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Lattice-Subspaces and Positive Bases in Function Spaces

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Abstract. Let $x_1, \dots, x_n$ be linearly independent, positive elements of the space $\mathbb{R}^\Omega$ of the real valued functions defined on a set Ω and let X be the vector subspace of $\mathbb{R}^\Omega$ generated by the functions x_i. We study the problem: Does a finite-dimensional minimal lattice-subspace (or equivalently a finite-dimensional minimal subspace with a positive basis) of $\mathbb{R}^\Omega$ which contains X exist? To this end we define the function $\beta(t) = \frac{1}{z(t)}(x_1(t), x_2(t), \dots, x_n(t))$, where $z(t) = x_1(t)+x_2(t)+...+x_n(t)$, which we call basic function and takes values in the simplex Δ_n of $\mathbb{R}^n_+$. We prove that the answer to the problem is positive if and only if the convex hull K of the closure of the range of β is a polytope. Also we prove that X is a lattice-subspace (or equivalently X has positive basis) if and only if, K is an $(n-1)$-simplex. In both cases, using the vertices of K, we determine a positive basis of the minimal lattice-subspace. In the sequel, we study the case where Ω is a convex set and $x_1, x_2, \dots, x_n$ are linear functions. This includes the case where x_i are positive elements of a Banach lattice, or more general the case where x_i are positive elements of an ordered space Y. Based on the linearity of the functions x_i we prove some criteria by means of which we study if K is a polytope or not and also we determine the vertices of K. Finally note that finite dimensional lattice-subspaces and therefore also positive bases have applications in economics.

1. Introduction

Suppose E is a vector lattice and D is a subset of E_+. In the theory of ordered spaces we are interested in a lattice subspace Y of E that contains D and that is as 'close' as possible to the linear subspace $[D]$ generated by D. The sublattice $S(D)$ generated by D is a lattice-subspace which contains D but in general, it is a 'big' subspace which is 'very far' from $[D]$.[1] Since the intersection of lattice-subspaces is not always a lattice-subspace, we are looking for minimal lattice-subspaces containing D and not for a minimum one. Also the class of lattice-subspaces is larger than that of sublattices, therefore a minimal lattice-subspace, if it exists, is 'closer' to $[D]$ than $S(D)$.

In Theorem 2.5 of [15] it is proved that if τ is a Lebesgue linear topology on E and the positive cone E_+ of E is τ-closed (especially if E is a Banach lattice

* This article is dedicated to the memory of my Father Andreas.

[1] In Example 3.18 of [15], $[D]$ is three-dimensional subspace of $C(\Omega)$, $[D]$ is not a lattice-subspace, $S(D)$ is dense in $C(\Omega)$ and a four-dimensional lattice-subspace containing D exists.

with order continuous norm) then the set of minimal lattice-subspaces of E, which contain D and have τ-closed positive cone, has minimal elements.

An important question is 'how far' a minimal lattice-subspace is from $[D]$. Motivated by this question we suppose that the set D is finite and we study the existence of finite-dimensional minimal lattice-subspaces containing D. In the framework of this problem we study also whether the sublattice $S(D)$ generated by D is a finite-dimensional subspace of E. So in the first part of this article we suppose that $D = \{x_1, x_2, \ldots, x_n\}$ is a subset of the positive cone $\mathbb{R}^{\Omega}_{+}$ of the space $\mathbb{R}^{\Omega}$ of the real valued functions defined in a set Ω, where the functions x_i are linearly independent and we study the existence of a finite-dimensional minimal lattice-subspace of $\mathbb{R}^{\Omega}$ which contains D. Since a finite dimensional ordered space is a vector lattice if and only if it has a positive basis, this problem is equivalent with the existence of a finite dimensional minimal subspace of $\mathbb{R}^{\Omega}$ with a positive basis which contains D. To study this problem we define the function $\beta(t) = \frac{1}{z(t)}(x_1(t), x_2(t), \ldots, x_n(t))$, where $z(t) = x_1(t) + x_2(t) + \ldots + x_n(t)$, which we call basic function and takes values in the simplex Δ_n of $\mathbb{R}^n_+$. The study of the range $R(\beta)$ of β and also the study of the convex hull K of the closure of $R(\beta)$ are very important in this article. In Theorem 7, we prove that the sublattice $S(D)$ generated by D is finite dimensional if and only if the range $R(\beta)$ of β is finite and also a positive basis of $S(D)$ is determined by means of the elements of $R(\beta)$. Note that, by using the known result that $S(D) = D^{\vee} - D^{\vee}$, where $D^{\vee}$ is the set of finite supremum of the elements of $[D]$, we cannot conclude if $S(D)$ is finite dimensional and if it is, we cannot also determine $S(D)$.

In Theorem 9 we prove that an m-dimensional minimal lattice-subspace Y which contains D exists, if and only if the convex hull K of the closure of $R(\beta)$ is a polytope with m vertices. Then a positive basis of Y is determined by the vertices of K. In general it is difficult to study whether K is a polytope and if it is, it is also difficult to determine its vertices. In the derivative criterion we prove that if K is a polytope, $\beta(t_0)$ is a vertex of K and $c : t = \sigma(u), u \in (-\epsilon, \epsilon)$ is a curve of Ω with $t_0 = \sigma(0)$ and if $\varphi = \beta o \sigma$ is the composition of σ, β, then $\varphi'(0) = 0$, whenever the derivative exists. In the case where the restriction of β in the curves of Ω is not differentiable we give a more general result, Theorem 11.

