Research Article

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An overview of financial mathematics with Python codes

https://doi.org/10.1515/mcma-2025-2020
Received July 1, 2025; revised October 31, 2025; accepted November 2, 2025

Abstract: In this note we will discuss and review some recent results concerning well-known financial mathematical problems such as portfolio construction, dynamic trading, option pricing, etc. We will use some Python codes concerning the above and we will compare the results with the existing methods and techniques. We propose also a new type of multi-asset options; the options on correlation. Using this kind of options one can refine more effectively the profit function of his/her portfolio when this contain two or more assets. It is mathematically certain that, in practice, someone will eventually apply the techniques described in this paper. This is because, regarding the portfolio construction problem, we allow the investor to employ any forecasting technique – e.g., statistical methods, machine learning, behavioral finance, intuition, etc. Subsequently, the investor can enhance both the return and safety of their portfolio by incorporating call and put options. In contrast, for the derivative pricing problem, it is evident that there is no room for forecasts (see volatility for example), as pricing involves two counterparties – the seller and the buyer – making it, metaphorically, a dance for two. For this reason, the pricing methodology proposed in this paper is model-free, ensuring that the resulting prices are fully consistent with the market values of available contracts. Moreover, the investor decides at which price to buy or sell a contract based on practical, statically implementable hedging strategies that we propose in this work. That is, any pricing method should justify why an investor ought to buy or sell a derivative at the proposed price. In other words, the method must provide a clear, economically sound rationale-typically grounded in no-arbitrage principles, replication arguments, or explicit hedging strategies – that links the quoted price to actionable, implementable decisions for market participants. What remains to be explored are advanced forecasting techniques that account for events affecting the stocks of interest to the investor, as well as the documentation of hedging strategies for path-dependent options.

Keywords: Prediction, portfolio construction, option pricing and hedging, liquidity, options on correlation

MSC 2020: 91G10, 91G20, 91-04

1 Introduction

In modern financial markets, understanding the basic building blocks of investment instruments is essential for both individual and institutional investors. Among these instruments, *stocks* and *options* play a central role in portfolio construction, risk management, and speculative strategies.

A *stock*, also known as an equity, represents ownership in a company. When an investor purchases shares of a stock, they acquire a claim on part of the company's assets and earnings. The value of a stock fluctuates based on various factors, including company performance, macroeconomic conditions, market sentiment, and geopolitical events. Investing in stocks allows individuals to participate in the growth and profitability of firms, typically through capital appreciation and dividend payments.

On the other hand, *options* are derivative securities whose value is derived from the price of an underlying asset – often a stock. There are two primary types of options: *call options* and *put options*.

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A *call option* gives the holder the right, but not the obligation, to buy the underlying asset at a predetermined price (the strike price) on or before a specified expiration date. Investors typically use call options when they expect the price of the underlying asset to rise. The payoff of this option is the following:

$$P_T = \max\{S_T - K, 0\} \quad \text{(call option payoff)},$$

where S_T is the price of the underlying stock at the time T. Conversely, a *put option* gives the holder the right, but not the obligation, to sell the underlying asset at the strike price before or on the expiration date. Put options are commonly used to hedge against downside risk or to profit from a decline in the asset's price. The payoff of this option is the following:

$$P_T = \max\{K - S_T, 0\} \quad \text{(put option payoff)},$$

where S_T is the price of the underlying stock at the time T.

Options offer flexibility and can be used for various purposes such as hedging, speculation, income generation, and leveraging capital. Their payoff structure makes them powerful tools in both traditional investing and advanced quantitative finance. More information can be found in [8].

Building upon the above concepts, an investor can construct a portfolio by combining various financial instruments such as stocks, risk-free assets, and options.

Suppose an investor has a total amount Y available for investment. A general form of such a portfolio can be expressed as

$$aS_0 + b + \sum_{i=1}^{n} \gamma_i C(K_i) + \delta_i P(K_i) = Y,$$

where

- a is the number of shares of the underlying stock,
- S_0 is the initial price of the stock,
- b is the amount invested in a risk-free asset (e.g., government bonds),
- $C(K_i)$ is the price of a call option with strike price K_i,
- $P(K_i)$ is the price of a put option with strike price K_i,
- γ_i is the number of call options purchased or sold at strike K_i,
- δ_i is the number of put options purchased or sold at strike K_i.

The objective of this portfolio construction is to achieve specific payoff characteristics at the expiration time T, tailored to the investor's preferences regarding risk, return, and market outlook. This involves selecting the parameters a, b, γ_i, δ_i in accordance with the investor's strategic goals – such as hedging, speculation, income generation, or capital protection. The total payoff of the above portfolio is the following:

$$\Pi(x) = ax + be^{rT} + \sum_{i=1}^{n} (\gamma_i (x - K_i)^+ + \delta_i (K_i - x)^+) - Y,$$

where x is the price of the asset at the time T. It is easy to see that we can construct portfolios containing two or more assets together with the call and put options.

This formulation serves as a foundation for more advanced strategies in portfolio management and derivative-based investment design.

At the following link one can find several Python codes concerning almost all the financial mathematics problems of this paper: https://github.com/nikoshalidias/Financial-Engineering.

2 Portfolio construction

The Modern Portfolio Theory (MPT), introduced by Harry Markowitz in 1952, revolutionized the field of financial economics by providing a formal framework for optimal portfolio selection (see [9]). In his seminal work, Markowitz proposed that investors should not only consider the expected return of an asset but also its risk and how it correlates with other assets in the portfolio.

At the core of Markowitz's theory lies the idea that risk-averse investors can construct portfolios to optimize the expected return based on a given level of market risk. This is achieved through diversification, which reduces unsystematic risk – the risk specific to individual assets – without sacrificing potential returns.

Markowitz introduced key concepts such as the *efficient frontier*, which represents the set of optimal portfolios that offer the highest expected return for a defined level of risk or the lowest risk for a given level of expected return. He also emphasized the importance of covariance between asset returns in determining the overall portfolio risk, expressed mathematically as

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij},$$

where
- σ_p^2 is the portfolio variance (a measure of risk),
- w_i, w_j is the weights of assets i and j in the portfolio,
- σ_{ij} is the covariance between the returns of assets i and j.

2.1 Minimum variance portfolio optimization

A central problem in portfolio optimization is finding the allocation of capital across assets that minimizes portfolio risk (variance) while satisfying certain constraints. The minimum variance portfolio is obtained by solving the following quadratic optimization problem:

$$\min_{\mathbf{w}} \mathbf{w}^\top \Sigma \mathbf{w}$$

subject to

$$\sum_{i=1}^{n} w_i = 1$$

Here:
- $\mathbf{w}$ is the vector of portfolio weights,
- Σ is the covariance matrix of asset returns.

This formulation ensures that all weights sum to one (i.e., full investment), while minimizing the total portfolio variance. The solution to this problem provides the optimal weights for constructing the least risky portfolio under the assumption of no expected return constraint. It is easy to see that we can construct similarly portfolios containing two or more assets and their call and put options.

Markowitz's theory is not a forecasting machine

This theory does not take into account:
- company-specific news (e.g., earnings, innovations),
- macroeconomic shocks or crises,
- interest rate or policy changes.

Future stock prices will be shaped by both recent and forthcoming events; thus, a sound prediction requires careful assessment of these factors. Analyzing past numerical data offers little value, since the path of a stock's price in the future unfolds only once – the future will happen only once.

The same holds for any method (for example CAPM) that uses only past numerical data.

2.2 Value at risk (VaR)

In addition to mean-variance analysis, modern risk management often employs the *Value at Risk* (VaR) as a widely used risk measure. VaR estimates the maximum potential loss in value of a portfolio over a specified time period for a given confidence interval.

Mathematically, for a given confidence level α (e.g., 95 %), the VaR is defined as

$$\text{VaR}_\alpha = -\inf\{x \in \mathbb{R} : P(L \leq x) \geq \alpha\},$$

where L denotes the portfolio loss distribution.

VaR provides a straightforward way to quantify downside risk and is commonly used in financial institutions for regulatory reporting and internal risk control. However, it has limitations, such as not accounting for losses beyond the VaR threshold (i.e., tail risk), which has led to the development of alternative measures like Conditional Value at Risk (CVaR).

In Markowitz's Modern Portfolio Theory (MPT), the construction of an optimal portfolio relies heavily on accurate estimation of two key statistical measures: the *expected returns* of individual assets and their *covariances*. These quantities are used to determine the expected return and risk (variance) of any given portfolio.

The expected return of each asset is typically estimated as the historical average of its past returns over a chosen time period. If we denote the vector of historical returns for n assets as $R_i = (r_{i1}, r_{i2}, \dots, r_{iT})$, where $i = 1, 2, \dots, n$, then the expected return μ_i of asset i is calculated as

$$\mu_i = \frac{1}{T} \sum_{t=1}^{T} r_{it}.$$

The covariance matrix, which captures how the returns of different assets move together, is computed using the deviations of each asset's returns from their respective means. The sample covariance between asset i and asset j is given by

$$\sigma_{ij} = \frac{1}{T-1} \sum_{t=1}^{T} (r_{it} - \mu_i)(r_{jt} - \mu_j).$$

This results in an $n \times n$ covariance matrix Σ, where diagonal elements represent the variances of individual assets, and off-diagonal elements represent the pairwise covariances.

It should be noted that these estimates depend heavily on the number of historical data points used in their calculation. One can use the code `Markowitz2.ipynb` using real data from Yahoo Finance.

Within the framework of Markowitz's theory, portfolios are constructed with a specific investment horizon in mind – typically denoted by a future time point T. The goal is to select a combination of assets that optimizes the trade-off between expected return and risk at this target time.

2.3 Improving forecasting through historical context

While the classical approach in Markowitz's framework relies on historical averages and covariances, a more sophisticated forecasting method would incorporate not only numerical data but also the *contextual factors* that influenced those data points. The intuition of a seasoned investor is, by far, more efficient and insightful than purely relying on retrospective numerical data analysis. Specifically, an enhanced predictive model could link historical asset returns with corresponding past events – such as macroeconomic shifts, geopolitical developments, or changes in monetary policy – to better understand the underlying drivers of market behavior.

By analyzing how markets have historically reacted to specific types of events, investors can move beyond purely statistical estimates and develop more informed expectations about future asset performance. This contextual analysis allows for the construction of scenarios that reflect potential market responses under similar future conditions, thereby improving the accuracy of expected returns and risk measures.

In this sense, integrating qualitative and quantitative insights leads to a more robust forecasting mechanism. Such an approach aligns with the broader objective of portfolio optimization: to construct investment strategies that are not only based on past performance, but also attuned to the evolving economic and financial landscape.

Example 1 (An airline stock). A 10-year history suggests that when crude oil falls by more than 15 % over a quarter and bookings recover, the sector tends to show positive returns over the next 6–9 months. A golden cross (50DMA > 200DMA) further reinforces a mechanical buy signal.

Recent events (qualitative information):
- Unexpected *fleet grounding* by a regulator after a safety incident.
- *Spike in fuel costs* due to a new geopolitical shock.
- *Strike risk* after a fresh union warning.

These immediately alter revenue, costs, and liquidity, breaking the historical pattern that the statistical model relied on.

Example 2 (A semiconductor stock.). Inventory-cycle patterns and a falling gross margin imply that after three consecutive quarters of rising inventory-to-sales, the 12-month forward return is, on average, negative. Price sits below the 200-day moving average, triggering a mechanical sell/avoid signal.

Recent events (qualitative information):
- A *multi-year supply agreement* with a top AI cloud provider including minimum purchase commitments.
- *Government incentives* and tax credits for a new fab that lower effective CAPEX per unit.
- *Export restrictions lifted* in a key market, reopening a high-margin demand channel.

These updates increase revenue visibility and fundamentally change the distribution of future cash flows, overturning the historical "average" inventory signal.

Why the mismatch happens. Signals from past data capture *average historical relationships*. Fresh, non-stationary information (regulatory, geopolitical, contractual, or technological) can rapidly change the *expected cash-flow landscape*, which the original sample never saw. Best practice is to separate *forecasting* from *portfolio construction* and to use instruments (e.g., options) or constraints that cap maximum loss at a known budget D, ensuring the position adapts to the *current* information set rather than only historical means and covariances. At the forecasting stage one should also use new information from the market, using machine learning, behavioral finance, etc.

2.4 Separating forecasting from portfolio construction

An important methodological consideration in portfolio design is the distinction between the *forecasting phase* and the *portfolio construction phase*. While Markowitz's theory assumes that expected returns and covariances are derived directly from historical averages, the framework itself remains valid regardless of how these inputs are obtained.

This framework enables the incorporation of more advanced forecasting techniques – such as machine learning algorithms, macroeconomic scenario analysis, or sentiment-based indicators – to produce more informed predictions regarding future asset prices. These methods can significantly enhance the quality of input parameters used in portfolio construction, moving beyond simple historical averages.

Types of predictions. In this context, we can distinguish two main types of financial predictions:
- The first type involves specifying a set $A \subseteq \mathbb{R}_+^n$, representing the investor's belief that the vector of asset prices at time T, i.e., $(S_1^T, \ldots, S_n^T)$, will lie within this set.
- The second approach assumes that the random vector $(S_1^T, \ldots, S_n^T)$ follows a known probability distribution with specified parameters. This allows for a probabilistic characterization of future outcomes and facilitates risk assessment within the optimization process.

These predictive approaches provide a richer foundation for portfolio selection, while remaining compatible with the principles of mean-variance optimization.

2.5 Core option strategies

2.5.1 Main results on portfolio construction

Below, we are going to generalize all the above strategies into a single one optimization problem. Extending standard strategies like butterflies affects both pricing and hedging as we shall see at the next sections. Any pricing method must be arbitrage-free and consistent with traded options, but why trade at its price? Only a feasible hedge provides the answer.

