domain.

6. No, since $g(x) = \csc x + 2\cot x \Rightarrow g'(x) = -\csc x \cot x - 2\csc^2 x = -\frac{\cos x}{\sin^2 x} - \frac{2}{\sin^2 x} = -\frac{1}{\sin^2 x}(\cos x + 2) < 0$ $\Rightarrow g(x)$ is always decreasing on its domain.

7. No absolute minimum because $\lim_{x\to\infty} (7+x)(11-3x)^{1/3} = -\infty$. Next $f'(x) =$ $(11-3x)^{1/3} - (7+x)(11-3x)^{-2/3} = \frac{(11-3x)-(7+x)}{(11-3x)^{2/3}} = \frac{4(1-x)}{(11-3x)^{2/3}} \Rightarrow x = 1$ and $x = \frac{11}{3}$ are critical points.

Since $f' > 0$ if $x < 1$ and $f' < 0$ if $x > 1$, $f(1) = 16$ is the absolute maximum.

8. $f(x) = \frac{ax+b}{x^2-1} \Rightarrow f'(x) = \frac{a(x^2-1)-2x(ax+b)}{(x^2-1)^2} = -\frac{ax^2+2bx+a}{(x^2-1)^2}$; $f(3) = 1 \Rightarrow \frac{1}{8}(3a+b) = 1$ and $f'(3) = 0$ $\Rightarrow -\frac{1}{64}(9a+6b+a) = 0$. Solving simultaneously, $a = 6$ and $b = -10$. These values mean $f'(x) = -\frac{6x^2-20x+6}{(x^2-1)^2} \Rightarrow f' > 0$ if $2 \le x < 3$ and $f' < 0$ if $3 < x \le 4 \Rightarrow$ local maximum value of $f(3) = 1$.

9. Yes, because at each point of $[0,1)$ except $x = 0$, the function's value is a local minimum value as well as a local maximum value. At $x = 0$ the function's value, 0, is not a local minimum value because each open interval around $x = 0$ on the x-axis contains points to the left of 0 where f equals -1.

10. (a) The first derivative of the function $f(x) = x^3$ is zero at $x = 0$ even though f has no local extreme value at $x = 0$.
    (b) Theorem 2 says only that if f is differentiable and f has a local extreme at $x = c$ then $f'(c) = 0$. It does not assert the (false) reverse implication $f'(c) = 0 \Rightarrow$ f has a local extreme at $x = c$.

11. No, because the interval $0 < x < 1$ fails to be closed. The Extreme Value Theorem for Continuous Functions says that if the function is continuous throughout a finite closed interval $a \le x \le b$ then the existence of absolute extrema is guaranteed on that interval.

12. The absolute maximum is $|-1| = 1$ and the absolute minimum is $|0| = 0$. The result is consistent because the Extreme Value Theorem for Continuous Functions does not require the interval be closed. However, if it is not closed, absolute extrema may not exist, as Exercise 11 shows. That the interval be closed is a sufficient condition (together with continuity of the function), but it is not necessary for absolute extrema to exist.

13. (a) $g(t) = \sin^2 t - 3t \Rightarrow g'(t) = 2 \sin t \cos t - 3 = \sin(2t) - 3 \Rightarrow g' < 0 \Rightarrow g(t)$ is always falling and hence must decrease on every interval in its domain.
    (b) One, since $\sin^2 t - 3t - 5 = 0$ and $\sin^2 t - 3t = 5$ have the same solutions: $f(t) = \sin^2 t - 3t - 5$ has the same derivative as $g(t)$ in part (a) and is always decreasing with $f(-3) > 0$ and $f(0) < 0$. The Intermediate Value Theorem guarantees the continuous function f has a root in $[-3, 0]$.

14. (a) $y = \tan\theta \Rightarrow \frac{dy}{d\theta} = \sec^2\theta > 0 \Rightarrow y = \tan\theta$ is always rising on its domain $\Rightarrow y = \tan\theta$ increases on every interval in its domain
    (b) The interval $\left[\frac{\pi}{4}, \pi\right]$ is not in the tangent's domain because $\tan\theta$ is undefined at $\theta = \frac{\pi}{2}$. Thus the tangent need not increase on this interval.

15. (a) $f(x) = x^4 + 2x^2 - 2 \Rightarrow f'(x) = 4x^3 + 4x$. Since $f(0) = -2 < 0$, $f(1) = 1 > 0$ and $f'(x) \ge 0$ for $0 \le x \le 1$, we may conclude from the Intermediate Value Theorem that f(x) has exactly one solution when $0 \le x \le 1$.
    (b) $x^2 = \frac{-2 \pm \sqrt{4+8}}{2} > 0 \Rightarrow x^2 = \sqrt{3} - 1$ and $x \ge 0 \Rightarrow x \approx \sqrt{.7320508076} \approx .8555996772$

16. (a) $y = \frac{x}{x+1} \Rightarrow y' = \frac{1}{(x+1)^2} > 0$, for all x in the domain of $\frac{x}{x+1} \Rightarrow y = \frac{x}{x+1}$ is increasing in every interval in its domain
    (b) $y = x^3 + 2x \Rightarrow y' = 3x^2 + 2 > 0$ for all $x \Rightarrow$ the graph of $y = x^3 + 2x$ is always increasing and can never have a local maximum or minimum

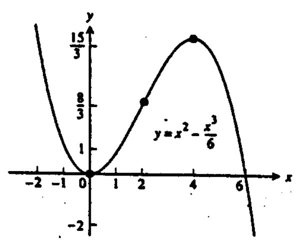
17. Let V(t) represent the volume of the water in the reservoir at time t, in minutes, let $V(0) = a_0$ be the initial amount and $V(1440) = a_0 + (1400)(43{,}560)(7.48)$ gallons be the amount of water contained in the reservoir after the rain, where 24 hr = 1440 min. Assume that V(t) is continuous on [0, 1440] and differentiable on (0, 1440). The Mean Value Theorem says that for some t_0 in (0, 1440) we have $V'(t_0) = \frac{V(1440) - V(0)}{1440 - 0}$ $= \frac{a_0 + (1400)(43{,}560)(7.48) - a_0}{1440} = \frac{456{,}160{,}320 \text{ gal}}{1440 \text{ min}} = 316{,}778$ gal/min. Therefore at t_0 the reservoir's volume was increasing at a rate in excess of 225,000 gal/min.

18. Yes, all differentiable functions g(x) having 3 as a derivative differ by only a constant. Consequently, the difference $3x - g(x)$ is a constant K because $g'(x) = 3 = \frac{d}{dx}(3x)$. Thus $g(x) = 3x + K$, the same form as F(x).

19. No, $\frac{x}{x+1} = 1 + \frac{-1}{x+1} \Rightarrow \frac{x}{x+1}$ differs from $\frac{-1}{x+1}$ by the constant 1. Both functions have the same derivative $\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2} = \frac{d}{dx}\left(\frac{-1}{x+1}\right)$.

20. $f'(x) = g'(x) = \frac{2x}{(x^2+1)^2} \Rightarrow f(x) - g(x) = C$ for some constant $C \Rightarrow$ the graphs differ by a vertical shift.

21.



