for

$$\mathbf{P} \equiv \mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1}.$$

The proof of this is lengthy and technical and we do not show it here; it can be found in VC p. 452.

M.5 DIFFERENTIAL CALCULUS

a. Definition

Differentiation with respect to elements of a vector $\mathbf{x}=\{_c\ x_i\}_{i=1}^{k}$ is defined by the notation

$$\frac{\partial}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_k} \end{bmatrix}.$$

b. Scalars

Thus

$$\frac{\partial}{\partial \mathbf{x}}(\mathbf{a}'\mathbf{x}) = \frac{\partial}{\partial \mathbf{x}}(\mathbf{x}'\mathbf{a}) = \mathbf{a}. \tag{M.13}$$

c. Vectors

For $\mathbf{y}' = [\ y_1 \quad y_2 \quad \cdots \quad y_p\ ]$

$$\frac{\partial \mathbf{y}'}{\partial \mathbf{x}} = \left\{_m\ \frac{\partial y_j}{\partial x_i}\right\}_{i=1,j=1}^{k\quad p}, \text{ a matrix of order } k \times p.$$

Then

$$\frac{\partial \mathbf{x}'}{\partial \mathbf{x}} = \mathbf{I} \tag{M.14}$$

and for $\mathbf{A}$ not involving $\mathbf{x}$

$$\frac{\partial}{\partial \mathbf{x}}(\mathbf{x}'\mathbf{A}) = \frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\mathbf{A} = \mathbf{A}. \tag{M.15}$$

d. Inner products

Consider $\mathbf{u}$ and $\mathbf{v}$, of the same order, each having elements that are functions of the elements of $\mathbf{x}$. Then $\mathbf{u}'\mathbf{v}$ is a scalar, and so by (M.13)

$\partial(\mathbf{u}'\mathbf{v})/\partial\mathbf{x}$ is a column. Therefore, because differentiating the $\mathbf{u}'$ part of $\mathbf{u}'\mathbf{v}$ gives $(\partial\mathbf{u}'/\partial\mathbf{x})\mathbf{v}$ and because $\mathbf{u}'\mathbf{v} = \mathbf{v}'\mathbf{u}$, we have

$$\frac{\partial\mathbf{u}'\mathbf{v}}{\partial\mathbf{x}} = \frac{\partial\mathbf{u}'}{\partial\mathbf{x}}\mathbf{v} + \frac{\partial\mathbf{v}'}{\partial\mathbf{x}}\mathbf{u}. \tag{M.16}$$

e. Quadratic forms

To differentiate $\mathbf{x}'\mathbf{A}\mathbf{x}$ with respect to $\mathbf{x}$, use (M.16) with $\mathbf{u}'$ and $\mathbf{v}$ being $\mathbf{x}'$ and $\mathbf{A}\mathbf{x}$ respectively. This gives

$$\begin{aligned}\frac{\partial}{\partial\mathbf{x}}\mathbf{x}'\mathbf{A}\mathbf{x} &= \frac{\partial\mathbf{x}'}{\partial\mathbf{x}}\mathbf{A}\mathbf{x} + \frac{\partial\mathbf{A}\mathbf{x}}{\partial\mathbf{x}}\mathbf{x} \\ &= \mathbf{A}\mathbf{x} + \mathbf{A}'\mathbf{x} \\ &= 2\mathbf{A}\mathbf{x} \quad \text{when } \mathbf{A} \text{ is symmetric,} \end{aligned} \tag{M.17}$$

which it usually is.

f. Inverse matrices

If $\mathbf{V}$ is non-singular of order n and has elements which are functions of a scalar w, differentiating $\mathbf{V}^{-1}$ with respect to w comes from differentiating the identity $\mathbf{V}^{-1}\mathbf{V} = \mathbf{I}$. Thus

$$\frac{\partial\mathbf{V}^{-1}}{\partial w}\mathbf{V} + \mathbf{V}^{-1}\frac{\partial\mathbf{V}}{\partial w} = \mathbf{0}$$

and so

$$\frac{\partial\mathbf{V}^{-1}}{\partial w} = -\mathbf{V}^{-1}\frac{\partial\mathbf{V}}{\partial w}\mathbf{V}^{-1} \tag{M.18}$$

where

$$\frac{\partial\mathbf{V}}{\partial w} = \left\{_m \frac{\partial v_{ij}}{\partial w}\right\}_{i,j=1}^{n}.$$

Note that (M.18) is a special case of (6.75) for generalized inverses.

Finally, using $\mathbf{P} = \mathbf{K}(\mathbf{K}'\mathbf{V}\mathbf{K})^{-1}\mathbf{K}'$ note that

$$\begin{aligned}\frac{\partial\mathbf{P}}{\partial w} &= -\mathbf{K}(\mathbf{K}'\mathbf{V}\mathbf{K})^{-1}\mathbf{K}'\frac{\partial\mathbf{V}}{\partial w}\mathbf{K}(\mathbf{K}'\mathbf{V}\mathbf{K})^{-1}\mathbf{K}' \\ &= -\mathbf{P}\frac{\partial\mathbf{V}}{\partial w}\mathbf{P}. \end{aligned} \tag{M.19}$$