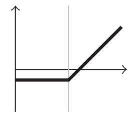
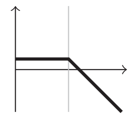
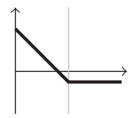
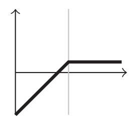
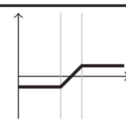
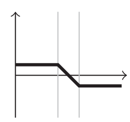
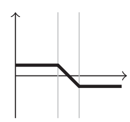
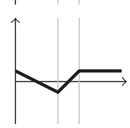
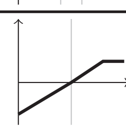
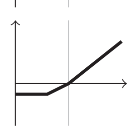
Strategy	Bias	Structure	Profit function
Long Call	Bullish	Buy Call (K), debit	
Short Call	Bearish/Neutral	Sell Call (K), credit	
Long Put	Bearish	Buy Put (K), debit	
Short Put	Bullish/Neutral	Sell Put (K), credit	
Bull Call Spread	Bullish	+Call(K_1), -Call(K_2), $K_1<K_2$	
Bear Put Spread	Bearish	+Put(K_1), -Put(K_2), $K_1>K_2$	
Bull Put Spread	Bullish/Neutral	-Put(K_1), +Put(K_2), $K_1>K_2$	
Bear Call Spread	Bearish/Neutral	-Call(K_1), +Call(K_2), $K_1<K_2$	
Covered Call	Mildly Bullish	Long stock S_0 & -Call(K)	
Protective Put	Bullish with Floor	Long stock S_0 & +Put(K)	

Table 1: Single-leg, vertical spreads, and stock combinations.

Let us see what we can do concerning portfolios that contain one asset and their derivatives. If we have a prediction of the first type, that is $S_T \in A \subseteq \mathbb{R}_+$, then we can solve a linear programming problem as we will see below. We seek the best portfolio which is such that $\Pi(x) \geq 0$ for any $x \in A$ and moreover $\Pi(x) \geq -D$ for all $x \geq 0$. The first requirement is for the profit on the prediction set while the second is the maximum acceptable loss in the worst case scenario. Our aim is to translate these two requirements into a finitely many linear constraints arriving to a linear programming problem. This can be done since the payoff of the options are piecewise linear with finite many branches.

The following theorems illustrates the above for the one asset case and the multi asset case as well.

Theorem 1 (Static superhedging on a finite grid (one-asset case)). *Fix strikes $\{K_1, \ldots, K_m\}$ for calls and $\{K'_1, \ldots, K'_{\tilde{m}}\}$ for puts on a single stock. Consider a static portfolio (in discounted units) with profit function*

$$\Pi(x) = b + \phi(x) - V, \quad x \in \mathbb{R}_+,$$

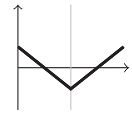
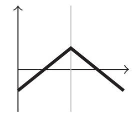
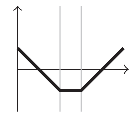
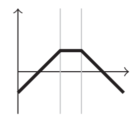
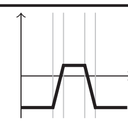
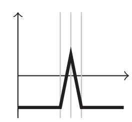
Strategy	Vol view	Structure	Profit function
Long Straddle	High Vol	+Call(K) & +Put(K), debit	
Short Straddle	Low Vol	−Call(K) & −Put(K), credit	
Long Strangle	High Vol	+Put(K_p), +Call(K_c), K_p<K_c	
Short Strangle	Low Vol	−Put(K_p), −Call(K_c), K_p<K_c	
Iron Condor (short)	Low Vol	−Put(K_2), +Put(K_1), −Call(K_3), +Call(K_4)	
Iron Butterfly (short)	Low Vol	−Call(K), −Put(K), +Call(K_1), + Put(K_2) at $K_1 < K < K_2$	

Table 2: Volatility/neutral, iron structures.

where

$$\phi(x) = a\,x + \sum_{j=1}^{m} c_j\,(x - K_j)^+ + \sum_{k=1}^{\tilde m} d_k\,(K'_k - x)^+$$

for real coefficients b, a, c_j, d_k (long > 0, short < 0). Let $V \in \mathbb{R}$ be the (discounted) initial setup cost and fix $D \in \mathbb{R}$. Define the node set

$$S := \{0\} \cup \{K_1, \dots, K_m, K'_1, \dots, K'_{\tilde m}\}.$$

Then the following are equivalent:

(i) $\Pi(x) \geq D$ *for all* $x \in \mathbb{R}_+$.

(ii) *The following finite family of linear inequalities holds:*

$$a + \sum_{j=1}^{m} c_j \geq 0, \tag{2.1}$$

$$(b - V) + \phi(s) \geq D \quad \textit{for every } s \in S. \tag{2.2}$$

Proof. Write the profit function as $\Pi(x) = (b - V) + \phi(x)$. We have

$$\inf_{x\geq 0}\big(\Pi(x)\big) = (b - V) + \inf_{x\geq 0} \phi(x). \tag{2.3}$$

Step 1: Tail slope and boundedness. On $[0, \infty)$ the function ϕ is piecewise affine with nodes at S, and its right slope for $x \to \infty$ equals $a + \sum_j c_j$. If this slope is negative then $\phi(x) \to -\infty$ as $x \to \infty$, so no uniform lower bound D can hold; hence (2.1) is necessary. Assuming (2.1), the function ϕ is bounded below on $[0, \infty)$.

Step 2: Minimum occurs at a node (or at 0). Between consecutive points of S, the function ϕ is affine; thus its minimum over any closed interval is attained at an endpoint. Under (2.1), the global minimum over $[0, \infty)$ is attained at some $s \in S$, i.e.,

$$\inf_{x\geq 0} \phi(x) = \min_{s\in S} \phi(s). \tag{2.4}$$

Step 3: Equivalence. Combining (2.3) and (2.4) yields

$$\inf_{x\geq 0}(\Pi(x)) = \min_{s\in S}[(b-V)+\phi(s)].$$

Therefore, $\Pi(x) \geq D$ for all $x \geq 0$ holds if and only if (2.2) holds for every $s \in S$, together with the necessary tail condition (2.1) to preclude $-\infty$ along the ray $x \to \infty$. This proves the equivalence of (i) and (ii). □

Theorem 2 (Finite-point certification on a subset of $\mathbb{R}_+$). *Fix finite strike sets $\{K_1,\dots,K_m\}$ for calls and $\{K'_1,\dots,K'_{\tilde m}\}$ for puts on a single stock. Consider a static portfolio (in discounted units) with profit function*

$$\Pi(x) = b - V + \phi(x), \quad \phi(x) = a\,x + \sum_{j=1}^{m} c_j\,(x-K_j)^+ + \sum_{k=1}^{\tilde m} d_k\,(K'_k - x)^+,$$

for real coefficients b, a, c_j, d_k (long > 0, short < 0). Let $V \in \mathbb{R}$ be the (discounted) setup cost.

Let $G \subseteq \mathbb{R}_+$ be a set that is a finite union of intervals and/or singletons (hence with finite boundary). Define the node set $S := \{0\} \cup \{K_1,\dots,K_m,K'_1,\dots,K'_{\tilde m}\}$ and

$$E_G := (S \cap \overline{G}) \cup \partial G,$$

where $\overline{G}$ is the closure and ∂G the topological boundary of G in $\mathbb{R}_+$. Then the following are equivalent:

(i) $\Pi(x) \geq 0$ *for all $x \in G$.*

(ii) *The following finite family of linear conditions holds:*

$$\text{(Tail, only if } G \text{ is unbounded above)} \qquad a + \sum_{j=1}^{m} c_j \geq 0, \tag{2.5}$$

$$\text{(Grid on } G) \qquad (b-V)+\phi(s) \geq 0 \quad \text{for every } s \in E_G. \tag{2.6}$$

Equivalently, there exist points $x_i \in E_G$ and numbers $D_i \geq 0$ (one for each x_i) such that

$$(b-V)+\phi(x_i) \geq D_i \quad \text{for all } i,$$

and in particular one may take $D_i = 0$ for all i.

Proof. On $[0,\infty)$ the function ϕ is piecewise affine with nodes at S.

Step 1: Tail boundedness. If G is unbounded above (i.e., $\sup G = +\infty$) and $a + \sum_j c_j < 0$, then $\phi(x) \to -\infty$ along G, so (i) fails. Thus (2.5) is necessary whenever G is unbounded above. Under (2.5) (or if G is bounded), ϕ is bounded below on G.

Step 2: Where the minimum over G can occur. Between consecutive nodes in S, ϕ is affine. On any connected component I of G lying inside such an affine segment, the minimum of the continuous affine function $(b-V)+\phi$ over I is attained at an endpoint of I (possibly an endpoint not belonging to G if I is half-open). Hence the global minimum of $(b-V)+\phi$ on G is attained at a point in $(S \cap \overline{G}) \cup \partial G = E_G$.

Step 3: Equivalence. Therefore,

$$\inf_{x\in G}(\Pi(x)) = \min_{s\in E_G}((b-V)+\phi(s)),$$

provided the tail is nonnegative when G is unbounded. Consequently, $\Pi(x) \geq 0$ for all $x \in G$ holds if and only if $(b-V)+\phi(s) \geq 0$ for all $s \in E_G$, together with the necessary tail condition when $\sup G = +\infty$. This is precisely (2.6)-(2.5). Finally, since E_G is finite, (2.6) is equivalent to the existence of nonnegative margins $D_i \geq 0$ at finitely many points $x_i \in E_G$ with $(b-V)+\phi(x_i) \geq D_i$ (e.g., take $D_i = 0$). □

Remark 3. If G is bounded, the tail condition (2.5) is not needed. If $G = [\alpha,\infty)$, then $E_G = (S \cap [\alpha,\infty)) \cup \{\alpha\}$ and the single tail inequality $a + \sum_j c_j \geq 0$ together with $(b-V)+\phi(s) \geq 0$ for $s \in E_G$ is necessary and sufficient.

Theorem 4 (Static superhedging on a finite grid (d-asset case)). *Fix $d \in \mathbb{N}$ and, for each asset $i = 1,\dots,d$, finite strike sets $\{K_{i,1},\dots,K_{i,m_i}\}$ for calls and $\{K'_{i,1},\dots,K'_{i,\tilde m_i}\}$ for puts. Consider a static portfolio (in discounted units)* *with profit function*

$$\Pi(x) = b - V + \sum_{i=1}^{d} \phi_i(x_i), \quad x = (x_1, \dots, x_d) \in \mathbb{R}_+^d,$$

where, for each i,

$$\phi_i(u) = a_i\, u + \sum_{j=1}^{m_i} c_{i,j}\,(u - K_{i,j})^+ + \sum_{k=1}^{\tilde{m}_i} d_{i,k}\,(K'_{i,k} - u)^+,$$

with real coefficients $b, a_i, c_{i,j}, d_{i,k}$ *(long* > 0*, short* < 0*). Let* $V \in \mathbb{R}$ *be the (discounted) initial setup cost and fix* $D \in \mathbb{R}$*. For each i define the node set*

$$S_i := \{0\} \cup \{K_{i,1}, \dots, K_{i,m_i}, K'_{i,1}, \dots, K'_{i,\tilde{m}_i}\}.$$

Then the following are equivalent:

(i) $\Pi(x) \geq D$ *for all* $x \in \mathbb{R}_+^d$.

(ii) *The following* finite *system of linear inequalities holds:*

$$a_i + \sum_{j=1}^{m_i} c_{i,j} \geq 0 \quad \textit{for each } i = 1, \dots, d, \tag{2.7}$$

$$(b - V) + \sum_{i=1}^{d} \phi_i(s_i) \geq D \quad \textit{for every } (s_1, \dots, s_d) \in S_1 \times \cdots \times S_d. \tag{2.8}$$

Proof. By separability,

$$\inf_{x \in \mathbb{R}_+^d} \big(\Pi(x)\big) = (b - V) + \sum_{i=1}^{d} \inf_{u \geq 0} \phi_i(u). \tag{2.9}$$

Step 1: Tail slopes and boundedness. For each i, the function ϕ_i is piecewise affine on $[0, \infty)$ with nodes at S_i. Its right slope as $u \to \infty$ equals $a_i + \sum_j c_{i,j}$. If this slope is negative for some i, then $\phi_i(u) \to -\infty$ and no uniform lower bound D for $\Pi(x)$ can hold. Hence the tail conditions (2.7) are necessary. Under (2.7), each ϕ_i is bounded below on $[0, \infty)$.

Step 2: Minima occur at nodes (or at 0). Between consecutive points of S_i, the function ϕ_i is affine; therefore its minimum over any closed interval is attained at an endpoint. Using (2.7), the global minimum over $[0, \infty)$ satisfies

$$\inf_{u \geq 0} \phi_i(u) = \min_{s_i \in S_i} \phi_i(s_i), \quad i = 1, \dots, d. \tag{2.10}$$

Step 3: Equivalence. Combining (2.9) and (2.10) yields

$$\inf_{x \in \mathbb{R}_+^d} \big(\Pi(x)\big) = \min_{(s_1, \dots, s_d) \in S_1 \times \cdots \times S_d} \left[(b - V) + \sum_{i=1}^{d} \phi_i(s_i) \right].$$

Thus $\Pi(x) \geq D$ for all $x \in \mathbb{R}_+^d$ holds if and only if (2.8) holds for every grid point, together with the necessary tail conditions (2.7) to preclude $-\infty$ along the rays $x_i \to \infty$. This proves the equivalence of (i) and (ii). □

Theorem 5 (Finite-point certification on a subset of $\mathbb{R}_+^d$). *Fix* $d \in \mathbb{N}$ *and, for each asset* $i = 1, \dots, d$*, finite strike sets* $\{K_{i,1}, \dots, K_{i,m_i}\}$ *for calls and* $\{K'_{i,1}, \dots, K'_{i,\tilde{m}_i}\}$ *for puts. Consider a static portfolio (in discounted units) with profit function*

$$\Pi(x) = b - V + \sum_{i=1}^{d} \phi_i(x_i), \quad x = (x_1, \dots, x_d) \in \mathbb{R}_+^d,$$

where

$$\phi_i(u) = a_i\, u + \sum_{j=1}^{m_i} c_{i,j}(u - K_{i,j})^+ + \sum_{k=1}^{\tilde{m}_i} d_{i,k}(K'_{i,k} - u)^+,$$

with real coefficients $b, a_i, c_{i,j}, d_{i,k}$ *(long* > 0*, short* < 0*). Let* $V \in \mathbb{R}$ *be the (discounted) setup cost.*

Let $G \subseteq \mathbb{R}_+^d$ *be a* finite union of axis-aligned boxes, *i.e.,*

$$G = \bigcup_{\ell=1}^{L} \prod_{i=1}^{d} I_{i,\ell},$$

where each $I_{i,\ell}$ is one of $[\alpha,\beta]$, $[\alpha,\infty)$, or the singleton $\{\alpha\}$ with $0 \le \alpha \le \beta < \infty$. For each i, define

$$S_i := \{0\} \cup \{K_{i,1},\dots,K_{i,m_i},K'_{i,1},\dots,K'_{i,\tilde{m}_i}\}, \quad B_i := \{\textit{all finite endpoints appearing among the } I_{i,\ell}\},$$

and set

$$E_i := \big(S_i \cap \overline{\pi_i(G)}\big) \cup B_i, \quad E := E_1 \times \cdots \times E_d,$$

where $\pi_i(G)$ is the projection of G onto the i-th axis and $\bar{\cdot}$ denotes closure. Let $I_\infty := \{\, i : \sup \pi_i(G) = +\infty \,\}$ be the set of coordinates along which G is unbounded above.