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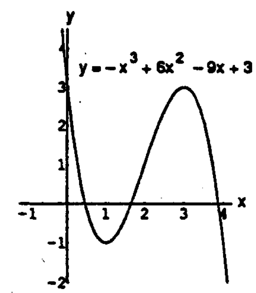

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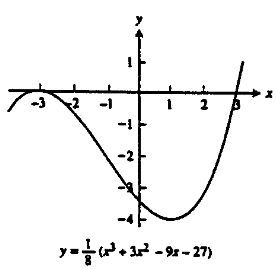

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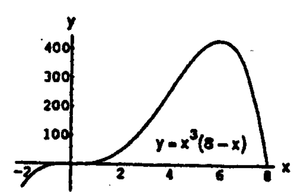

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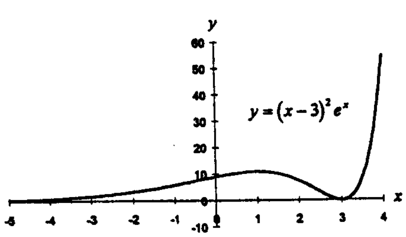

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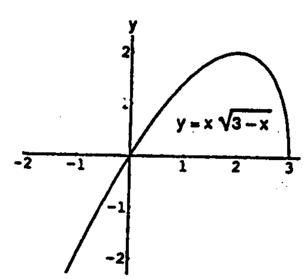
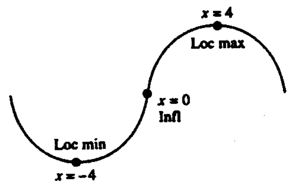


27. (a) $y' = 16 - x^2 \Rightarrow y' = \underset{-4}{---|}\underset{4}{+++|}---$ $\Rightarrow$ the curve is rising on $(-4,4)$, falling on $(-\infty,-4)$ and $(4,\infty)$ $\Rightarrow$ a local maximum at $x = 4$ and a local minimum at $x = -4$; $y'' = -2x \Rightarrow y'' = \underset{0}{+++|}---$ $\Rightarrow$ the curve is concave up on $(-\infty,0)$, concave down on $(0,\infty)$ $\Rightarrow$ a point of inflection at $x = 0$

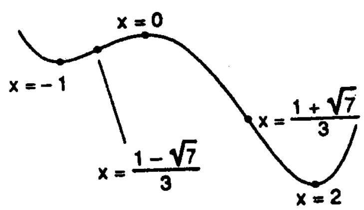
(b)



28. (a) $y' = 6x(x+1)(x-2) = 6x^3 - 6x^2 - 12x \Rightarrow y' = \underset{-1}{---|}\underset{0}{+++|}\underset{2}{---|}+++$ $\Rightarrow$ the graph is rising on $(-1,0)$ and $(2,\infty)$, falling on $(-\infty,-1)$ and $(0,2)$ $\Rightarrow$ a local maximum at $x = 0$, local minima at $x = -1$ and $x = 2$; $y'' = 18x^2 - 12x - 12 = 6(3x^2 - 2x - 2) = 6\left(x - \frac{1-\sqrt{7}}{3}\right)\left(x - \frac{1+\sqrt{7}}{3}\right) \Rightarrow$

$y'' = +++ \underset{\frac{1-\sqrt{7}}{3}}{|} --- \underset{\frac{1+\sqrt{7}}{3}}{|} +++$ $\Rightarrow$ the curve is concave up on $\left(-\infty, \frac{1-\sqrt{7}}{3}\right)$ and $\left(\frac{1+\sqrt{7}}{3}, \infty\right)$, concave down on $\left(\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right)$ $\Rightarrow$ points of inflection at $x = \frac{1 \pm \sqrt{7}}{3}$

(b)



29. $f(x) = e^{x/\sqrt{x^4+1}}$ for all $x \in (-\infty,\infty)$;

$$f'(x) = \left[\frac{(\sqrt{x^4+1})\cdot 1 - x\left(\frac{2x^3}{\sqrt{x^4+1}}\right)}{(\sqrt{x^4+1})^2}\right] e^{x/\sqrt{x^4+1}} = \frac{1-x^4}{(\sqrt{x^4+1})^3} e^{x/\sqrt{x^4+1}}$$

$$= \frac{(1-x^2)(1+x^2)}{(x^4+1)^{3/2}} e^{x/\sqrt{x^4+1}} = 0$$

$\Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$ are the critical points. Consider the behavior of f as $x \to \pm\infty$;

$\lim_{x\to\infty} e^{x/\sqrt{x^4+1}} = \lim_{x\to-\infty} e^{x/\sqrt{x^4+1}} = 1$ as suggested by the following table (14 digit precision, 12 digits displayed):

x	$x/\sqrt{x^4+1}$	$e^{x/\sqrt{x^4+1}}$
$-\infty$	0	1
⋮	⋮	⋮
−100000	−0.0000 10000 0000 0000	0.9999 9000 0050
−10000	−0.0001 0000 0000 000	0.9999 0000 5000
−1000	−0.0010 0000 0000 00	0.9990 0049 9833
−100	−0.0099 9999 9950 00	0.9900 4983 3799
−10	−0.0999 9500 0375 0	0.9048 4194 1895
0	0	1
10	0.0999 9500 0375 0	1.1051 6539 265
100	0.0099 9999 9950 00	1.0100 5016 703
1000	0.0010 0000 0000 00	1.0010 0050 017
10000	0.0001 0000 0000 000	1.0001 0000 500
100000	0.0000 1000 0000 0000	1.0000 1000 005
⋮	⋮	⋮
∞	0	1

Therefore, $y = 1$ is a horizontal asymptote in both directions. Check the critical points for absolute extreme values: $f(-1) = e^{-\sqrt{2}/2} \approx 0.4931$, $f(1) = e^{\sqrt{2}/2} \approx 2.0281 \Rightarrow$ the absolute minimum value of the function is $e^{-\sqrt{2}/2}$ at $x = -1$, and the absolute maximum value is $e^{\sqrt{2}/2}$ at $x = 1$.

30. $g(x) = e^{\sqrt{3-2x-x^2}}$;

The domain of g is all x such that $3 - 2x - x^2 \geq 0$. The parabola $y = 3 - 2x - x^2$ is concave down with x-intercepts at $x = -3$ and $x = 1$, therefore, $3 - 2x - x^2 \geq 0$ if $-3 \leq x \leq 1$, and this interval is the domain of g; $g'(x) = -\frac{1+x}{\sqrt{3-2x-x^2}} e^{\sqrt{3-2x-x^2}} = 0 \Rightarrow 1 + x = 0 \Rightarrow x = -1$ is a critical point; $g(-3) = g(1) = e^0 = 1$, $g(-1) = e^2 \approx 7.3891 \Rightarrow$ the absolute minimum value of the function is 1 at $x = -3$ and at $x = 1$, and the absolute maximum value is e^2 at $x = -1$.

31. (a) $t = 0, 6, 12$ (b) $t = 3, 9$ (c) $6 < t < 12$ (d) $0 < t < 6$, $12 < t < 14$

32. (a) $t = 4$ (b) at no time (c) $0 < t < 4$ (d) $4 < t < 8$

33. (a) $v(t) = s'(t) = 4 - 6t - 3t^2$
(b) $a(t) = v'(t) = -6 - 6t$
(c) The particle starts at position 3 moving in the positive direction, but decelerating. At approximately $t = 0.528$, it reaches a position 4.128 and changes direction, beginning to move in the negative direction. After that, it continues to accelerate while moving in the negative direction.