Then the following are equivalent:

(i) $\Pi(x) \ge 0$ *for all* $x \in G$.

(ii) *The following finite system of linear conditions holds:*

$$a_i + \sum_{j=1}^{m_i} c_{i,j} \ge 0 \quad \textit{for every } i \in I_\infty, \tag{2.11}$$

$$(b - V) + \sum_{i=1}^{d} \phi_i(s_i) \ge 0 \quad \textit{for every } s = (s_1,\dots,s_d) \in E \cap \overline{G}. \tag{2.12}$$

Equivalently, there exist finitely many points $x^{(1)},\dots,x^{(N)} \in E \cap \overline{G}$ and numbers $D_1,\dots,D_N \ge 0$ (e.g., one can take all $D_\ell = 0$) such that

$$\Pi(x^{(\ell)}) \ge D_\ell, \quad \ell = 1,\dots,N,$$

and (2.11) *holds.*

Proof. Each ϕ_i is piecewise affine on $[0,\infty)$ with nodes at S_i.

Step 1: Tail boundedness. For $i \in I_\infty$, the right slope of ϕ_i as $u \to \infty$ equals $a_i + \sum_j c_{i,j}$. If this slope is negative for some $i \in I_\infty$, then along G we can send $x_i \to \infty$ keeping the other coordinates fixed, and obtain $\phi_i(x_i) \to -\infty$; hence (i) fails. Thus (2.11) is necessary. When (2.11) holds (or if $i \notin I_\infty$), each ϕ_i is bounded below on $\pi_i(G)$, so $\Pi(\cdot)$ is bounded below on G.

Step 2: Reduction to a finite grid. For each i, order the finite set E_i increasingly and use it to partition $[0,\infty)$ into intervals with endpoints in E_i. On the resulting *grid* of products of such intervals, each ϕ_i is affine on every interval factor, hence the sum $(b - V) + \sum_i \phi_i$ is affine on each grid cell. Intersecting this finite grid with G (a finite union of boxes) yields a finite union of convex polytopes, on each of which $(b - V) + \sum_i \phi_i$ is affine. Therefore the minimum of an affine function over each such polytope is attained at one of its vertices. Every such vertex has coordinates in E_i, hence belongs to $E \cap \overline{G}$. Consequently,

$$\inf_{x\in G}\big(\Pi(x)\big) = \min_{s\in E\cap\overline{G}}\left[(b - V) + \sum_{i=1}^{d} \phi_i(s_i)\right],$$

provided (2.11) holds on I_∞.

Step 3: Equivalence. From Step 2, (i) holds iff the right-hand side above is ≥ 0, i.e., iff (2.12) holds for all $s \in E \cap \overline{G}$, together with the necessary tail conditions (2.11). Since $E \cap \overline{G}$ is finite, this is equivalent to the existence of finitely many points $x^{(\ell)}$ in $E \cap \overline{G}$ and nonnegative margins $D_\ell \ge 0$ with $\Pi(x^{(\ell)}) - V \ge D_\ell$ (for instance $D_\ell = 0$), completing the proof. □

Remark 6. If G is bounded, no tail condition is required (i.e., $I_\infty = \varnothing$). When $G = \prod_{i=1}^{d}[\alpha_i,\infty)$, one has

$$E_i = (S_i \cap [\alpha_i,\infty)) \cup \{\alpha_i\}$$

and the single family of inequalities (2.11)–(2.12) is necessary and sufficient.

Example 3 (Constructing portfolios under the first type of prediction). Suppose an investor wants to invest an amount Y in a stock and the corresponding calls and puts. Suppose they predict that the stock price will lie in the interval (c, v) at time T. Then they can allocate Y after solving the following linear program.

Given an acceptable loss D, find the coefficients $a, b, \gamma_i, \delta_i, D_c, D_v, D_i, M$ such that

$$\begin{aligned} \text{maximize} \quad & w_c D_c + w_1 D_1 + \cdots + w_l D_l + w_v D_v + w_M M \\ \text{subject to} \quad & \Pi(x) \geq -D \quad \text{for every } x \geq 0, \\ & \Pi(x) \geq 0 \quad \text{for every } x \in (c, v), \\ & ax + be^{rT} + \sum_{i=1}^{n} \gamma_i C(K_i) + \delta_i P(K_i) = Y, \\ & \sum |\gamma_i| + |\delta_i| \leq N, \\ & a, b, \gamma_i, \quad \delta_i \in [-N_i, N_i], \end{aligned} \tag{2.13}$$

where w_c, w_v, w_i, w_M are investor-chosen weights. Note that the set of constraints arising from $\Pi(x) \geq -D$ for all $x \geq 0$ are not necessary while some restrictions on a, b, γ_i, δ_i are. Limits on the number of options are determined by option liquidity and the investor's financial resources.

If $A \subseteq \mathbb{R}_+$ is a union of intervals we can simply choose some points and their weights in these intervals and solve the above optimization problem. The designing of linear programming problems for the construction of portfolios containing d assets and their derivatives are straightforward.

It is clear that the above setting is a generalization of the strategies that we have described before since we can design more complex strategies.

Instead of $\Pi(x) \geq -D$ for all $x \geq 0$ we can use a weaker constraint such the following: $\Pi(x) \geq -D$ for all $x \in (H, L)$ for some $0 < H < L$ chosen by the investor.

Remark 7 (Protecting a portfolio from excessive loss). An investor who does not wish to incur a loss greater than the amount D during their investment in a portfolio (which does not contain options) can simply redeem it once the loss reaches that limit. However, with this strategy, the investor effectively locks in the loss, potentially missing out on profit opportunities if the portfolio were to recover and reach profitable scenarios. Moreover, there is a risk that the loss could increase further if the portfolio continues to lose value.

Instead, by equipping the portfolio with appropriate options, it is ensured that the maximum loss will never exceed the amount D. Thus, the investor can redeem the portfolio whenever they wish without being exposed to undesirable losses.

Theorem 8 (Low loss–low profit). *Consider problem* (2.13) *with objective*

$$Z(D) = \max\left(w_c D_c + w_1 D_1 + \cdots + w_l D_l + w_v D_v + w_M M\right)$$

subject to

$$\Pi(x) \geq -D \quad \textit{for all } x \geq 0, \qquad \Pi(x) \geq 0 \quad \textit{for all } x \in (c, v),$$

together with the linear cost/position constraints stated in (2.13). *Then $Z(D)$ is nondecreasing in D: if $0 \leq D_1 \leq D_2$, then $Z(D_1) \leq Z(D_2)$.*

Proof. Let $D_1 < D_2$. Consider an optimal solution $(a^*, b^*, \gamma_i^*, \delta_i^*, D_c^*, D_1^*, \ldots, D_l^*, D_v^*, M^*)$ to the portfolio construction problem with maximum acceptable loss $D = D_1$. This solution satisfies all constraints for $D = D_1$. Since $D_1 < D_2$, the constraint $\Pi(x) \geq -D_1$ is stricter than $\Pi(x) \geq -D_2$. Therefore, any solution that satisfies the constraints for $D = D_1$ also satisfies the constraints for $D = D_2$. Thus, the solution $(a^*, b^*, \gamma_i^*, \delta_i^*, D_c^*, D_1^*, \ldots, D_l^*, D_v^*, M^*)$ is also feasible for the problem with $D = D_2$. By definition, $Z(D_1)$ is the value of the objective function for this solution, that is,

$$Z(D_1) = wcD_c^* + w_1 D_1^* + \cdots + w_l D_l^* + w_v D_v^* + w_M M^*.$$

The value $Z(D_2)$ is the maximum possible value of the objective function for the problem with $D = D_2$. Since the solution above is feasible for $D = D_2$, we have

$$Z(D_2) \geq wcD_c^* + w_1 D_1^* + \cdots + w_l D_l^* + w_v D_v^* + w_M M^* = Z(D_1).$$

Therefore, for any $D_1 < D_2$, we have $Z(D_1) \leq Z(D_2)$, which means that $Z(D)$ is an increasing function of D. □

This result has important financial implications: as the maximum acceptable loss D increases, the investor can construct portfolios with potentially higher expected returns while still maintaining a level of risk control. This aligns with the intuitive understanding that more risk tolerance generally allows for higher potential returns in portfolio optimization problems.

Theorem 9 (Low risk–low profit). *Let $Z(A)$ denote the optimal value of the maximization problem* (2.13) *when the profit constraint is imposed on the set $A \subseteq \mathbb{R}_+$, i.e., $\Pi(x) \geq 0$ for all $x \in A$ (with the rest of the constraints unchanged). If $A_1 \subseteq A_2$, then $Z(A_1) \geq Z(A_2)$.*

Proof. For each A, let $\mathcal{F}(A)$ be the feasible set of 2.13 when the constraint $\Pi(x) \geq 0$ is enforced on A. If $A_1 \subseteq A_2$, then requiring $\Pi(x) \geq 0$ on the larger set A_2 is stricter, hence $\mathcal{F}(A_2) \subseteq \mathcal{F}(A_1)$. Since we are maximizing and the objective function does not depend on A, the maximum over the smaller feasible set cannot be larger:

$$Z(A_2) = \max_{z \in \mathcal{F}(A_2)} \operatorname{obj}(z) \leq \max_{z \in \mathcal{F}(A_1)} \operatorname{obj}(z) = Z(A_1).$$

Therefore $Z(A_1) \geq Z(A_2)$. □

Remark 10. If instead $Z(A)$ were defined as a *minimum* cost (e.g., writer's cost for super-hedging on A), then the inequality reverses and $Z(A_1) \leq Z(A_2)$ when $A_1 \subseteq A_2$.

Remark 11 (Profit realization in option portfolios). Recall that the value of a put option increases as the price of the underlying asset decreases, whereas the value of a call option increases as the asset price rises. Consider a portfolio consisting exclusively of long positions in American-style options. For any time $t \in [0, T]$ such that the underlying asset price S_t belongs to the prediction set A corresponding to favorable price levels, the investor has the opportunity to either exercise the options or liquidate the entire position immediately, thereby realizing a profit.

In contrast, if the portfolio comprises only long positions in European-style options – whose early exercise is not permitted – a profit may still be realized by selling the options in the secondary market whenever $S_t \in A$. In this case, the realization of profit does not necessitate the execution of transactions involving the underlying asset at the predetermined strike price.

Example 4 (Constructing portfolios under the second type of prediction). Suppose the investor want to invest the amount Y in d-stocks and their derivatives. Let us assume that the investor's prediction is that $(S_1^T, \dots, S_d^T)$ follows a known probability distribution. Then they can allocate Y after solving the following optimization problem.

Given an acceptable loss D, find the coefficients $a, b, \gamma_i, \delta_i, D_c, D_v, D_i, M$ such that

$$\begin{aligned} &\text{maximize} && F(\mathbb{P}(\Pi(S_1^T, \dots, S_d^T), \operatorname{Var}(\Pi(S_1^T, \dots, S_d^T)), \dots) \\ &\text{subject to} && \Pi(x_1, \dots, x_d) \geq -D \quad \text{for any } (x_1, \dots, x_d) \in \mathbb{R}_+^d, \\ &&& \sum |\gamma_i| + |\delta_i| \leq N, \\ &&& \gamma_i, \delta_i \in [-N_i, N_i], \end{aligned} \tag{2.14}$$

where F is a function chosen by the investor.

Using the Python code `PCUP.ipynb` (to solve problem (2.13)) the user can give some predictions and their weights about the future price of the option, the maximum acceptable loss D and the maximum call and put options to buy/sell (maybe zero). A similar Python code for portfolio construction with two assets is the `Multi-Asset4.ipynb`. We have take into account the bid/ask spread in these Python codes and we use real data taken from yahoo finance.

So, the above method of portfolio construction allows the investor to use sophisticated prediction mechanisms and also to include call and put options (if they wish) at their portfolios. Therefore it is clearly an extension of the Markowitz portfolio construction theory.

The first four images in Figure 1 are for the profit function of a portfolio containing one asset and some derivatives. Under any prediction we can construct a portfolio with profit at this scenario while the maximum

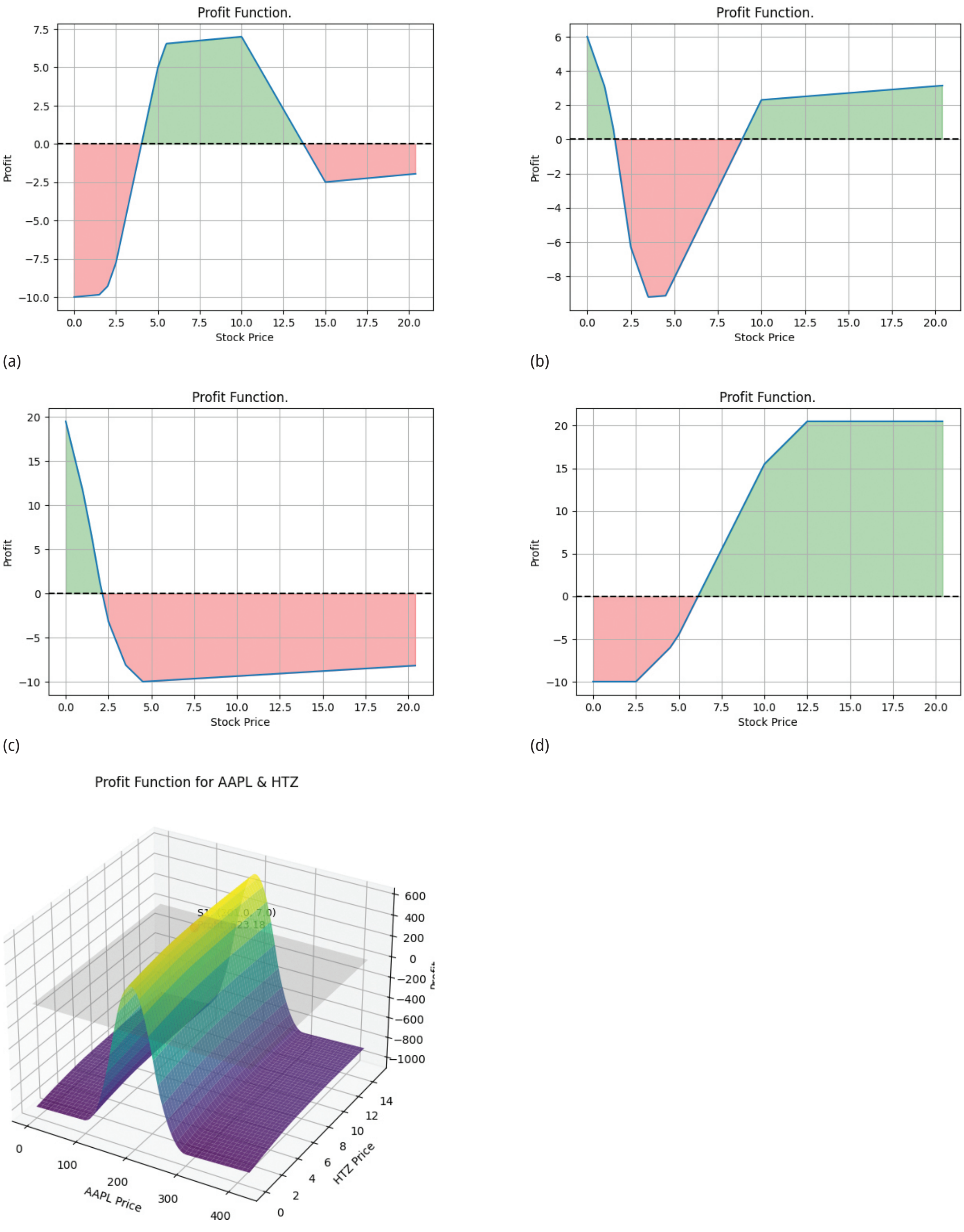


Figure 1

possible loss is the number D. The fifth image is the plot of the profit function of a portfolio containing two assets and some of their derivatives. In this case also we can construct a portfolio with profit at any scenario we want and the maximum possible loss is again the given amount D. We have used real data from the yahoo finance taking into account the bid/ask spread.

The investor can submit a single order to buy or sell multiple derivative contracts simultaneously through an *all-or-none* instruction. This means that the transaction will be executed only if all components of the order – across different strike prices and types (call/put) – can be fulfilled at the specified prices and quantities. Otherwise, the entire order is rejected, ensuring that the portfolio is constructed exactly as intended without partial executions that could compromise the strategy.

This type of order is particularly useful in algorithmic trading and portfolio hedging, where maintaining the integrity of the portfolio structure is crucial for risk management and payoff replication.

2.6 A new type of multi-asset option: Options on correlation

Let two assets S_1, S_2 and the corresponding call and put options with strike prices $K_1 < \cdots < K_n$ and $L_1 \cdots L_m$ respectively. As we can see in Figure 1 (a)–(d) the call and put options are enough to manipulate the profit function as we want. But in two dimensions (see Figure 1 (e)) they are not. If we want to refine the profit function more efficiently, it would be easier if the following contracts (options on correlation) existed,

$$\begin{aligned}
P_{CC}(K_i, L_j) &= \min\{(S_1^T - K_i)^+, (S_2^T - L_j)^+\} \quad \text{for } i = 1, \ldots, n,\ j = 1, \ldots, m,\\
P_{CP}(K_i, L_j) &= \min\{(S_1^T - K_i)^+, (L_j - S_2^T)^+\} \quad \text{for } i = 1, \ldots, n,\ j = 1, \ldots, m,\\
P_{PC}(K_i, L_j) &= \min\{(K_i - S_1^T)^+, (S_2^T - L_j)^+\} \quad \text{for } i = 1, \ldots, n,\ j = 1, \ldots, m,\\
P_{PP}(K_i, L_j) &= \min\{(K_i - S_1^T)^+, (L_j - S_2^T)^+\} \quad \text{for } i = 1, \ldots, n,\ j = 1, \ldots, m,
\end{aligned}$$

where P_{CC}, etc., are the payoff functions.

To be more precise, suppose the investor places a bet on the event $(S_1^T, S_2^T) \in G \subseteq \mathbb{R}_+^2$. If the investor only has access to call and put options on the underlying assets, the resulting profit function will be positive in the region G, but it may also be positive in other regions. This implies that the profit generated within G will be diluted compared to what could be achieved if the profit function were concentrated exclusively on G.

By utilizing options on correlation, however, we can construct a portfolio such that the profit function is strictly positive within G while maximizing the potential profit. Thus, options on correlation are not only of financial and economic interest for trading purposes but also present a mathematical challenge related to the optimal design of a portfolio.

2.6.1 Arbitrage considerations and pricing

How should such contracts be priced? Using standard no-arbitrage reasoning, we can easily deduce that, for example, the price of a contract with payoff $P_{CC}(K_i, L_j)$ must be lower than the prices of the individual call options with strike prices K_i and L_j. Otherwise, the writer of the contract could construct a portfolio that guarantees a riskless profit. For a more accurate bounds one should compute the arbitrage-free interval that we describe in Section 5.1.

It is important to note that if one attempts to price such an option using any of the classical option pricing models, the resulting value will, in general, lead to an arbitrage opportunity. This is because these models do not take into account the existing prices of related call and put options in the market. This limitation applies broadly to all multi-asset options.

The only method that can consistently yield arbitrage-free prices is the one we propose in [5]; see also Section 5 below for more details.

2.6.2 Motivation for writing these multi-asset contracts

Why would someone wish to write a contract with payoff $P_{CC}(K_i, L_j)$? A possible motivation is that the writer believes there is virtually no chance that the prices of the two underlying assets will increase simultaneously.

Conversely, an investor may believe that the two assets are sufficiently positively correlated to justify purchasing contracts with payoffs P_{PP} and P_{CC} for suitable strike prices, anticipating joint movements in the asset's values.

Of course, these type of options can be generalized concerning d-assets.

2.6.3 Constructing a portfolio

Let two options S_1, S_2 and suppose that the investor predicts that $(S_1^T, S_2^T) \in G \subseteq \mathbb{R}_+^2$. The domain G is not necessarily a rectangle taking into account the possible correlation of these assets. The investor can choose some points $(U_i, V_j) \in G$ and a weight w_{ij} for each point.

Given a maximum acceptable loss D, find the coefficients a, b, c, γ_i such that

$$\begin{aligned} \text{maximize} \quad & \sum w_{ij} D_{ij} \\ \text{subject to} \quad & \Pi(x,y) \geq -D \quad \text{for all } x, y \geq 0, \\ & \Pi(U_i, V_j) \geq D_{ij} \geq 0, \\ & ax + by + ce^{rT} + \sum \gamma_i P_i = Y, \end{aligned} \tag{2.15}$$

where P_i is the price of each contract (call, put, option on correlation). Here, the profit function $\Pi(x,y)$ contain all the payoffs of all the contracts (i.e., call, put and options on correlation). We should add also some constraints on the number of options to buy/sell.

3 Arbitrage

Definition 1 (Arbitrage). Let a market consists of one asset and the corresponding call and put options. We say that there exists an opportunity for risk-free profit (*arbitrage*) in that market if we can construct a portfolio with a payoff function satisfying the following:

$$\begin{aligned} \Pi(x) \geq 0 \quad & \text{for all } x \geq 0, \\ \Pi(x) > 0 \quad & \text{for some } x \geq 0. \end{aligned}$$

For better results, all available call and put options should be taken into account, leading to the following linear programming problem: Find parameters $a, b, \gamma_i, \delta_i, D$ such that the following conditions are satisfied:

$$\begin{aligned} \text{minimize} \quad & D \\ \text{subject to} \quad & aS_0 + b + \sum_{i=1}^{n} (\gamma_i C(K_i) + \delta_i P(K_i)) = Y, \\ & \Pi(x) \geq -D \quad \text{for all } x \geq 0, \\ & a, \gamma_i, \delta_i \in [-M, M], \end{aligned} \tag{3.1}$$

where $C(K_i), P(K_i)$ are the current prices of the options.

Theorem 12 (More options–better portfolios). *In problem* (3.1) *(minimization of the worst-case loss D under the linear budget and super-hedging constraints), let I denote the set of tradable options that are allowed in the portfolio, and let $D^*(I)$ be the optimal value. If $I_1 \subseteq I_2$ (i.e., more options are made available), then*

$$D^*(I_2) \leq D^*(I_1).$$

Hence, the optimal worst-case loss is a nonincreasing function of the option set.

Proof. For each I, let $\mathcal{F}(I)$ be the feasible set of (3.1) in the variables $a, b, \{\gamma_i\}_{i \in I}, \{\delta_i\}_{i \in I}, D$, with the same market parameters and bounds. If $I_1 \subseteq I_2$, every feasible solution for I_1 remains feasible for I_2 by setting the positions of the additional options to zero. Thus $\mathcal{F}(I_1) \subseteq \mathcal{F}(I_2)$. Since (3.1) minimizes D and the objective does not change with I except through the feasible set, the minimum over a superset of feasible points cannot be larger:

$$D^*(I_2) = \min_{z \in \mathcal{F}(I_2)} D(z) \le \min_{z \in \mathcal{F}(I_1)} D(z) = D^*(I_1).$$

Therefore $D^*(I)$ is nonincreasing in I. □

Remark 13. The inequality can be strict when the additional options enlarge the span/cone of attainable payoff adjustments; otherwise equality may hold if the new options are redundant relative to those already available.

Assume there is one stock in the market with price S_0, a call option priced at $C(K)$, a put option priced at $P(K)$, and a risk-free bank account with interest rate r. These contracts have the same underlying asset and expiration date. Then, if the following relation (put-call parity) does not hold:

$$C(K) + Ke^{-rT} = S_0 + P(K) \quad \text{(put-call parity)}$$

there exists an arbitrage opportunity. The proof is straightforward. Assume that

$$C(K) + Ke^{-rT} > S_0 + P(K).$$

Then one can buy a put option and the stock, and sell a call option. The resulting amount (either positive or negative) can be deposited or borrowed from the bank. At time T, a risk-free profit will be available. If

$$C(K) + Ke^{-rT} < S_0 + P(K),$$

then one can buy a call option and the stock, and sell a put option. Again, the resulting amount can be deposited or borrowed from the bank. At time T, a risk-free profit will be available.

Conclusion 1. How is the put-call parity related to the solution of the linear programming problem (3.1)? This linear programming problem generalizes the concept of put-call parity, as it identifies the portfolio with the greatest arbitrage opportunity, considering all available option contracts.

In practice, attempting to repeat this process infinitely would lead to changes in prices and thus eliminate the arbitrage opportunity. Additionally, in practice, transaction costs, bid-ask spreads, dividends, and other market frictions must be included in the analysis.

The Python code `findingArbitrage1.ipynb` solves the above linear programming problem taking into account the bid/ask spread. The investor can choose how many call and put options to buy/sell having in mind the liquidity of the market. Besides the obvious interest of finding arbitrage there is one more interesting application of this notion. It is very useful in the option pricing problem in order to compute a unique fair and arbitrage free price.

Figure 2 shows the profit function of a portfolio with arbitrage opportunity allowing the possible number of call/put options to buy/sell per strike price to be 4.

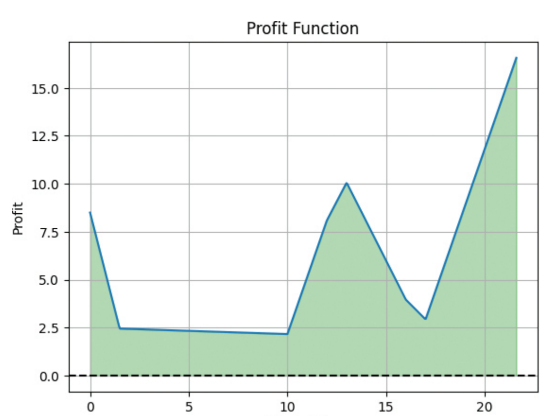


Figure 2

4 Dynamic trading strategies

In this section, we describe a dynamic trading strategy based on the principle of *“sell high–buy low”* (and also *“borrow high–repay low”*). Suppose an investor has decided to spend an amount Y by purchasing a shares of a company, borrowing an amount b from their own risk-free bank account at an interest rate r. Assume further that they have decided to rebalance their portfolio at discrete times $0 < t_1 < t_2 < \cdots < t_N = T$, and that the current price of the asset is S_0. We will construct a sequence a_k representing the number of shares held at each time t_k, with $a_0 = a$. If $a > 0$, the investor can buy or sell shares at time t_k so that

$$a_k = a_{k-1} - a_{k-1}\delta\Big(\frac{S_{t_k}}{S_0}\Big) + (a - a_{k-1})\mathbb{I}_{\{S_0 > S_{t_k}\}} \tag{4.1}$$

while if $a < 0$,

$$a_k = a_{k-1} - a_{k-1}\delta\Big(\frac{S_0}{S_{t_k}}\Big) + (a - a_{k-1})\mathbb{I}_{\{S_{t_k} > S_0\}}. \tag{4.2}$$

For example, we can choose the function $\delta(x)$ as follows:

$$\delta(x) = \begin{cases} \dfrac{z_1(x-1)^{z_2}}{z_1(x-1)^{z_2} + z_3} & \text{when } x > 1, \\ 0 & \text{otherwise,} \end{cases}$$

for some constants $z_1, z_2, z_3 > 0$. Naturally, there are infinitely many possible choices for the function δ, and the selection should be made by the investor. For instance, one might define δ as

$$\delta(x) = \begin{cases} 1 & \text{for } x > M, \\ p(x) & \text{for } x \in (1, M), \\ 0 & \text{for } x \le 1, \end{cases}$$

where $p(x) = a_n x^n + \cdots + a_0$ is chosen such that $p(M) = 1$ and $p(1) = 0$.