34. $s(t) = \frac{1}{2}t^4 - 4t^3 + 6t^2,\ t \geq 0 \Rightarrow v(t) = 2t^3 - 12t^2 + 12t;\ v(t) = 0 \Rightarrow t\left(t^2 - 6t + 6\right) = 0 \Rightarrow t = 0,\ t = 3 - \sqrt{3}$ ≈ 1.268, and $t = 3 + \sqrt{3} \approx 4.732$. For $0 < t < 3 - \sqrt{3}$, $v(t) > 0$, for $3 - \sqrt{3} < t < 3 + \sqrt{3}$, $v(t) < 0$, and for $t > 3 + \sqrt{3}$, $v(t) > 0$, therefore, the particle moves forward during the time intervals, $\left(0, 3 - \sqrt{3}\right)$ and $\left(3 + \sqrt{3}, \infty\right)$.

35. Since $\frac{d}{dx}\left(-\frac{1}{4}x^{-4} - e^{-x}\right) = x^{-5} + e^{-x}$, $f(x) = -\frac{1}{4}x^{-4} - e^{-x} + C$.

36. Since $\frac{d}{dx}$ sec x = sec x tan x, f(x) = sec x + C.

37. Since $\frac{d}{dx}\left(-\frac{2}{x} + \frac{1}{3}x^3 + x\right) = \frac{2}{x^2} + x^2 + 1$, $f(x) = -\frac{2}{x} + \frac{1}{3}x^3 + x + C$ for $x > 0$.

38. Since $\frac{d}{dx}\left(\frac{2}{3}x^{3/2} + 2x^{1/2}\right) = \sqrt{x} + \frac{1}{\sqrt{x}}$, $f(x) = \frac{2}{3}x^{3/2} + 2x^{1/2} + C$.

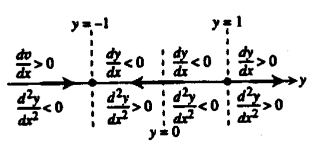
39. $v(t) = s'(t) = 9.8t + 5 \Rightarrow s(t) = 4.9t^2 + 5t + C;\ s(0) = 10 \Rightarrow C = 10 \Rightarrow s(t) = 4.9t^2 + 5t + 10$

40. $a(t) = v'(t) = 32 \Rightarrow v(t) = 32t + C_1;\ v(0) = 20 \Rightarrow C_1 = 20 \Rightarrow v(t) = s'(t) = 32t + 20$
$s(t) = 16t^2 + 20t + C_2;\ s(0) = 5 \Rightarrow C_2 = 5 \Rightarrow s(t) = 16t^2 + 20t + 5$

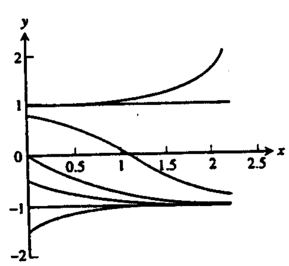
41. $\frac{dy}{dx} = y^2 - 1$

(a) $\frac{dy}{dx} = y^2 - 1 = 0 \Rightarrow y = \pm 1;\ y < -1 \Rightarrow \frac{dy}{dx} > 0,\ -1 < y < 1 \Rightarrow \frac{dy}{dx} < 0,\ y > 1 \Rightarrow \frac{dy}{dx} > 0$. Therefore, $y = -1$ is stable and $y = 1$ is unstable.

(b) $\frac{d^2y}{dx^2} = 2y\,\frac{dy}{dx} = 2y\left(y^2 - 1\right)$

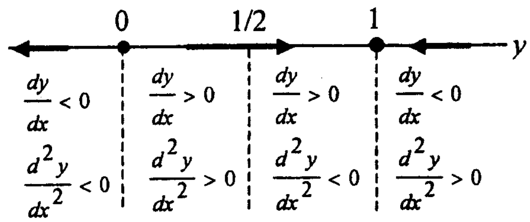


(c)

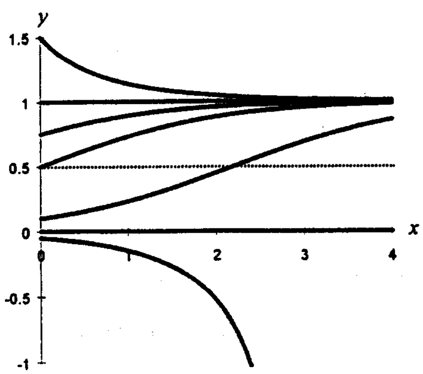


42. (a) $\frac{dy}{dx} = y - y^2 = 0 \Rightarrow y = 0$ or 1; $y < 0 \Rightarrow \frac{dy}{dx} < 0$, $0 < y < 1 \Rightarrow \frac{dy}{dx} > 0$, $y > 1 \Rightarrow \frac{dy}{dx} < 0$.
Therefore, $y = 0$ is unstable and $y = 1$ is stable.

(b) $\frac{d^2y}{dx^2} = (1-2y)\frac{dy}{dx} = (1-2y)(y-y^2) = y(1-2y)(1-y)$

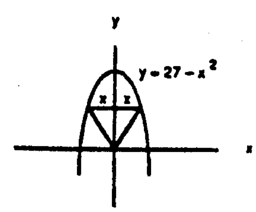


(c)



43. Note that $s = 100 - 2r$ and the sector area is given by $A = \pi r^2\left(\frac{s}{2\pi r}\right) = \frac{1}{2}rs = \frac{1}{2}r(100-2r) = 50r - r^2$. To find the domain of $A(r) = 50r - r^2$, note that $r > 0$ and $0 < s < 2\pi r$, which gives $12.1 \approx \frac{50}{\pi+1} < r < 50$. Since $A'(r) = 50 - 2r$, the critical point occurs at $r = 25$. This value is in the domain and corresponds to the maximum area because $A''(r) = -2$, which is negative for all r. The greatest area is attained when $r = 25$ ft and $s = 50$ ft.

44. $A(x) = \frac{1}{2}(2x)(27 - x^2)$ for $0 \le x \le \sqrt{27}$
$\Rightarrow A'(x) = 3(3+x)(3-x)$ and $A''(x) = -6x$.
The critical points are -3 and 3, but -3 is not in the domain. Since $A''(3) = -18 < 0$ and $A(\sqrt{27}) = 0$, the maximum occurs at $x = 3 \Rightarrow$ the largest area is $A(3) = 54$ sq units.



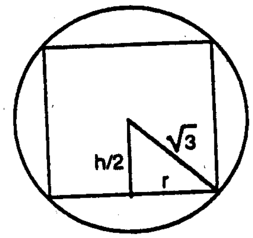
45. From the diagram we have $\left(\frac{h}{2}\right)^2 + r^2 = \left(\sqrt{3}\right)^2$

$\Rightarrow r^2 = \frac{12 - h^2}{4}$. The volume of the cylinder is

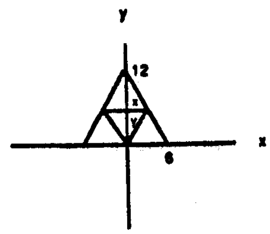
$V = \pi r^2 h = \pi\left(\frac{12 - h^2}{4}\right)h = \frac{\pi}{4}\left(12h - h^3\right)$, where

$0 \le h \le 2\sqrt{3}$. Then $V'(h) = \frac{3\pi}{4}(2 + h)(2 - h)$

$\Rightarrow$ the critical points are -2 and 2, but -2 is not in the domain. At $h = 2$ there is a maximum since $V''(2) = -3\pi < 0$. The dimensions of the largest cylinder are radius $= \sqrt{2}$ and height $= 2$.



46. From the diagram we have $y = 12 - 2x$ and $V(x) = \frac{1}{3}\pi x^2(12 - 2x)$, where $0 \le x \le 6$ $\Rightarrow V'(x) = 2\pi x(4 - x)$ and $V''(4) = -4\pi$. The critical points are 0 and 4; $V(0) = V(6) = 0 \Rightarrow x = 4$ gives the maximum. Thus the values of $r = 4$ and $h = 4$ yield the largest volume for the smaller cone.

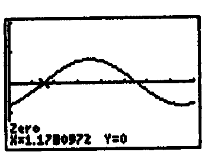


47. The profit $P = 2px + py = 2px + p\left(\frac{40 - 10x}{5 - x}\right)$, where p is the profit on grade B tires and $0 \le x \le 4$. Thus $P'(x) = \frac{2p}{(5 - x)^2}\left(x^2 - 10x + 20\right) \Rightarrow$ the critical points are $\left(5 - \sqrt{5}\right)$, 5, and $\left(5 + \sqrt{5}\right)$, but only $\left(5 - \sqrt{5}\right)$ is in the domain. Now $P'(x) > 0$ for $0 < x < \left(5 - \sqrt{5}\right)$ and $P'(x) < 0$ for $\left(5 - \sqrt{5}\right) < x < 4 \Rightarrow$ at $x = \left(5 - \sqrt{5}\right)$ there is a local maximum. Also $P(0) = 8p$, $P\left(5 - \sqrt{5}\right) = 4p\left(5 - \sqrt{5}\right) \approx 11p$, and $P(4) = 8p \Rightarrow$ at $x = \left(5 - \sqrt{5}\right)$ there is an absolute maximum. The maximum occurs when $x = \left(5 - \sqrt{5}\right)$ and $y = 2\left(5 - \sqrt{5}\right)$, the units are hundreds of tires, i.e., $x \approx 276$ tires and $y \approx 553$ tires.

48. (a) The distance between the particles is $|f(t)|$ where $f(t) = -\cos t + \cos\left(t + \frac{\pi}{4}\right)$.

Then $f'(t) = \sin t - \sin\left(t + \frac{\pi}{4}\right)$

Solving $f'(t) = 0$ graphically, we obtain $t \approx 1.178$, $t \approx 4.320$, and so on.



[0, 2π] by [−2, 2]

Alternatively, $f'(t) = 0$ may be solved analytically as follows.

$f'(t) = \sin\left[\left(t+\frac{\pi}{8}\right)-\frac{\pi}{8}\right] - \sin\left[\left(t+\frac{\pi}{8}\right)+\frac{\pi}{8}\right]$

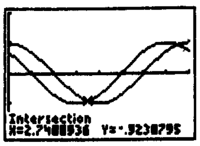
$= \left[\sin\left(t+\frac{\pi}{8}\right)\cos\frac{\pi}{8} - \cos\left(t+\frac{\pi}{8}\right)\sin\frac{\pi}{8}\right] - \left[\sin\left(t+\frac{\pi}{8}\right)\cos\frac{\pi}{8} + \cos\left(t+\frac{\pi}{8}\right)\sin\frac{\pi}{8}\right]$

$= -2\sin\frac{\pi}{8}\cos\left(t+\frac{\pi}{8}\right),$

so the critical points occur when

$\cos\left(t+\frac{\pi}{8}\right) = 0$, or $t = \frac{3\pi}{8} + k\pi$. At each of these values, $f(t) = \pm 2\cos\frac{3\pi}{8} \approx \pm 0.765$ units, so the maximum distance between the particles is 0.765 units.

(b) Solving $\cos t = \cos\left(t+\frac{\pi}{4}\right)$ graphically, we obtain $t \approx 2.749$, $t \approx 5.890$, and so on.



[0, 2π] by [−2, 2]

Alternatively, this problem can be solved analytically as follows.

$\cos t = \cos\left(t+\frac{\pi}{4}\right) \Rightarrow \cos\left[\left(t+\frac{\pi}{8}\right)-\frac{\pi}{8}\right] = \cos\left[\left(t+\frac{\pi}{8}\right)+\frac{\pi}{8}\right]$

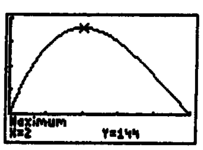
$\Rightarrow \cos\left(t+\frac{\pi}{8}\right)\cos\frac{\pi}{8} + \sin\left(t+\frac{\pi}{8}\right)\sin\frac{\pi}{8} = \cos\left(t+\frac{\pi}{8}\right)\cos\frac{\pi}{8} - \sin\left(t+\frac{\pi}{8}\right)\cos\frac{\pi}{8}$

$\Rightarrow 2\sin\left(t+\frac{\pi}{8}\right)\sin\frac{\pi}{8} = 0 \Rightarrow \sin\left(t+\frac{\pi}{8}\right) = 0 \Rightarrow t = \frac{7\pi}{8} + k\pi$

The particles collide when $t = \frac{7\pi}{8} \approx 2.749$ (plus multiples of π if they keep going.)

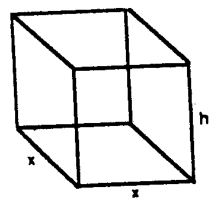
49. The dimensions will be x in. by $10-2x$ in. by $16-2x$ in., so $V(x) = x(10-2x)(16-2x) = 4x^3 - 52x^2 + 160x$ for $0 < x < 5$. Then $V'(x) = 12x^2 - 104x + 160 = 4(x-2)(3x-20)$, so the critical point in the correct domain is $x = 2$. This critical point corresponds to the maximum possible volume because $V'(x) > 0$ for $0 < x < 2$ and $V'(x) < 0$ for $2 < x < 5$. The box of largest volume has a height of 2 in. and a base measuring 6 in. by 12 in., and its volume is 144 in.^{3}

Graphical support:

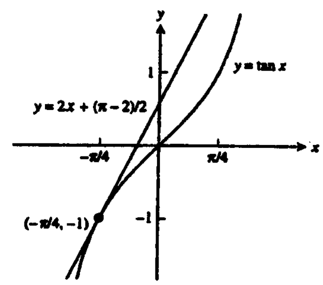


[0, 5] by [−40, 160]

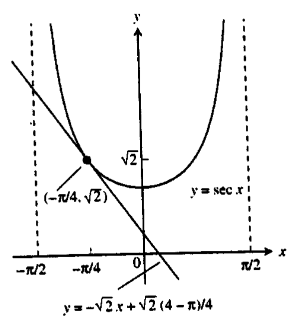
50. The volume is $V = x^2 h = 32 \Rightarrow h = \frac{32}{x^2}$. The surface area is $S(x) = x^2 + 4x\left(\frac{32}{x^2}\right) = x^2 + \frac{128}{x}$, where $x > 0 \Rightarrow S'(x) = \frac{2x^3 - 128}{x^2}$ $\Rightarrow$ the critical points are 0 and 4, but 0 is not in the domain. Now $S''(4) = 2 + \frac{256}{4^3} > 0 \Rightarrow$ at $x = 4$ there is a minimum. The dimensions 4 ft by 4 ft by 2 ft minimize the surface area.