If transaction costs are taken into account, we should find an appropriate $\varepsilon > 0$ and redefine the function $\delta(x)$ as follows:

$$\delta(x) = \begin{cases} \dfrac{z_1(x-(1+\varepsilon))^{z_2}}{z_1(x-(1+\varepsilon))^{z_2} + z_3} & \text{when } x > 1+\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that at time t, the price S_t is below S_0, i.e., $S_t < S_0$. If the investor buys back shares they previously sold, they may reset S_0 to a lower value accordingly. Alternatively, we can construct a different trading strategy by setting, for $a > 0$,

$$a_k = a_{k-1} - a_{k-1}\delta\Big(\frac{S_{t_k}}{S_0}\Big) \tag{4.3}$$

and for $a < 0$,

$$a_k = a_{k-1} - a_{k-1}\delta\Big(\frac{S_0}{S_{t_k}}\Big). \tag{4.4}$$

Finally, another similar trading strategy arises by defining, for $a > 0$,

$$a_k = a_{k-1} - a_{k-1}\delta\Big(\frac{S_{t_k}}{S_0}\Big) + (a - a_{k-1})\delta\Big(\frac{S_0}{S_{t_k}}\Big)$$

and for $a < 0$,

$$a_k = a_{k-1} - a_{k-1}\delta\Big(\frac{S_0}{S_{t_k}}\Big) + (a - a_{k-1})\delta\Big(\frac{S_{t_k}}{S_0}\Big).$$

In this case, different functions δ can be used depending on the situation.

Therefore, at time T, the value of the portfolio – assuming the simple trading strategy defined in (4.3) and (4.4), and incorporating transaction costs – is given by

$$\Pi = a_N S_T + \sum_{k=1}^{N} S_{t_k}(1-\varepsilon)(a_k - a_{k-1})e^{r(T-t_k)} + be^{rT}.$$

To select the optimal trading strategy, one could assume that the asset price follows a stochastic differential equation (see [7]). The use of numerical schemes that preserve positivity would be useful for solving such an equation. Then we must construct an appropriate optimization problem to determine the optimal trading strategy – that is, to find the best parameters of the function δ.

This analysis can be extended to a portfolio consisting of n assets, assuming that each asset's price follows a stochastic differential equation, taking into account possible cross-covariances. Our objective is not only to find the best parameters for the function δ, but also to determine how frequently the portfolio should be rebalanced.

An interesting mathematical problem involves considering dynamic trading in a portfolio containing n assets along with all available call and put options. Solving this problem would require tools from stochastic differential equations, stochastic calculus, and numerical methods for stochastic differential equations.

Such kind of dynamic trading strategy can be used to price and hedge path-dependent options as we will see later.

5 Option pricing

Let an option with payoff

$$f(S_T),$$

which expires at the time T. Suppose that there exist n call and put options at the market with strike prices

$$0 < K_1 \cdots < K_n.$$

We will examine the problem of option pricing and hedging. An investor interested in buying or selling a specific option seeks to estimate a fair price, which is valuable during negotiations, as well as to determine a suitable hedging strategy. If the market has already established the option price, the investor's primary concern shifts to evaluating and implementing an effective hedging strategy at that price, and deciding whether entering the position is in their best interest. Accurately forecasting the price of an option or an asset involves, among other factors, insights from behavioral finance. The process is far more complex than simply computing the volatility of the underlying stock! One can see [1, 2, 4] for classic works on this subject.

5.1 Arbitrage-free price interval

Standing assumptions. We work in a frictionless market with a single riskless bank account accruing at rate r over $[0, T]$. The variable b denotes the *time-0* cash position (positive for a deposit, negative for borrowing), which grows to $b\,e^{rT}$ at maturity. Short sales and static positions in the listed options are allowed as specified in each linear program.

Suppose that a contract is to be bought or sold with a payoff function $f(x)$ which is piecewise linear with finite many branches. For example, for a call option we have $f(x) = (x - K)^+$, while for a put option $f(x) = (K - x)^+$. Clearly, the price of the contract is determined by the law of supply and demand. However, a decisive role will be played by the presence of an arbitrage opportunity. Therefore, in practice, to identify a no-arbitrage price interval one should at least take into account the available call and put options and solve the following linear programming problems:

$$\begin{aligned}
\text{minimize} \quad & Y \\
\text{subject to} \quad & aS_0 + b + \sum_{i=1}^{n} \big(\gamma_i C(K_i) + \delta_i P(K_i)\big) = Y, \\
& \Pi^{\text{writer}}(x) \ge 0 \quad \text{for all } x \ge 0, \\
& \sum |\gamma_i| + |\delta_i| \le N, \\
& a, \gamma_i, \delta_i \in [-N_i, N_i]
\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}&\text{maximize} \quad Y\\&\text{subject to} \quad aS_0 + b + \sum_{i=1}^{n}\left(\gamma_i C(K_i) + \delta_i P(K_i)\right) = -Y,\\&\qquad \Pi^{\text{buyer}}(x) \geq 0 \quad \text{for all } x \geq 0,\\&\qquad \sum |\gamma_i| + |\delta_i| \leq N,\\&\qquad a, \gamma_i, \delta_i \in [-N_i, N_i],\end{aligned} \tag{5.2}$$

where

$$\Pi^{\text{writer}}(x) = ax + be^{rT} + \sum_{i=1}^{n} \gamma_i (x - K_i)^+ + \delta_i (K_i - x)^+ - f(x),$$

$$\Pi^{\text{buyer}}(x) = ax + be^{rT} + \sum_{i=1}^{n} \gamma_i (x - K_i)^+ + \delta_i (K_i - x)^+ + f(x).$$

The solution to the first problem provides the smallest required amount to construct a portfolio (by the option writer) with no possible loss in any state. Similarly, solving the second problem we find the largest amount the buyer can pay while maintaining a portfolio with no possible loss.

Let the solution to the first problem be Y^{writer}, and the solution to the second be Y^{buyer}. Then the no-arbitrage price interval is $(Y^{\text{buyer}}, Y^{\text{writer}})$, provided $Y^{\text{buyer}} < Y^{\text{writer}}$.

Definition 2. We call the interval $(Y^{\text{buyer}}, Y^{\text{writer}})$ the *arbitrage-free price interval* for the option with payoff $f(x)$.

Conclusion 2. Any transaction price outside $(Y^{\text{buyer}}, Y^{\text{writer}})$ creates an arbitrage for one of the parties and is therefore *not acceptable* for the side unable to exploit that arbitrage. Within the interval, no such riskless profit is available to either side.

Model prices vs. no-arbitrage bounds

Prices output by *uncalibrated* probabilistic models (e.g., binomial or Black–Scholes with historical parameters) need not fall inside the arbitrage-free interval implied by currently traded options and may therefore be incompatible with observed markets. In practice, the no-arbitrage interval provides a robust bargaining range; any model used for valuation should respect it.

5.2 Model-free fair prices of an option

We study an option with maturity $T > 0$ and payoff $f : [0, \infty) \to \mathbb{R}$ that is piecewise linear with finitely many branches. We write

$$z^+ := \max\{z, 0\} \quad \text{and} \quad z^- := \max\{-z, 0\}.$$

A frictionless cash account with continuously compounded rate r is available, and the cash position $b \in \mathbb{R}$ is unconstrained. Let S_0 denote the current underlying price, and $C(K_i), P(K_i)$ the current call/put prices at strikes $\{K_i\}_{i=1}^n$.

Writer's and buyer's linear programs. Given a budget level $Y \in \mathbb{R}$, consider the writer's problem

$$\begin{aligned}&\text{minimize} \quad D\\&\text{subject to} \quad aS_0 + b + \sum_{i=1}^{n}\left(\gamma_i\, C(K_i) + \delta_i\, P(K_i)\right) = Y,\\&\qquad \Pi^{\text{writer}}(x) \geq -D \quad \text{for all } x \geq 0,\\&\qquad \sum |\gamma_i| + |\delta_i| \leq N,\\&\qquad a, \gamma_i, \delta_i \in [-N_i, N_i],\end{aligned} \tag{5.3}$$

where the profit (terminal wealth relative to the liability) is

$$\Pi^{\text{writer}}(x) = ax + b\,e^{rT} + \sum_{i=1}^{n} \gamma_i (x - K_i)^+ + \sum_{i=1}^{n} \delta_i (K_i - x)^+ - f(x).$$

Similarly, the buyer's problem is

$$\begin{aligned} &\text{minimize} && D \\ &\text{subject to} && aS_0 + b + \sum_{i=1}^{n} \big(\gamma_i\, C(K_i) + \delta_i\, P(K_i)\big) = -Y, \\ &&& \Pi^{\text{buyer}}(x) \ge -D \quad \text{for all } x \ge 0, \\ &&& \sum |\gamma_i| + |\delta_i| \le N, \\ &&& a, \gamma_i, \delta_i \in [-N_i, N_i], \end{aligned} \tag{5.4}$$

with

$$\Pi^{\text{buyer}}(x) = ax + b\,e^{rT} + \sum_{i=1}^{n} \gamma_i (x - K_i)^+ + \sum_{i=1}^{n} \delta_i (K_i - x)^+ + f(x).$$

Definition 3. We define the function $D^{\text{writer}}(Y)$ as the minimum of problem (5.3) and the function $D^{\text{buyer}}(Y)$ as the minimum of problem (5.4). We define also the super/sub-hedging bounds Y^{writer} as the minimum of 5.1 and Y^{buyer} as the maximum of 5.2.

Including bid-ask spreads. If quotes have bid-ask spreads, replace the budget constraint by

$$aS_0 + b + \sum_{i=1}^{n} \big(\gamma_i^{\text{ask}} C^{\text{ask}}(K_i) - \gamma_i^{\text{bid}} C^{\text{bid}}(K_i) + \delta_i^{\text{ask}} P^{\text{ask}}(K_i) - \delta_i^{\text{bid}} P^{\text{bid}}(K_i)\big) = Y,$$

and take $\gamma_i^{\text{bid}}, \gamma_i^{\text{ask}}, \delta_i^{\text{bid}}, \delta_i^{\text{ask}} \ge 0$, with writer's profit

$$\Pi^{\text{writer}}(x) = ax + be^{rT} + \sum_{i=1}^{n} (\gamma_i^{\text{ask}} - \gamma_i^{\text{bid}})(x - K_i)^+ + \sum_{i=1}^{n} (\delta_i^{\text{ask}} - \delta_i^{\text{bid}})(K_i - x)^+ - f(x).$$

Similarly we obtain the buyer's profit function.

Lemma 14 (Monotonicity and continuity of D^{writer}). *Under the standing assumptions (frictionless cash at rate r and free $b \in \mathbb{R}$), for any $Y_2 > Y_1$,*

$$\begin{aligned} D^{\text{writer}}(Y_2) &\le D^{\text{writer}}(Y_1) - e^{rT}(Y_2 - Y_1), \\ D^{\text{writer}}(Y_1) &\le D^{\text{writer}}(Y_2) + e^{rT}(Y_2 - Y_1). \end{aligned} \tag{5.5}$$

Thus the following equality holds:

$$D^{\text{writer}}(Y_2) - D^{\text{writer}}(Y_1) = -e^{rT}(Y_2 - Y_1).$$

That means that the function $D^{\text{writer}}(Y)$ is continuous and nonincreasing.

Moreover, $D^{\text{writer}}(Y^{\text{writer}}) = 0$ in the case where $Y^{\text{writer}} > -\infty$. In particular, for $0 < Y \le Y^{\text{writer}}$,

$$D^{\text{writer}}(Y) = e^{rT}\,(Y^{\text{writer}} - Y).$$

Proof. Fix $Y_1 < Y_2$ and let $(a, b, \gamma, \delta, D)$ be feasible for the writer's problem (5.3) at budget Y_1, i.e.,

$$aS_0 + b + \sum_{i=1}^{n} (\gamma_i C(K_i) + \delta_i P(K_i)) = Y_1, \qquad \Pi^{\text{writer}}(x) \ge -D \quad \text{for all } x \ge 0.$$

Define $\tilde{b} := b + (Y_2 - Y_1)$ and keep (a, γ, δ) the same. Then the budget equals Y_2, and the profit shifts by the cash account:

$$\tilde{\Pi}^{\text{writer}}(x) = ax + \tilde{b}\,e^{rT} + \sum_i \gamma_i (x - K_i)^+ + \sum_i \delta_i (K_i - x)^+ - f(x) = \Pi^{\text{writer}}(x) + e^{rT}(Y_2 - Y_1).$$

Therefore $\tilde{\Pi}^{\text{writer}}(x) \ge -(D - e^{rT}(Y_2 - Y_1))$ for all x, showing feasibility at Y_2 with objective $\tilde{D} := D - e^{rT}(Y_2 - Y_1)$. Taking infima over feasible solutions,

$$D^{\text{writer}}(Y_2) \le D^{\text{writer}}(Y_1) - e^{rT}(Y_2 - Y_1).$$

Interchanging the roles of Y_1, Y_2 yields $D^{\text{writer}}(Y_1) \le D^{\text{writer}}(Y_2) - e^{rT}(Y_1 - Y_2)$. Combining the two inequalities gives the identity

$$D^{\text{writer}}(Y_2) - D^{\text{writer}}(Y_1) = -e^{rT}(Y_2 - Y_1),$$

so D^{writer} is affine with slope $-e^{rT}$, hence continuous and nonincreasing.

By the definition of Y^{writer}, $D^{\text{writer}}(Y^{\text{writer}}) = 0$. Indeed, let a sequence $Y_n \uparrow Y^{\text{writer}}$ so that $D^{\text{writer}}(Y_n) \ge 0$. Using the equality

$$D^{\text{writer}}(Y_2) - D^{\text{writer}}(Y_1) = -e^{rT}(Y_2 - Y_1)$$

with $Y_1 = Y_n$ and $Y_2 = Y^{\text{writer}}$ we see that $D^{\text{writer}}(Y^{\text{writer}}) \ge 0$. Take now a sequence $Y_n \downarrow Y^{\text{writer}}$ and use again the equality

$$D^{\text{writer}}(Y_2) - D^{\text{writer}}(Y_1) = -e^{rT}(Y_2 - Y_1)$$

with $Y_1 = Y^{\text{writer}}$ and $Y_2 = Y_n$ now. Noting that $D^{\text{writer}}(Y_n) \le 0$ we arrive at $D^{\text{writer}}(Y^{\text{writer}}) \le 0$.