51. (a) If $f(x) = \tan x$ and $x = -\frac{\pi}{4}$, then $f'(x) = \sec^2 x$, $f\left(-\frac{\pi}{4}\right) = -1$ and $f'\left(-\frac{\pi}{4}\right) = 2$. The linearization of $f(x)$ is $L(x) = 2\left(x + \frac{\pi}{4}\right) + (-1) = 2x + \frac{\pi - 2}{2}$.



(b) If $f(x) = \sec x$ and $x = -\frac{\pi}{4}$, then $f'(x) = \sec x \tan x$, $f\left(-\frac{\pi}{4}\right) = \sqrt{2}$ and $f'\left(-\frac{\pi}{4}\right) = -\sqrt{2}$. The linearization of $f(x)$ is $L(x) = -\sqrt{2}\left(x + \frac{\pi}{4}\right) + \sqrt{2} = -\sqrt{2}x + \frac{\sqrt{2}(4 - \pi)}{4}$.



52. $f(x) = \frac{1}{1 + \tan x} \Rightarrow f'(x) = \frac{-\sec^2 x}{(1 + \tan x)^2}$. The linearization at $x = 0$ is $L(x) = f'(0)(x - 0) + f(0) = 1 - x$.

53. $f(x) = e^x + \sin x - 0.5 \Rightarrow f'(x) = e^x + \cos x \Rightarrow L(x) = f(0) + f'(0)(x - 0) \Rightarrow L(x) = 0.5 + 2x$

54. $f(x) = \frac{2}{1 - x} + \sqrt{1 + x} - 3.1 = 2(1 - x)^{-1} + (1 + x)^{1/2} - 3.1 \Rightarrow f'(x) = -2(1 - x)^{-2}(-1) + \frac{1}{2}(1 + x)^{-1/2}$
$= \frac{2}{(1 - x)^2} + \frac{1}{2\sqrt{1 + x}} \Rightarrow L(x) = f'(0)(x - 0) + f(0) = 2.5x - 0.1$

55. When the volume is $V = \frac{1}{3}\pi r^2 h$, then $dV = \frac{2}{3}\pi r_0 h\ dr$ estimates the change in the volume for fixed h.

56. (a) $S = 6r^2 \Rightarrow dS = 12r\ dr$. We want $|dS| \le (2\%)S \Rightarrow |12r\ dr| \le \frac{12r^2}{100} \Rightarrow |dr| \le \frac{r}{100}$. The measurement of the edge r must have an error less than 1%.

(b) When $V = r^3$, then $dV = 3r^2\ dr$. The accuracy of the volume is $\left(\frac{dV}{V}\right)(100\%) = \left(\frac{3r^2\ dr}{r^3}\right)(100\%)$
$= \left(\frac{3}{r}\right)(dr)(100\%) = \left(\frac{3}{r}\right)\left(\frac{r}{100}\right)(100\%) = 3\%$

57. $C = 2\pi r \Rightarrow r = \frac{C}{2\pi}$, $S = 4\pi r^2 = \frac{C^2}{\pi}$, and $V = \frac{4}{3}\pi r^3 = \frac{C^3}{6\pi^2}$. It also follows that $dr = \frac{1}{2\pi}\ dC$, $dS = \frac{2C}{\pi}\ dC$ and $dV = \frac{C^2}{2\pi^2}\ dC$. Recall that $C = 10$ cm and $dC = 0.4$ cm.

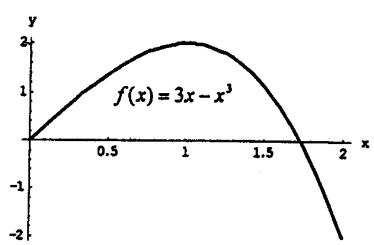
(a) $dr = \frac{0.4}{2\pi} = \frac{0.2}{\pi}$ cm $\Rightarrow \left(\frac{dr}{r}\right)(100\%) = \left(\frac{0.2}{\pi}\right)\left(\frac{2\pi}{10}\right)(100\%) = (.04)(100\%) = 4\%$

(b) $dS = \frac{20}{\pi}(0.4) = \frac{8}{\pi}$ cm $\Rightarrow \left(\frac{dS}{S}\right)(100\%) = \left(\frac{8}{\pi}\right)\left(\frac{\pi}{100}\right)(100\%) = 8\%$

(c) $dV = \frac{10^2}{2\pi^2}(0.4) = \frac{20}{\pi^2}$ cm $\Rightarrow \left(\frac{dV}{V}\right)(100\%) = \left(\frac{20}{\pi^2}\right)\left(\frac{6\pi^2}{1000}\right)(100\%) = 12\%$

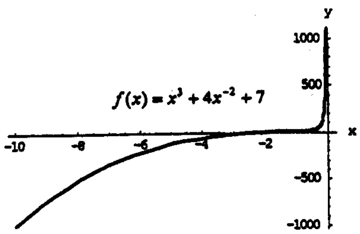
58. Similar triangles yield $\frac{35}{h} = \frac{15}{6} \Rightarrow h = 14$ ft. The same triangles imply that $\frac{20+a}{h} = \frac{a}{6} \Rightarrow h = 120a^{-1} + 6$
$\Rightarrow dh = -120a^{-2}\ da = -\frac{120}{a^2}\ da = \left(-\frac{120}{a^2}\right)\left(\pm\frac{1}{12}\right) = \pm\frac{2}{45} \approx \pm 0.0444$ ft $= \pm 0.53$ inches.

59. The graph of f(x) shows that for $1 \le x \le 2$, $f(x) = 0$ has one solution near $x = 1.7$. (Note: The exact solution is $x = \sqrt{3} \approx 1.732051$. Nonetheless, we use Newton's method to find an estimate for this solution.)



$f(x) = 3x - x^3 \Rightarrow f'(x) = 3 - 3x^2 \Rightarrow x_{n+1} = x_n - \frac{3x_n - x_n^3}{3 - 3x_n^2} \Rightarrow x_0 = 1.7$, $x_1 = 1.732981$, $x_2 = 1.732052$, $x_3 = 1.732051$, $x_4 = 1.732051$. Solution: $x \approx 1.732051$.

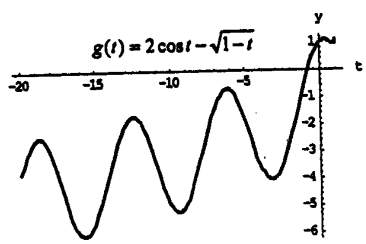
60. The graph of f(x) shows that for $x < 0$, $f(x) = 0$ has one solution near $x = -2$.



$$f(x) = x^3 + 4x^{-2} + 7 \Rightarrow f'(x) = 3x^2 - 8x^{-3} \Rightarrow x_{n+1} = x_n - \frac{x_x^3 + 4x_n^{-2} + 7}{3x_n^2 - 8x_n^{-3}} = x_n - \frac{x_x^6 + 4x_n + 7x_n^3}{3x_n^5 - 8} \Rightarrow x_0 = -2,$$

$x_1 = -2$. This is because $x = -2$ is a root, the one we are looking for.