For any $0 < Y \le Y^{\text{writer}}$, the affine formula gives

$$D^{\text{writer}}(Y) = D^{\text{writer}}(Y^{\text{writer}}) + e^{rT}(Y^{\text{writer}} - Y) = e^{rT}(Y^{\text{writer}} - Y).$$ □

Lemma 15 (Monotonicity and continuity of D^{buyer}). *Under the standing assumptions (frictionless cash at rate r and free $b \in \mathbb{R}$), for any $Y_2 > Y_1$,*

$$\begin{aligned} D^{\text{buyer}}(Y_2) &\le D^{\text{buyer}}(Y_1) + e^{rT}(Y_2 - Y_1), \\ D^{\text{buyer}}(Y_1) &\le D^{\text{buyer}}(Y_2) - e^{rT}(Y_2 - Y_1). \end{aligned} \tag{5.6}$$

Thus the following equality holds:

$$D^{\text{buyer}}(Y_2) - D^{\text{buyer}}(Y_1) = e^{rT}(Y_2 - Y_1),$$

which means that the function $D^{\text{buyer}}(Y)$ is affine with slope $+e^{rT}$, hence continuous and nondecreasing.

Moreover, $D^{\text{buyer}}(Y^{\text{buyer}}) = 0$ if $Y^{\text{buyer}} > -\infty$. In particular, for $Y \ge Y^{\text{buyer}} > 0$,

$$D^{\text{buyer}}(Y) = e^{rT}\,(Y - Y^{\text{buyer}}).$$

Proof. Fix $Y_1 < Y_2$ and let $(a, b, \gamma, \delta, D)$ be feasible for the buyer's problem (5.4) at budget Y_1, i.e.,

$$aS_0 + b + \sum_{i=1}^{n} (\gamma_i C(K_i) + \delta_i P(K_i)) = -Y_1, \qquad \Pi^{\text{buyer}}(x) \ge -D \quad \text{for all } x \ge 0.$$

Define $\tilde{b} := b - (Y_2 - Y_1)$ and keep (a, γ, δ) the same. Then the budget equals $-Y_2$, and the profit shifts by the cash account:

$$\tilde{\Pi}^{\text{buyer}}(x) = ax + \tilde{b}\, e^{rT} + \sum_i \gamma_i (x - K_i)^+ + \sum_i \delta_i (K_i - x)^+ + f(x) = \Pi^{\text{buyer}}(x) - e^{rT}(Y_2 - Y_1).$$

Therefore $\tilde{\Pi}^{\text{buyer}}(x) \ge -(D + e^{rT}(Y_2 - Y_1))$ for all x, showing feasibility at Y_2 with objective $\tilde{D} := D + e^{rT}(Y_2 - Y_1)$. Taking infima over feasible solutions,

$$D^{\text{buyer}}(Y_2) \le D^{\text{buyer}}(Y_1) + e^{rT}(Y_2 - Y_1).$$

Interchanging the roles of Y_1, Y_2 and repeating the same argument (now shifting the cash position by $+(Y_2 - Y_1)$) yields

$$D^{\text{buyer}}(Y_1) \le D^{\text{buyer}}(Y_2) - e^{rT}(Y_2 - Y_1).$$

Combining the two inequalities gives the identity

$$D^{\text{buyer}}(Y_2) - D^{\text{buyer}}(Y_1) = e^{rT}(Y_2 - Y_1),$$

so D^{buyer} is affine with slope $+e^{rT}$, hence continuous and nondecreasing. Using sequences $Y_n \uparrow Y^{\text{buyer}}$ and $Y_n \downarrow Y^{\text{buyer}}$, we arrive as before at the equality $D^{\text{buyer}}(Y^{\text{buyer}}) = 0$.

Finally, for any $Y \ge Y^{\text{buyer}} > 0$, the affine identity with $Y_1 = Y^{\text{buyer}}$ gives

$$D^{\text{buyer}}(Y) = D^{\text{buyer}}(Y^{\text{buyer}}) + e^{rT}(Y - Y^{\text{buyer}}) = e^{rT}\,(Y - Y^{\text{buyer}}).$$ □

Theorem 16 (Fair price and midpoint in the frictionless case). *Assume $Y^{\text{buyer}} < Y^{\text{writer}}$. Then there exists a unique price $Y^{D^*} \in (Y^{\text{buyer}}, Y^{\text{writer}})$ such that*

$$D^{\text{writer}}(Y^{D^*}) = D^{\text{buyer}}(Y^{D^*}) =: D^* > 0.$$

So

$$Y^{D^*} = \frac{Y^{\text{buyer}} + Y^{\text{writer}}}{2}, \quad D^* = \frac{e^{rT}}{2}(Y^{\text{writer}} - Y^{\text{buyer}}).$$

Proof. Let $G(Y) := D^{\text{writer}}(Y) - D^{\text{buyer}}(Y)$. By Lemmas 14–15, D^{writer} and D^{buyer} are continuous and affine with slopes $-e^{rT}$ and $+e^{rT}$, respectively. Hence G is continuous and strictly decreasing on $[Y^{\text{buyer}}, Y^{\text{writer}}]$, with

$$G(Y^{\text{buyer}}) = e^{rT}(Y^{\text{writer}} - Y^{\text{buyer}}) > 0, \quad G(Y^{\text{writer}}) = -e^{rT}(Y^{\text{writer}} - Y^{\text{buyer}}) < 0.$$

By the intermediate value theorem there is a unique Y^{D^*} where G vanishes, and then

$$D^{\text{writer}}(Y^{D^*}) = D^{\text{buyer}}(Y^{D^*}) =: D^* > 0.$$

In the frictionless case, Lemma 14 yields, for any $Y \leq Y^{\text{writer}}$, $D^{\text{writer}}(Y) = e^{rT}(Y^{\text{writer}} - Y)$, and Lemma 15 yields, for any $Y \geq Y^{\text{buyer}}$, $D^{\text{buyer}}(Y) = e^{rT}(Y - Y^{\text{buyer}})$. In particular, on $[Y^{\text{buyer}}, Y^{\text{writer}}]$ these equalities hold, so equating them gives $Y^{D^*} = \frac{1}{2}(Y^{\text{buyer}} + Y^{\text{writer}})$ and then $D^* = \frac{e^{rT}}{2}(Y^{\text{writer}} - Y^{\text{buyer}})$. □

Remark 17. Note that

$$Y^{\text{writer}} = e^{-rT} D^{\text{writer}}(0), \quad Y^{\text{buyer}} = -e^{-rT} D^{\text{buyer}}(0).$$

Therefore

$$Y^{D^*} = e^{-rT}\frac{D^{\text{writer}}(0) - D^{\text{buyer}}(0)}{2}, \quad D^* = \frac{D^{\text{writer}}(0) + D^{\text{buyer}}(0)}{2}.$$

Theorem 18 (Arbitrage-free interval). *Assume $Y^{\text{buyer}} \leq Y^{\text{writer}}$. Then every price*

$$p \notin [Y^{\text{buyer}}, Y^{\text{writer}}]$$

is not arbitrage-free while, by construction, any price inside this interval it is. In particular, the fair price Y^{D^*} of Theorem 16 is arbitrage-free.

Proof. If $p > Y^{\text{writer}}$, short one option at price p, buy the writer's superhedge costing Y^{writer}, and invest the surplus $p - Y^{\text{writer}}$ at rate r: terminal wealth is $(p - Y^{\text{writer}})e^{rT} > 0$. If $p < Y^{\text{buyer}}$, buy the option at p, combine it with the buyer's superhedge which generates an initial inflow Y^{buyer}, and invest the surplus: terminal wealth is $(Y^{\text{buyer}} - p)e^{rT} > 0$. Hence no-arbitrage requires $p \in [Y^{\text{buyer}}, Y^{\text{writer}}]$, and all such p are arbitrage-free. □

There are many notions of a fair price. Given a multivariable functional F, define

$$\mathcal{F}^{\text{writer}}(a, b, \gamma, \delta) = F\left(D, \int_0^\infty (\Pi^{\text{writer}}(x))^- \, dx, \int_0^M (\Pi^{\text{writer}}(x))^+ \, dx, \int_0^M \Pi^{\text{writer}}(x)\, dx, \ldots\right),$$

for some $M > 0$, and analogously $\mathcal{F}^{\text{buyer}}$. For a given Y, the writer solves

$$\min \mathcal{F}^{\text{writer}}(a, b, \gamma, \delta) \quad \text{subject to (5.3)},$$

while the buyer solves

$$\min \mathcal{F}^{\text{buyer}}(a, b, \gamma, \delta) \quad \text{subject to (5.4)}.$$

We may declare $Y^{\mathcal{F}}$ a (model-free) fair price if

$$\min \mathcal{F}^{\text{writer}}(a, b, \gamma, \delta) = \min \mathcal{F}^{\text{buyer}}(a, b, \gamma, \delta).$$

Corollary 1 (Uniqueness in a limited sense). *Suppose $Y^{\text{buyer}} \leq Y^{\text{writer}}$. Among model-free fair values defined by equal minimal criteria as above, the equal-radius price Y^{D^*} satisfying $D^{\text{writer}}(Y^{D^*}) = D^{\text{buyer}}(Y^{D^*})$ is always arbitrage-free (by Theorem 18). Computing the fair price of an option is a dance for two; for that reason, there's no room for any kind of prediction in the methodology.*

Remarks and extensions. Traders can compute Y^{D^*} to obtain an order-of-magnitude value useful at the bargaining stage. The market price will be set by the law of supply and demand and need not be fair or even arbitrage-free. The analysis extends to multi-asset underlyings; for American-style contracts, American calls and puts may be included among the traded instruments.

Caution on model-based "fair" values. The "fair" values produced by stochastic models (e.g., Black–Scholes, binomial trees) are not necessarily fair in practice (the proposed replicating strategies are not implementable exactly) and need not be arbitrage-free in the model-free sense above, since they may fall outside the arbitrage-free interval $[Y^{\text{buyer}}, Y^{\text{writer}}]$ inferred from traded quotes.

Theorem 19 (Call bounds and optimal static hedges in a stock – bank market). *Consider a frictionless market with one risky asset S (spot $S_0 > 0$) and a bank account accruing at continuously compounded rate r up to time $T > 0$. No other options are traded. Let $f(x) = (x - K)^+$ be the payoff of a European call with strike $K > 0$.*

Let Y^{writer} be the writer's superhedging bound (5.1) *and Y^{buyer} the buyer's subhedging bound* (5.2)*. Then*

$$Y^{\text{writer}} = S_0, \quad Y^{\text{buyer}} = \max\{0, S_0 - Ke^{-rT}\}.$$

Consequently, every premium Y in the interval $[Y^{\text{buyer}}, Y^{\text{writer}}] = [\max\{0, S_0 - Ke^{-rT}\}, S_0]$ is arbitrage-free.

Proof. We consider the writer and buyer cases.

Writer case. A static hedge with stock/cash has terminal profit relative to the liability

$$\Pi_{\text{writer}}(x) = a\,x + b\,e^{rT} - (x - K)^+.$$

For $x \geq K$, $\Pi_{\text{writer}}(x) = (a-1)x + b\,e^{rT} + K$, so to keep $\Pi_{\text{writer}} \geq 0$ for large x we must have $a \geq 1$. For $x \in [0, K)$, $\Pi_{\text{writer}}(x) = a\,x + b\,e^{rT}$ attains its minimum at $x = 0$, hence $b \geq 0$. Therefore any feasible portfolio costs

$$Y = aS_0 + b \geq S_0.$$

The choice $(a, b) = (1, 0)$ yields $\Pi_{\text{writer}}(x) = x - (x-K)^+ = \min\{x, K\} \geq 0$ with cost $Y = S_0$, so $Y^{\text{writer}} = S_0$.

Buyer case. We work in a market with one stock (spot S_0) and a bank account at rate r until T, and consider a European call $f(x) = (x - K)^+$. A static stock-cash position (a, b) that the buyer picks must satisfy the budget

$$aS_0 + b = -Y$$

and yields terminal profit

$$\Pi_{\text{buyer}}(x) = a\,x + b\,e^{rT} + (x - K)^+, \quad x \geq 0.$$

Finite check (nodes and tail). The function Π_{buyer} is piecewise affine with nodes at $x = 0$, $x = K$. Thus $\Pi_{\text{buyer}}(x) \geq 0$ for all $x \geq 0$ iff the following hold:

$$\text{(Tail)} \quad a + 1 \geq 0, \qquad \text{(Grid)} \quad b\,e^{rT} \geq 0, \quad aK + b\,e^{rT} \geq 0.$$

The buyer aims to *maximize* $Y = -(aS_0 + b)$, i.e., to *minimize* $aS_0 + b$ subject to these linear constraints.

Case 1: $S_0 > Ke^{-rT}$. Given that $b = -Y - aS_0$ and the above constraints, we obtain that

$$a \in [-1, 0] \quad \text{and} \quad b \geq \max\{0, -aKe^{-rT}\}.$$

For a fixed $a \in [-1, 0)$, the smallest feasible b is

$$b^*(a) = \max\{0, -aKe^{-rT}\}.$$

Since $a < 0$, we have $-aKe^{-rT} \geq 0$, hence

$$b^*(a) = -aKe^{-rT}.$$

The objective therefore reduces to a function of a:

$$f(a) = aS_0 + b^*(a) = aS_0 - aKe^{-rT} = a(S_0 - Ke^{-rT}).$$

Because $S_0 - Ke^{-rT} > 0$ by assumption, $f(a)$ is linear and strictly increasing in a on $[-1, 0)$. Thus its minimum over $[-1, 0)$ is attained at the left endpoint:

$$a^* = -1, \quad b^* = -a^* Ke^{-rT} = Ke^{-rT}.$$

The minimal value is

$$\min(aS_0 + b) = a^* S_0 + b^* = -S_0 + Ke^{-rT},$$
$$a^* = -1, \quad b^* = Ke^{-rT}, \quad \min = Ke^{-rT} - S_0.$$

Case 2: $S_0 \le Ke^{-rT}$. In this case $a \ge 0$ and $b \ge 0$. Thus $\inf\{aS_0 + b\} = 0$, attained at $(a, b) = (0, 0)$, which yields $Y = 0$ and $\Pi_{\text{buyer}}(x) = (x - K)^+ \ge 0$. No $Y > 0$ is feasible, because $aS_0 + b = -Y$ would then be negative, contradicting the previous lower bound.