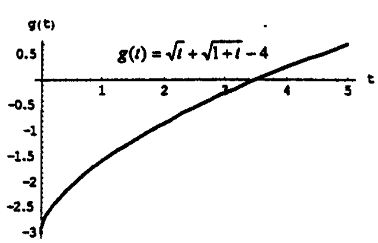
61. The domain of g(t) is $(-\infty, 1]$, and the graph of g(t) shows that $g(t) = 0$ has one solution near $t = -1$.



$$g(t) = 2 \cos t - \sqrt{1-t} \Rightarrow g'(t) = -2 \sin t + \frac{1}{2\sqrt{1-t}} \Rightarrow t_{n+1} = t_n - \frac{2 \cos t_n - \sqrt{1-t_n}}{-2 \sin t_n + \frac{1}{2\sqrt{1-t_n}}} \Rightarrow t_0 = -1,$$

$t_1 = -0.836185$, $t_2 = -0.828381$, $t_3 = -0.828361$, $t_4 = -0.828361$. Solution: $t \approx -0.828361$.

62. The graph of g(t) shows that for $t > 0$, $g(t) = 0$ has one solution between $t = 3$ and $t = 4$.



$g(t) = \sqrt{t} + \sqrt{1+t} - 4 \Rightarrow g'(t) = \frac{1}{2\sqrt{t}} + \frac{1}{2\sqrt{1+t}} \Rightarrow t_{n+1} = t_n - \frac{\sqrt{t_n} + \sqrt{1+t_n} - 4}{\frac{1}{2\sqrt{t_n}} + \frac{1}{2\sqrt{1+t_n}}} \Rightarrow x_1 = 3,\ x_2 \approx 3.497423,$

$x_2 \approx 3.515604,\ x_3 \approx 3.515625,\ x_4 \approx 3.515625$

Solution: $x \approx 3.515625$

CHAPTER 3 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. If M and m are the maximum and minimum values, respectively, then $m \le f(x) \le M$ for all $x \in I$. If $m = M$ then f is constant on I.

2. No, the function $f(x) = \begin{cases} 3x+6, & -2 \le x < 0 \\ 9 - x^2, & 0 \le x \le 2 \end{cases}$ has an absolute minimum value of 0 at $x = -2$ and an absolute maximum value of 9 at $x = 0$, but it is discontinuous at $x = 0$.

3. On an open interval the extreme values of a continuous function (if any) must occur at an interior critical point. On a half-open interval the extreme values of a continuous function may be at a critical point or at the closed endpoint. Extreme values occur only where $f' = 0$, f' does not exist, or at the endpoints of the interval. Thus the extreme points will not be at the open ends of an open interval.

4. The pattern $f' = +++\underset{1}{|}----\underset{2}{|}----\underset{3}{|}++++\underset{4}{|}+++$ indicates a local maximum at $x = 1$ and a local minimum at $x = 3$.

5. (a) If $y' = 6(x+1)(x-2)^2$, then $y' < 0$ for $x < -1$ and $y' > 0$ for $x > -1$. The sign pattern is $f' = ---\underset{-1}{|}+++\underset{2}{|}+++ \Rightarrow$ f has a local minimum at $x = -1$. Also $y'' = 6(x-2)^2 + 12(x+1)(x-2)$ $= 6(x-2)(3x) \Rightarrow y'' > 0$ for $x < 0$ or $x > 2$, while $y'' < 0$ for $0 < x < 2$. Therefore f has points of inflection at $x = 0$ and $x = 2$.

   (b) If $y' = 6x(x+1)(x-2)$, then $y' < 0$ for $x < -1$ and $0 < x < 2$; $y' > 0$ for $-1 < x < 0$ and $x > 2$. The sign sign pattern is $y' = ---\underset{-1}{|}+++\underset{0}{|}---\underset{2}{|}+++$. Therefore f has a local maximum at $x = 0$ and local minima at $x = -1$ and $x = 2$. Also, $y'' = 6\left[x - \left(\frac{1-\sqrt{7}}{3}\right)\right]\left[x - \left(\frac{1+\sqrt{7}}{3}\right)\right]$, so $y'' < 0$ for $\frac{1-\sqrt{7}}{3} < x < \frac{1+\sqrt{7}}{3}$ and $y'' > 0$ for all other $x \Rightarrow$ f has points of inflection at $x = \frac{1 \pm \sqrt{7}}{3}$.

6. The Mean Value Theorem indicates that $\frac{f(6) - f(0)}{6 - 0} = f'(c) \le 2$ for some c in $(0, 6)$. Then $f(6) - f(0) \le 12$ indicates the most that f can increase is 12.

7. If f is continuous on $[a, c)$ and $f'(x) \le 0$ on $[a, c)$, then by the Mean Value Theorem for all $x \in [a, c)$ we have $\frac{f(c) - f(x)}{c - x} \le 0 \Rightarrow f(c) - f(x) \le 0 \Rightarrow f(x) \ge f(c)$. Also if f is continuous on $(c, b]$ and $f'(x) \ge 0$ on $(c, b]$, then for all $x \in (c, b]$ we have $\frac{f(x) - f(c)}{x - c} \ge 0 \Rightarrow f(x) - f(c) \ge 0 \Rightarrow f(x) \ge f(c)$. Therefore $f(x) \ge f(c)$ for all $x \in [a, b]$.

8. (a) For all x, $-(x+1)^2 \le 0 \le (x-1)^2 \Rightarrow -(1+x^2) \le 2x \le (1+x^2) \Rightarrow -\frac{1}{2} \le \frac{x}{1+x^2} \le \frac{1}{2}$.

   (b) There exists $c \in (a,b)$ such that $\frac{c}{1+c^2} = \frac{f(b)-f(a)}{b-a} \Rightarrow \left|\frac{f(b)-f(a)}{b-a}\right| = \left|\frac{c}{1+c^2}\right| \le \frac{1}{2}$, from part (a) $\Rightarrow |f(b)-f(a)| \le \frac{1}{2}|b-a|$.

9. No. Corollary 1 requires that $f'(x) = 0$ for all x in some interval I, not $f'(x) = 0$ at a single point in I.

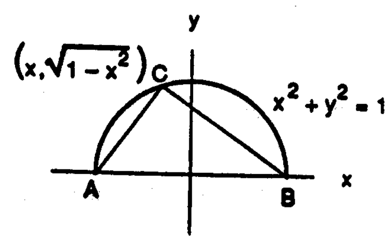
10. (a) $h(x) = f(x)g(x) \Rightarrow h'(x) = f'(x)g(x) + f(x)g'(x)$ which changes signs at $x = a$ since $f'(x), g'(x) > 0$ when $x < a$, $f'(x), g'(x) < 0$ when $x > a$ and $f(x), g(x) > 0$ for all x. Therefore h(x) does have a local maximum at $x = a$.

    (b) No, let $f(x) = g(x) = x^3$ which have points of inflection at $x = 0$, but $h(x) = x^6$ has no point of inflection (it has a local minimum at $x = 0$).