Conclusion. Combining the two cases,

$$Y^{\text{buyer}} = \max\{0,\ S_0 - Ke^{-rT}\}.$$

The stated arbitrage-free interval follows immediately. □

Theorem 20 (Monotonicity of the no-arbitrage interval under enlargement of the option set). *Fix $S_0 > 0$, maturity $T > 0$, risk-free rate r, and a piecewise-linear payoff f with finite many branches. Let $\mathcal{I}_1 \subseteq \mathcal{I}_2$ be two sets of traded European calls/puts (with given quotes). Write $Y_{\text{writer}}(\mathcal{I})$ and $Y_{\text{buyer}}(\mathcal{I})$ for the writer-/buyer-hedging bounds defined directly via problems* (5.1)–(5.2)*. Then*

$$Y_{\text{buyer}}(\mathcal{I}_1) \le Y_{\text{buyer}}(\mathcal{I}_2) \quad \textit{and} \quad Y_{\text{writer}}(\mathcal{I}_2) \le Y_{\text{writer}}(\mathcal{I}_1).$$

Consequently,

$$\big[Y_{\text{buyer}}(\mathcal{I}_2), Y_{\text{writer}}(\mathcal{I}_2)\big] \subseteq \big[Y_{\text{buyer}}(\mathcal{I}_1), Y_{\text{writer}}(\mathcal{I}_1)\big].$$

Proof. Let $\mathcal{H}_\mathcal{I}$ be the set of static payoffs implementable with stock, cash, and the options in $\mathcal{I}$:

$$\mathcal{H}_\mathcal{I} = \Big\{h(x) = a\,x + b\,e^{rT} + \sum_i \gamma_i (x - K_i)^+ + \sum_i \delta_i (K_i - x)^+\Big\},$$

with setup cost $\operatorname{cost}(h) = aS_0 + b + \sum_i \gamma_i C(K_i) + \sum_i \delta_i P(K_i)$. Define the writer- and buyer-hedging feasible sets

$$\mathcal{F}_\mathcal{I}^{\text{writer}} = \{h \in \mathcal{H}_\mathcal{I} : h(x) - f(x) = \Pi^{\text{writer}}(x) \ge 0 \text{ for all } x \ge 0\},$$
$$\mathcal{F}_\mathcal{I}^{\text{buyer}} = \{h \in \mathcal{H}_\mathcal{I} : h(x) + f(x) = \Pi^{\text{buyer}}(x) \ge 0 \text{ for all } x \ge 0\}.$$

Since the payoff f is piecewise linear with finite many branches, it is easy to see that the above feasible sets are non-empty. Let us see how to prove this below.

Writer case. Since f is piecewise linear with finitely many breakpoints, its right derivative is bounded above: there exists $M := \sup_{x\ge0} f'_+(x) < \infty$. Choose any $a > M$ and consider the payoff using only stock and bank account

$$h(x) := a\,x + b\,e^{rT}.$$

Set $g(x) := h(x) - f(x) = a\,x + b\,e^{rT} - f(x)$. Because $a - M > 0$, g has positive “tail slope” and thus $g(x) \to +\infty$ as $x \to \infty$. Hence $\inf_{x\ge0} g(x) > -\infty$. Choose

$$b := e^{-rT}\left(-\inf_{x\ge0}(a\,x - f(x))\right)_+,$$

so that $\inf_{x\ge0} g(x) \ge 0$. Therefore $h \in F_I^{\text{writer}}$.

Buyer case. Because f is piecewise linear with finitely many breakpoints, there exists $X_0 < \infty$ and constants $\tau, c \in \mathbb{R}$ such that $f(x) = \tau x + c$ for all $x \ge X_0$ (i.e., f has a linear tail with slope τ). Pick $a > -\tau$ and again let $h(x) := a\,x + b\,e^{rT}$. Then, for $x \ge X_0$,

$$h(x) + f(x) = (a + \tau)x + \big(b\,e^{rT} + c\big),$$

whose tail slope $a + \tau > 0$ implies $h(x) + f(x) \to +\infty$ as $x \to \infty$. Thus $\inf_{x\ge0}(h(x) + f(x)) > -\infty$. Choose

$$b := e^{-rT}\Big(-\inf_{x\ge0}(a\,x + f(x))\Big)_+$$

to make $h(x) + f(x) \ge 0$ for all $x \ge 0$. Hence $h \in F_I^{\text{buyer}}$.

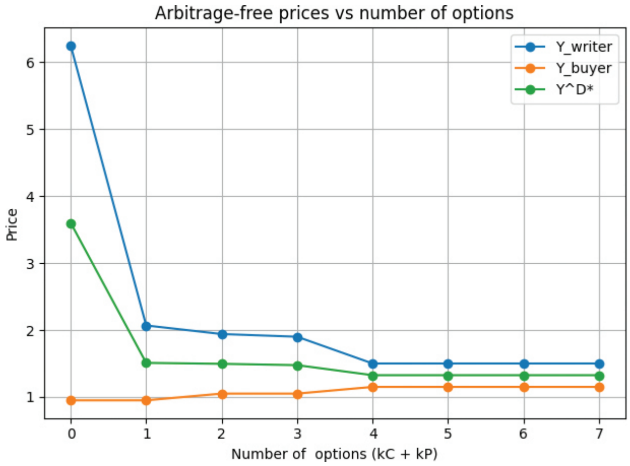


Figure 3: We can see how the arbitrage-free interval is getting smaller as more and more options are added. Note that the price of any stochastic model lies at the first arbitrage-free interval which is much bigger than the last.

In both cases we have constructed h using only stock and bank account, so the conclusion holds for any choice of the option set I (including $I = \varnothing$).

By the definition of the bounds,

$$Y_{\text{writer}}(\mathcal{I}) = \inf\{\text{cost}^{\text{writer}}(h) : h \in \mathcal{F}_{\mathcal{I}}^{\text{writer}}\}, \quad Y_{\text{buyer}}(\mathcal{I}) = \sup\{-\text{cost}^{\text{buyer}}(h) : h \in \mathcal{F}_{\mathcal{I}}^{\text{buyer}}\}.$$

Note that $\text{cost}^{\text{buyer}}(h) < 0$ since the buyer should pay a positive premium. If $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then $\mathcal{H}_{\mathcal{I}_1} \subseteq \mathcal{H}_{\mathcal{I}_2}$; hence $\mathcal{F}_{\mathcal{I}_1}^{\text{writer}} \subseteq \mathcal{F}_{\mathcal{I}_2}^{\text{writer}}$ and $\mathcal{F}_{\mathcal{I}_1}^{\text{buyer}} \subseteq \mathcal{F}_{\mathcal{I}_2}^{\text{buer}}$. Taking an infimum over a larger set cannot increase the value, while taking a supremum over a larger set cannot decrease it. Therefore,

$$Y_{\text{writer}}(\mathcal{I}_2) \le Y_{\text{writer}}(\mathcal{I}_1) \quad \text{and} \quad Y_{\text{buyer}}(\mathcal{I}_2) \ge Y_{\text{buyer}}(\mathcal{I}_1),$$

which yields the nesting of the no-arbitrage intervals. □

Example 5 (Model-free fair price and hedging for a single European call). Consider a frictionless market with one risky asset S (spot $S_0 > 0$) and a bank account accruing at continuously compounded rate r up to time $T > 0$. No other options are traded. Let $f(x) = (x - K)^+$ be the payoff of a European call with strike $K > 0$.

Arbitrage-free interval. With only the stock and the bank account available, the call's arbitrage bounds are

$$Y_{\text{buyer}} = \max\{0, S_0 - Ke^{-rT}\} \le Y \le Y_{\text{writer}} = S_0,$$

and every Y in this interval is arbitrage-free. Define $D_{\text{writer}}(Y)$ and $D_{\text{buyer}}(Y)$ as the minimum possible *maximum* loss at T for the writer and buyer, respectively, when they transact at premium Y and are allowed to form static positions in S and the bank account. In the frictionless case,

$$D_{\text{writer}}(Y) = e^{rT}(Y_{\text{writer}} - Y), \quad D_{\text{buyer}}(Y) = e^{rT}(Y - Y_{\text{buyer}}), \quad Y \in [Y_{\text{buyer}}, Y_{\text{writer}}].$$

Hence there is a unique price $Y^{D*} \in (Y_{\text{buyer}}, Y_{\text{writer}})$ such that $D_{\text{writer}}(Y^{D*}) = D_{\text{buyer}}(Y^{D*}) =: D^* > 0$, and

$$Y^{D*} = \frac{Y_{\text{buyer}} + Y_{\text{writer}}}{2}, \quad D^* = \frac{e^{rT}}{2}(Y_{\text{writer}} - Y_{\text{buyer}}).$$

Explicit formulas in the two elementary cases:

- Case A:

$$S_0 > Ke^{-rT}, \quad Y^{D*} = S_0 - \frac{1}{2}Ke^{-rT}, \quad D^* = \frac{1}{2}K,$$

- Case B:

$$S_0 \le Ke^{-rT}, \quad Y^{D*} = \frac{S_0}{2}, \quad D^* = \frac{1}{2}e^{rT}S_0.$$

Static hedges that attain the minima. Write a static hedging portfolio as a shares and bank position b (so terminal value $aS_T + b\,e^{rT}$).

Writer's hedge (any $Y \in [Y_{\text{buyer}}, Y_{\text{writer}}]$). Take

$$a = 1, \quad b = Y - S_0 \ (\le 0).$$

The writer's (terminal) profit is

$$\Pi_{\text{writer}}(x) = ax + be^{rT} - (x - K)^+ = x + (Y - S_0)e^{rT} - (x - K)^+.$$

Piecewise

$$\Pi_{\text{writer}}(x) = \begin{cases} x + (Y - S_0)e^{rT}, & x \le K, \\ (Y - S_0)e^{rT} + K, & x \ge K. \end{cases}$$

The worst case is $\min\{(Y - S_0)e^{rT}, (Y - S_0)e^{rT} + K\} = (Y - S_0)e^{rT}$, hence

$$\max\text{-loss} = D_{\text{writer}}(Y) = e^{rT}(S_0 - Y).$$

At $Y = Y^{D*}$ this yields $D^* = e^{rT}(S_0 - Y^{D*})$.

Buyer's hedge (any $Y \in [Y_{\text{buyer}}, Y_{\text{writer}}]$). Two subcases give the optimal static choice:

$$(a, b) = \begin{cases} (-1, S_0 - Y) & \text{if } S_0 > Ke^{-rT}, \\ (0, -Y) & \text{if } S_0 \le Ke^{-rT}. \end{cases}$$

- When $S_0 > Ke^{-rT}$,

$$\Pi_{\text{buyer}}(x) = -x + (S_0 - Y)e^{rT} + (x - K)^+ = \begin{cases} -x + (S_0 - Y)e^{rT}, & x \le K, \\ (S_0 - Y)e^{rT} - K, & x \ge K, \end{cases}$$

  so $\min_x \Pi_{\text{buyer}}(x) = (S_0 - Y)e^{rT} - K = -e^{rT}(Y - Y_{\text{buyer}})$ and $D_{\text{buyer}}(Y) = e^{rT}(Y - Y_{\text{buyer}})$.
- When $S_0 \le Ke^{-rT}$,

$$\Pi_{\text{buyer}}(x) = 0 \cdot x + (-Y)e^{rT} + (x - K)^+ = \begin{cases} -Ye^{rT}, & x \le K, \\ x - K - Ye^{rT}, & x \ge K, \end{cases}$$

  whose minimum is $-Ye^{rT} = -e^{rT}(Y - Y_{\text{buyer}})$ (because $Y_{\text{buyer}} = 0$). Thus $D_{\text{buyer}}(Y) = e^{rT}(Y - Y_{\text{buyer}})$.

Fair price and hedges at Y^{D*}. At the equal-radius price Y^{D*}, both parties' maximum losses match D^*, and the above static hedges (writer: $a = 1$, $b = Y^{D*} - S_0$; buyer: choose the optimal case based on the sign of $S_0 - Ke^{-rT}$) *attain* these minima.

The actual price of the option will be formulated by the law of supply and demand and can be, in general, any price in the arbitrage free interval or even out of it.

If you try to price the option using a stochastic model like Black–Scholes, you have to choose the volatility. The two parties are not going to agree on this volatility and therefore the bargaining begins. That is, the stochastic model does not produce a unique fair price, instead give a whole family of prices depending on the volatility. If we add more options at the market the arbitrage free interval is getting smaller. The stochastic model does not take into account this new information and therefore the price will not belong, in general, in the new arbitrage free interval. The model free option pricing that we have described gives a unique fair price which is always the middle of the arbitrage free interval.

Implied volatility and (why) to use it. Suppose the call is transacted at a premium $Y \in [Y_{\text{buyer}}, Y_{\text{writer}}]$. If one postulates the Black–Scholes model, the *implied volatility* σ_{imp} is defined as the *unique* volatility input for which the Black–Scholes price equals the observed premium:

$$C^{\text{BS}}(S_0, K, T; r, \sigma_{\text{imp}}) = Y.$$

When is this useful here? Extracting σ_{imp} is practically useful only if the writer intends to run a *discrete-time Black–Scholes delta hedge*, i.e.,

$$a_{t_i} = \partial_S C^{\text{BS}}(t_i, S_{t_i}; \sigma_{\text{imp}}), \quad i = 0, 1, \ldots, N.$$

As stochastic models for option pricing are applied only by implementing their hedging strategies- and only in discrete time-the development of new models should center on whether those strategies are feasible.

Why outcomes are unclear vs the static hedge. Continuous-time replication is infeasible; in practice hedging is necessarily *discrete* and subject to frictions. The discrete Black–Scholes/binomial rebalancing typically leads to buy-high/sell-low trades, and performance is sensitive to the (unknown) future volatility and to bid-ask/transaction costs. Consequently, the discrete BS hedge does *not* inherit the model-free, worst-case bound D^* achieved by the static hedge at Y^{D*}, and it need not dominate the static construction in realized outcomes.

If the underlying asset is not tradable, the arbitrage – free interval is undefined, and any pricing approach that relies on hedging arguments loses its foundation. In that case one should think differently, see, e.g., [6].

Of course, the writer and the buyer may adopt hedging strategies consistent with their own beliefs.

Remark 21 (Actuarial analogue). Similar issues arise in actuarial mathematics. A stochastic pricing model is useful only insofar as one intends to implement the *hedging* strategy it prescribes for the hedgeable (tradable) risks; any feasible hedge is necessarily in *discrete* time and subject to frictions. When the underlying risk is not tradable (e.g., mortality/longevity), the arbitrage-free interval is not defined and hedging-based pricing loses its operational meaning.

Consider for example a GMAB embedded in an equity-linked life policy. The guarantee behaves like an option on the reference fund.

Using the code `FairValueOption1.ipynb`, we compute the fair value Y^{D^*} as well as the arbitrage free interval, if it exists.

5.3 Hedging problem

The price of the option will be formed by the law of supply and demand and will be an amount Y not necessarily a fair or even an arbitrage free value. The writer/buyer of the option can invest this amount in various ways. For example:

- The writer can construct a portfolio with the minimum possible loss D by solving problem (5.1). This can be done by the Python code `WriterOption.ipynb`. Similar construction can be done by the buyer using the Python code `BuyerOptions.ipynb`.
- The writer can make a prediction about the future price of the underlying and solve a linear programming problem like (2.13). In this case the profit function is as follows

$$\Pi(x) = ax + be^{rT} + \sum_{i=1}^{d} \gamma_i (x - K_i)^+ + \delta_i (K_i - x)^+ - f(x).$$

  This can be done using the Python code `WriterHedgingOption1.ipynb`. Similar construction can be done by the buyer using the Python code `BuyerHedgingOption1.ipynb`.
- The investors may employ a sell-high/buy-low trading strategy (see Section 6 below). However, as a form of dynamic hedging, it also suffers from the uncertainty of the future volatility, bid-ask spread and from transaction costs, just like the binomial and Black–Scholes hedging approaches.

You can try also the code `Minimum-Loss-Writer-Buyer-Option-and-Hedging.ipynb` to see the whole problem using a single code.

Suppose that an option writer has reliable information or a strong predictive model. Using the Python code `WriterHedgingOption1.ipynb`, they can profitably sell the option at a premium Y even when $Y < Y^{\text{buyer}}$, as long as they prediction comes true. Symmetrically, a buyer with a predictive edge can, via the Python code `BuyerHedgingOption1.ipynb`, pay a premium Y exceeding the writer's no-loss bound Y^{writer} and still expect to profit.

Define $Y^{\text{writer}}_{\text{prediction}}$ as the lowest premium at which the writer remains profitable, conditional on their forecast being realized. If the market premium is below $Y^{\text{writer}}_{\text{prediction}}$, then some writers must either hold different beliefs about the asset's future or have miscalculated. Hence, it is unwise to transact at the market price without first producing one's own forecasts and calculations. From the above we deduce that it is not easy to predict the price of an option since we do not know the beliefs and the type of calculations made by the investors.

Corollary 2. *Do not try to forecast an option's price. With a model and a free parameter, anyone can fit the market. The real challenge is constructing a hedge once the option's price is given.*

Remark 22 (A theoretical example). Suppose you (the writer) price an option using a stochastic model, while the buyer knows the model-free bounds. Because the model is not calibrated to the prevailing option quotes, it may output a price Y outside the interval $[Y^{\text{buyer}}, Y^{\text{writer}}]$.

- *Case 1:* If $Y < Y^{\text{buyer}}$, a buyer can lock in a riskless profit by purchasing the option from you at Y; your model-based hedge cannot prevent this.
- *Case 2:* If $Y > Y^{\text{writer}}$, no rational buyer will pay Y, since a cheaper superhedge exists at cost Y^{writer}. In practice, your quote is not transaction-feasible. But even if you manage to sell at that price you can not construct a portfolio with sure profit using your stochastic model.

Thus, model prices that ignore traded quotes need not be arbitrage-free or market-consistent.

One can see for example the papers [3] and [10] for a closely related study of the option pricing problem.

6 Path dependent options

An interesting problem is the hedging of path dependent options. One possible approach to this problem is to apply an appropriate sell high–buy low strategy where the parameters of this strategy will depend on the form of the payoff function.

The replicating portfolio proposed by the Black–Scholes theory can only be implemented in discrete time. However, as you will notice, this leads us to a sell low–buy high strategy! Regarding the dynamic hedging problem of an option, one can use a similar strategy and choose the parameters based on the payoff function.

This can be very useful for the pricing and hedging problems of options that depend on the path (path dependent options).

Pricing problem. In the pricing problem, the writer, for a given $a \in (0,1)$, can compute the minimum amount Y for which there exist parameters in the dynamic sell high–buy low trading strategy such that

$$\mathbb{P}\big(V_n \geq f(S_1, \dots, S_n)\big) \geq a,$$

where $V_n = a_n S_n + b_n$ with a_n, b_n determined by the above strategy and $Y = a_0 S_0 + b_0$. By computing this amount, the writer obtains an order of magnitude for the option price, useful during negotiations.

Hedging problem. However, the actual price will be determined by the law of supply and demand. Therefore, for the given actual price Y of the option, the writer can find the maximum $a \in (0,1)$ for which there exist parameters in the sell high–buy low strategy such that

$$\mathbb{P}\big(V_n \geq f(S_1, \dots, S_n)\big) \geq a,$$

with $Y = a_0 S_0 + b_0$. In this way, the writer obtains a practically applicable hedging strategy.

At the above minimization problem we can employ the expected shortfall as well. One can add call and put options so the minimization problem can find the best possible choice about the actual number of these options. Finally, one can employ more complex trading strategies, not necessarily of sell high–buy low type, by choosing other sequences a_k in order to minimize the desired quantities (e.g., expected shortfall, etc.).

As can be seen, the pricing problem here is not to find a fair or arbitrage free price, but to estimate the order of magnitude of the price which depends on the hedging strategy that the investor will follow. Similar procedures can be followed by the buyer.

Remark 23. The pricing and hedging strategy described above is not limited to path-dependent options; it can also be applied to simpler instruments such as call and put options. However, because it does not incorporate the market prices of existing calls and puts, it will generally fail to produce arbitrage-free prices. Finally, this approach can also be applied to options with nonlinear payoff structures.

Using the Python code `Pricing-and-Hedging-Lookback-Options-Sell-High-and-Binomial.ipynb`, one can price and hedge a lookback option using the sell high–buy low strategy.

7 Open problems

In light of the above, we identify two central research problems that warrant further investigation.

Problem 1. A challenging research objective is the design of a predictive framework for stock price forecasting that moves beyond conventional statistical modeling of historical time series data. The proposed methodology should incorporate techniques from behavioral finance and machine learning to simulate the behavior and decision patterns of an expert investor, potentially leveraging cognitive modelling, reinforcement learning, or hybrid agent-based approaches.

Problem 2. Another important research challenge involves the pricing and hedging of path dependent options. Our interest is to find the best possible dynamic trading strategy for a given path dependent option.

A possible solution to this problem is to employ call and put options of American type. Thus the problem is to find, given the premium Y, how many shares, what amount b from/to a bank account, how many call and put options to buy/sell at the time zero but also how to rebalance this portfolio at a specified future times until the expiration, i.e., $0 < t_1 < \cdots < t_n < T$. Since we have buy call and put options of American type we can choose any time t_i if we want to exercise and how many of them. On the other hand, if we have sell some call and put options of American type we can choose to close our position any time we want by buying a similar option.

For example, we can design an optimization problem like the following: For a given premium Y find the parameters $a_0, b_0, \gamma_i^0, \delta_i$ and the parameters $z_1, z_2, \dots$ such that

$$\begin{aligned} &\text{minimize} \quad F(\mathbb{P}(V_n \geq f(S_1,\dots,S_n)), \mathbb{E}S_a(f(S_1,\dots,S_n) - V_n) \\ &\text{subject to} \quad a_0 S_0 + b + \sum_{i=1}^{N} \gamma_i^0 W_i + \sum_{i=1}^{m} \delta_i W_i^{\text{European}} = Y, \end{aligned}$$

where

$$V_n = a_n S_n + b_n + \sum_{i=1}^{C} \gamma_i^n (S_n - K_i)^+ + \sum_{i=C+1}^{N} \gamma_i^n (K_i - S_n)^+ + \sum_{i=1}^{m} \delta_i W_i^{\text{European}}(S_n).$$

Here, $a, b \in \mathbb{R}$ and $\gamma_i \in \mathbb{N}$ for simplicity $\delta_i \in \mathbb{Z}$. By W_i we denote American type call or put option while by W_i^{European} we denote the European type options with payoff $W_i(x)$. That is, our portfolio consists by two parts: the first part contain the assets, the bank account and the American options. We rebalance accordingly the first part while the second, containing European options, is a static portfolio.

The sequences can be the following. For $a_0 > 0$,

$$a_k = a_{k-1} - a_{k-1}\delta^{\text{asset-sell}}\left(\frac{S_{t_k}}{S_0}\right) + (a - a_{k-1})\delta^{\text{asset-buy}}\left(\frac{S_0}{S_{t_k}}\right)$$

and for $a_0 < 0$,

$$a_k = a_{k-1} - a_{k-1}\delta^{\text{asset-sell}}\left(\frac{S_0}{S_{t_k}}\right) + (a - a_{k-1})\delta^{\text{asset-buy}}\left(\frac{S_{t_k}}{S_0}\right).$$

Similarly

$$\gamma_i^k = \gamma_i^{k-1} - \left[\gamma_i^{k-1}\delta^{\text{call-option}}\left(\frac{S_{t_k}}{K_i}\right)\right] \quad \text{and} \quad \gamma_i^k = \gamma_i^{k-1} - \left[\gamma_i^{k-1}\delta^{\text{put-option}}\left(\frac{K_i}{S_{t_k}}\right)\right].$$

The sequence γ_i is a decreasing sequence since the investor can exercise a number of options each time t_n.

For better results one should think functions δ that depend also to a_{k-1}, γ_{k-1}, that is the current number of existing assets and options.

Of course one way to solve such an optimization problem is by Monte Carlo methods assuming that the price of the stock follows a stochastic differential equation. A different approach is by discretizing the $[0, T]$ into subintervals and for each interval make a prediction about the payoff $f(x)$. Next, choose the best parameters a, b, γ_i, δ_i in order to hedge the payoff at the end of this time interval, and so on.

For a given option, our objective is to identify the optimal discrete-time dynamic trading strategy according to our evaluation criteria. Standard hedging strategies (but remember; in discrete time only) derived from stochastic models may also be employed, provided they meet these criteria. However, we expect our techniques above to yield better results because they allow flexible selection of the trading policy, in contrast to the fixed strategies implied by stochastic models. That means of course that there is more room for research in this direction using stochastic methods!

8 Conclusions and discussion

By separating the prediction mechanism from the portfolio construction we are able to use more sophisticated prediction techniques to the portfolio construction problem. Moreover, we have described in detail how one can add call and put option in his/her portfolio if he/she wishes arriving to an extension of the Markowitz portfolio theory.

The option pricing problem should not be viewed as a prediction task aimed at determining the exact future price of the contract. Instead, its primary purpose is to provide the investor with an approximate order of magnitude of the contract's fair value, which serves as a crucial reference point during the negotiation phase. In addition to determining the order of magnitude of the contract's value – which serves as a crucial reference during negotiations – it is equally important to consider the investment methodology associated with this amount for effective risk hedging. The way an investor allocates and manages the capital used to construct the portfolio directly impacts the ability to mitigate potential losses and ensure robustness under various market scenarios.

Rather than focusing solely on the price of the derivative, the investor should also be equipped with a clear strategy for deploying the invested capital in a manner that aligns with their risk tolerance and market expectations. This includes considerations such as:
- the structure of the hedging portfolio,
- constraints on asset allocation,
- transaction costs and liquidity limitations,
- rebalancing frequency in dynamic environments.

Such a methodology not only enhances the practical relevance of the pricing mechanism but also ensures that the investor can actively manage exposure, rather than merely quoting a theoretical or statistical "fair" value.

The well-known option pricing methods, such as Black–Scholes, Binomial and all the extensions of them, do not produce natural results due to the unnatural hypotheses they made. Using them, the investor do not have the order of magnitude of the price of the option and even worst do not have a feasible hedging strategy.

In practice, an investor must take into account the *liquidity* of the market when constructing or rebalancing a portfolio. Liquidity refers to the ease with which assets – such as stocks and derivatives – can be bought or sold without significantly affecting their price.

To ensure realistic and executable trading strategies, the investor should quantify liquidity in some measurable way – for instance, by analyzing historical trading volumes, bid-ask spreads, or order book depth. This allows the investor to estimate the maximum number of shares or derivative contracts that can be traded within a given time period without incurring excessive transaction costs or market impact.

Specifically, liquidity constraints may impose limits on:
- The maximum number of shares $a_{\max}$ that can be bought or sold at any given time.
- The maximum number of call or put options $\gamma_i^{\max}, \delta_i^{\max}$ that can be acquired or written, depending on the strike price K_i.

These constraints are crucial for formulating feasible portfolio optimization problems, particularly in illiquid markets where large trades may distort prices or fail to execute altogether. By incorporating such bounds into the investment strategy, the investor ensures that the constructed portfolio remains both theoretically optimal and practically implementable.

This consideration is especially important in dynamic trading environments and arbitrage-based strategies, where frequent adjustments are necessary and execution risk becomes a key factor in achieving desired payoffs. In all the Python codes that we propose the investor can choose these kind of parameters.

Finally, by using dynamic hedging strategies we have proposed a way to price, i.e., to find an order of magnitude of the price, and hedge a path dependent option with feasible techniques.

Remark 24 (Python codes). Summarizing the Python codes for the model free computations, we have:
- `PCUP.ipynb` for the construction of optimal portfolio (one asset and options) under a prediction. The code `Multi-Asset4.ipynb` is for the construction of an optimal portfolio (two assets and options) under a prediction. Using AI it is easy to extend this code for portfolios with d-assets.
- `FindingArbitrage.ipynb` is for detection of an arbitrage opportunity.
- `Minimum-Loss-Writer-Buyer-Option-and-Hedging.ipynb` is for the option pricing problem. With this code you can compute the arbitrage free interval (if any) for any option written on one asset with payoff function which is piecewise linear with finite branches, the fair price Y^{D^*}, the D^* and hedging strategies for the writer and the buyer. These hedging strategies are static and therefore feasible. There is also the possibility to see how the arbitrage free interval is getting smaller as more options are added.
- Using the Python code `Pricing-and-Hedging-Lookback-Options-Sell-High-and-Binomial.ipynb`, one can price and hedge a lookback option using the sell high–buy low strategy.

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