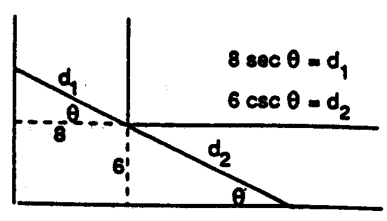
11. From (ii), $f(-1) = \frac{-1+a}{b-c+2} = 0 \Rightarrow a = 1$; from (iii), $1 = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+1}{bx^2+cx+2} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{bx+c+\frac{2}{x}}$ $\Rightarrow b = 0$ (because $b = 1 \Rightarrow \lim_{x\to\infty} f(x) = 0$). Also, if $c = 0$ then $\lim_{x\to\infty} f(x) = \infty$ so we must have $c = 1$. In summary, $a = 1$, $b = 0$, and $c = 1$.

12. $\frac{dy}{dx} = 3x^2 + 2kx + 3 = 0 \Rightarrow x = \frac{-2k \pm \sqrt{4k^2-36}}{6} \Rightarrow$ x has only one value when $4k^2 - 36 = 0 \Rightarrow k^2 = 9$ or $k = \pm 3$.

13. The area of the ΔABC is $A(x) = \frac{1}{2}(2)\sqrt{1-x^2} = (1-x^2)^{1/2}$, where $0 \le x \le 1$. Thus $A'(x) = \frac{-x}{\sqrt{1-x^2}} \Rightarrow 0$ and ± 1 are critical points. Also $A(\pm 1) = 0$ so $A(0) = 1$ is the maximum. When $x = 0$ the ΔABC is isosceles since $AC = BC = \sqrt{2}$.



14. The length of the ladder is $d_1 + d_2 = 8 \sec\theta + 6 \csc\theta$. We wish to maximize $I(\theta) = 8\sec\theta + 6\csc\theta \Rightarrow I'(\theta) = 8\sec\theta\tan\theta - 6\csc\theta\cot\theta$. Then $I'(\theta) = 0 \Rightarrow 8\sin^3\theta - 6\cos^3\theta = 0 \Rightarrow \tan\theta = \frac{\sqrt[3]{6}}{2} \Rightarrow$ $d_1 = 4\sqrt{4+\sqrt[3]{36}}$ and $d_2 = \sqrt[3]{36}\sqrt{4+\sqrt[3]{36}}$ $\Rightarrow$ the length of the ladder is about $(4+\sqrt[3]{36})\sqrt{4+\sqrt[3]{36}} = (4+\sqrt[3]{36})^{3/2} \approx 19$ ft (rounded down so that the ladder will make the corner).



15. The time it would take the water to hit the ground from height y is $\sqrt{\frac{2y}{g}}$, where g is the acceleration of gravity. The product of time and exit velocity (rate) yields the distance the water travels:

$D(y) = \sqrt{\frac{2y}{g}}\sqrt{64(h-y)} = 8\sqrt{\frac{2}{g}}\left(hy-y^2\right)^{1/2}$, $0 \le y \le h \Rightarrow D'(y) = 4\sqrt{\frac{2}{g}}\left(hy-y^2\right)^{-1/2}(h-2y) \Rightarrow 0, \frac{h}{2}$ and h are critical points. Now $D(0) = 0$, $D\left(\frac{h}{2}\right) = \frac{8h}{\sqrt{g}}$ and $D(h) = 0 \Rightarrow$ the best place to drill the hole is at $y = \frac{h}{2}$.

16. From the figure in the text, $\tan(\beta+\theta) = \frac{b+a}{h}$; $\tan(\beta+\theta) = \frac{\tan\beta + \tan\theta}{1-\tan\beta\tan\theta}$; and $\tan\theta = \frac{a}{h}$. These equations give $\frac{b+a}{h} = \frac{\tan\beta+\frac{a}{h}}{1-\frac{a}{h}\tan\beta} = \frac{h\tan\beta + a}{h - a\tan\beta}$. Solving for $\tan\beta$ gives $\tan\beta = \frac{bh}{h^2+a(b+a)}$ or

$\left(h^2+a(b+a)\right)\tan\beta = bh$. Differentiating both sides with respect to h gives

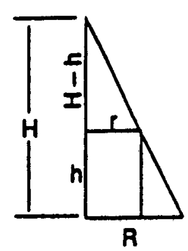
$2h\tan\beta + \left(h^2+a(b+a)\right)\sec^2\beta\,\frac{d\beta}{dh} = b$. Then $\frac{d\beta}{dh} = 0 \Rightarrow 2h\tan\beta = b \Rightarrow 2h\left(\frac{bh}{h^2+a(b+a)}\right) = b$

$\Rightarrow 2bh^2 = bh^2 + ab(b+a) \Rightarrow h^2 = a(b+a) \Rightarrow h = \sqrt{a(a+b)}$.

17. The surface area of the cylinder is $S = 2\pi r^2 + 2\pi rh$. From the diagram we have $\frac{r}{R} = \frac{H-h}{H} \Rightarrow h = \frac{RH-rH}{R}$ and

$S(r) = 2\pi r(r+h) = 2\pi r\left(r + H - r\frac{H}{R}\right) = 2\pi\left(1-\frac{H}{R}\right)r^2 + 2\pi Hr$,

where $0 \le r \le R$.



Case 1: $H < R \Rightarrow S(r)$ is a quadratic equation containing the origin and concave upward $\Rightarrow S(r)$ is maximum at $r = R$.

Case 2: $H = R \Rightarrow S(r)$ is a linear equation containing the origin with a positive slope $\Rightarrow S(r)$ is maximum at $r = R$.

Case 3: $H > R \Rightarrow S(r)$ is a quadratic equation containing the origin and concave downward. Then $\frac{dS}{dr} = 4\pi\left(1-\frac{H}{R}\right)r + 2\pi H$ and $\frac{dS}{dr} = 0 \Rightarrow 4\pi\left(1-\frac{H}{R}\right)r + 2\pi H = 0 \Rightarrow r = \frac{RH}{2(H-R)}$. For simplification we let $r^* = \frac{RH}{2(H-R)}$.

(a) If $R < H < 2R$, then $0 \ge H - 2R \Rightarrow H \ge 2(H-R) \Rightarrow \frac{RH}{2(H-R)} \ge R$ which is impossible.

(b) If $H = 2R$, then $r^* = \frac{2R^2}{2R} = R \Rightarrow S(r)$ is maximum at $r = R$.

(c) If $H > 2R$, then $2R + H \le 2H \Rightarrow H \le 2(H-R) \Rightarrow \frac{H}{2(H-R)} \le 1 \Rightarrow \frac{RH}{2(H-R)} \le R \Rightarrow r^* \le R$. Therefore, $S(r)$ is a maximum at $r = r^* = \frac{RH}{2(H-R)}$.

Conclusion: If $H \in (0,R]$ or $H = 2R$, then the maximum surface area is at $r = R$. If $H \in (R, 2R)$, then $r > R$ which is not possible. If $H \in (2R, \infty)$, then the maximum is at $r = r^* = \frac{RH}{2(H-R)}$.

18. $f(x) = mx - 1 + \frac{1}{x} \Rightarrow f'(x) = m - \frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3} > 0$ when $x > 0$. Then $f'(x) = 0 \Rightarrow x = \frac{1}{\sqrt{m}}$ yields a

minimum. If $f'\left(\frac{1}{\sqrt{m}}\right) \geq 0$, then $\sqrt{m} - 1 + \sqrt{m} = 2\sqrt{m} - 1 \geq 0 \Rightarrow m \geq \frac{1}{4}$. Thus the smallest acceptable value for m is $\frac{1}{4}$.

19. $\lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h} = f''(c) \Leftrightarrow$ for $\epsilon = \frac{1}{2}|f''(c)| > 0$ there exists a $\delta > 0$ such that $0 < |h| < \delta$

$\Rightarrow \left|\frac{f'(c+h) - f'(c)}{h} - f''(c)\right| < \frac{1}{2}|f''(c)|$. Then $f'(c) = 0 \Rightarrow -\frac{1}{2}|f''(c)| < \frac{f'(c+h)}{h} - f''(c) < \frac{1}{2}|f''(c)|$

$\Rightarrow f''(c) - \frac{1}{2}|f''(c)| < \frac{f'(c+h)}{h} < f''(c) + \frac{1}{2}|f''(c)|$. If $f''(c) < 0$, then $|f''(c)| = -f''(c)$

$\Rightarrow \frac{3}{2}f''(c) < \frac{f'(c+h)}{h} < \frac{1}{2}f''(c) < 0$; likewise if $f''(c) > 0$, then $0 < \frac{1}{2}f''(c) < \frac{f'(c+h)}{h} < \frac{3}{2}f''(c)$.

(a) If $f''(c) < 0$, then $-\delta < h < 0 \Rightarrow f'(c+h) > 0$ and $0 < h < \delta \Rightarrow f'(c+h) < 0$. Therefore, f(c) is a local maximum.

(b) If $f''(c) > 0$, then $-\delta < h < 0 \Rightarrow f'(c+h) < 0$ and $0 < h < \delta \Rightarrow f'(c+h) > 0$. Therefore, f(c) is a local minimum.

20. (a) By completing the square we have $f(x) = a\left(x + \frac{b}{a}\right)^2 + \frac{ac - b^2}{a} \geq 0$. If $a > 0$ and $f(x) \geq 0$, then $\frac{ac - b^2}{a} \geq 0$ $\Rightarrow ac - b^2 > 0 \Rightarrow ac > b^2$. If $ac > b^2$ and $a > 0$, then $\frac{ac - b^2}{a} > 0 \Rightarrow f(x) \geq 0$.

(b) If $f(x) = (a_1x + b_1)^2 + \ldots + (a_nx + b_n)^2$, then let $g(x) = Ax^2 + 2Bx + C$, where $A = \sum_{i=1}^{n} a_i^2$, $B = \sum_{i=1}^{n} a_ib_i$ and $C = \sum_{i=1}^{n} b_i^2$. Part (a) $\Rightarrow B^2 \leq AC$ or $\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$.

(c) $B^2 = AC \Rightarrow$there is a unique $x = x_0$ such that $g(x_0) = A\left(x_0 - \frac{B}{A}\right)^2 + \frac{AC - B^2}{A} = 0$, from part (b). Therefore $f(x_0) = 0 \Rightarrow$ that each $a_ix_0 + b_i = 0 \Rightarrow a_ix_0 = -b_i$ for $i = 1, 2, \ldots, n$.

21. (a) $(1)^2 = \frac{4\pi^2 L}{32.2} \Rightarrow L \approx 0.8156$ ft

(b) $2T\,dT = \frac{4\pi^2}{g}\,dL \Rightarrow dT = \frac{2\pi^2}{Tg}\,dL = \frac{2\pi^2}{\left(\frac{2\pi\sqrt{L}}{\sqrt{g}}\right)g}\,dL = \frac{\pi}{\sqrt{gL}}\,dL \approx \left(\frac{\pi}{\sqrt{32.2}\sqrt{0.8156}}\right)(0.01) = 0.00613$ sec.

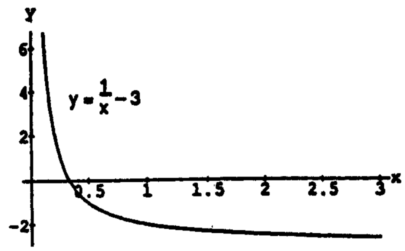
(c) The original clock completes 1 swing every second or $(24)(60)(60) = 86{,}400$ swings per day. The new clock completes 1 swing every 1.00613 seconds. Therefore it takes $(86{,}400)(1.00613) = 86{,}929.632$ seconds for the new clock to complete the same number of swings. Thus the new clock loses $\frac{529.632}{60} \approx 8.83$ min/day.

22. (a) If $\frac{1}{x} - 3 = 0$, then $\frac{1-3x}{x} = 0 \Rightarrow x = \frac{1}{3}$.

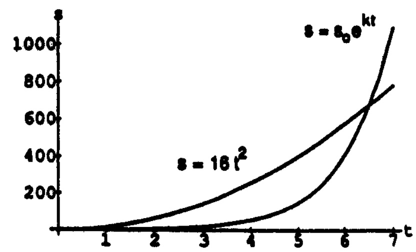
(b) $f(x) = \frac{1}{x} - 3$ and $f'(x) = -\frac{1}{x^2}$

$$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\frac{1}{x_n} - 3}{-\frac{1}{x_n^2}}$$

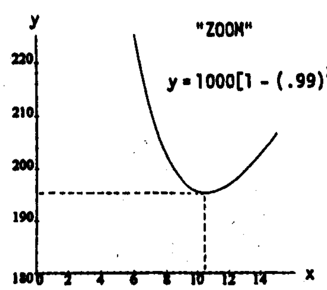
$$= 2x_n - 3x_n^2 = x_n(2 - 3x_n)$$

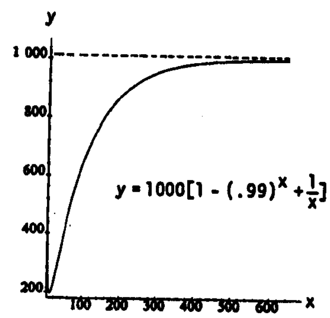


23. $\frac{ds}{dt} = ks \Rightarrow \frac{ds}{s} = k\,dt \Rightarrow \ln s = kt + C \Rightarrow s = s_0e^{kt}$ $\Rightarrow$ the 14th century model of free fall was exponential; note that the motion starts too slowly at first and then becomes too fast after about 7 seconds



24. Two views of the graph of $y = 1000\left[1 - (.99)^x + \frac{1}{x}\right]$ are shown below.





(a) At about $x = 11$ there is a minimum

(b) There is no maximum; however, the curve is asymptotic to $y = 1000$. The curve is near 1000 when $x \geq 643$.

25. (a) $L = k\left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4}\right) \Rightarrow \frac{dL}{d\theta} = k\left(\frac{b\csc^2\theta}{R^4} - \frac{b\csc\theta\cot\theta}{r^4}\right)$; solving $\frac{dL}{d\theta} = 0$

$\Rightarrow r^4b\csc^2\theta - bR^4\csc\theta\cot\theta = 0 \Rightarrow (b\csc\theta)\left(r^4\csc\theta - R^4\cot\theta\right) = 0$; but $b\csc\theta \neq 0$ since

$\theta \neq \frac{\pi}{2} \Rightarrow r^4\csc\theta - R^4\cot\theta = 0 \Rightarrow \cos\theta = \frac{r^4}{R^4} \Rightarrow \theta = \cos^{-1}\left(\frac{r^4}{R^4}\right)$, the critical value of θ

(b) $\theta = \cos^{-1}\left(\frac{5}{6}\right)^4 \approx \cos^{-1}(0.48225) \approx 61°$

NOTES.