In the second part of this article (Section 5) we generalize the previous results in ordered spaces. So we suppose that $\langle Y, G\rangle$ is an ordered dual system (see Def. 14) $D = \{x_1, x_2, \ldots, x_n\}$, and that x_i are linearly independent positive elements of G. Then G is a subspace of $\mathbb{R}^{\Omega}$, where Ω is an absorbing subset of Y_+.[1] Based on the linearity of the functions x_i we study if K is a polytope or not. In Corollary 2 we prove that if $\beta(t_0)$ is an extreme point of K then t_0 is an extreme point of the domain $D(\beta)$ of β, or the function β takes the constant value $\beta(t_0)$ on each line segment c of $D(\beta)$ that has t_0 as an interior point. In Theorem 16, we prove that if Y is a Banach lattice with a positive basis $\{d_i\}$, then $K \subseteq \overline{co}\{\beta(d_i)\}$, therefore

[1] For example Ω is the whole cone Y_+ or the positive part U^+_Y of the closed unit ball of Y whenever Y is a normed space.

in order to study if K is a polytope, we study if each $\beta(t)$ is a convex combination of the vectors $\beta(d_i)$. If Y is an m-dimensional space with a positive basis $\{d_1, d_2, \ldots, d_m\}$, then $K = co\{\beta(d_i)\}$, therefore K is a polytope and its vertices are among the elements of the set $\{\beta(d_i)\}$. Also if G is an ordered normed space and Y the topological dual of G, we have: (i) $K \subseteq \overline{co}\{\beta(t) \mid t \in ep(U_Y^+)\}$, and (ii) if e is an order unit of Y and Y_+ is normal, then $K \subseteq \overline{co}\{\beta(t) \mid t \in L_e\}$, where $ep(U_Y^+)$ is the set of the extreme points of U_Y^+ and L_e is the Boolean algebra of the components of e, Theorem 15.

For applications of lattice-subspaces in economics, see in [3] and [4] and for an application of finite dimensional minimal lattice-subspaces in optimization, see in [16]. Also we refer to the book of Aliprantis et al. [2].

2. Notations

Let E be a (partially) ordered vector space with positive cone E_+. Any subspace X of E, ordered by the induced ordering (i.e., by the cone $X_+ = X \cap E_+$) will be referred as an **ordered subspace** of E. An ordered subspace X of E which is also a vector lattice, i.e., for each $x, y \in X$ the supremum $x \vee_X y$ of $\{x, y\}$ in X exists,[1] is a **lattice-subspace** of E. We will denote also this supremum by $\sup_X\{x, y\}$. It is clear that

$$x \vee y \leqslant x \vee_X y,$$

whenever the supremum $x \vee y$ of $\{x, y\}$ in E, exists. If E is a vector lattice and $x \vee y = x \vee_X y$ for any $x, y \in X$, then X is a **sublattice** (Riesz subspace) of E. In lattice-subspaces, $x \vee_X y$ depends on the subspace. In other words, in this kind of subspaces we have the induced ordering and a lattice structure but not the induced one.[2] Suppose also that E is a Banach space. A sequence $\{e_n\}$ of E is a

[1] $x \vee_X y = z$ if and only if $z \in X$, $z \geqslant x, y$ and for each $w \in X$, $w \geqslant x, y$ implies $w \geqslant z$.

[2] Lattice-subspaces appear in the work of many authors in their attempt to study the subspaces X of a vector lattice E which are the range of a positive projection P, i.e., $X = P(E)$. Then it is easy to show that X is a lattice-subspace with $x \vee_X y = P(x \vee y)$, for each $x, y \in X$ but as it is remarked in [1], there are even finite-dimensional lattice-subspaces which are not the range of a positive projection. For results on this special class of lattice-subspaces (which are the range of a positive projection) see in [6, 7, 9]. The notion of the lattice-subspace was introduced by Polyrakis in [12] where it is proved that each infinite dimensional closed lattice-subspace of ℓ_1 is order-isomorphic to ℓ_1. At the same time the notion of the lattice-subspace was introduced independently by Miyajima in [11], where the term 'quasi sublattice' is used and it is proved that X is a lattice-subspace if and only if X is the range of a positive projection from the sublattice $S(X)$ generated by X onto X. In [13], it is proved that each separable Banach lattice is order-isomorphic to a closed lattice-subspace of $C[0, 1]$, therefore $C[0, 1]$ is also a universal Banach lattice. Since the class of sublattices is not enough for this representation this result shows the importance of lattice-subspaces in the geometry of Banach lattices. In 1992 C. Aliprantis and D. Brown understood the meaning of lattice-subspaces in economics and posed the problem of the study of finite-dimensional lattice-subspaces. This problem is interesting, even in $\mathbb{R}^n$, because many economic models, as the famous Arrow-Debreu model, are finite. This problem was the motivation for [1], where the lattice-subspaces of $\mathbb{R}^n$ are studied.

(Schauder) **basis** of E if each element x of E has a unique expression of the form $x = \sum_{i=1}^{\infty} \lambda_n e_n$. If moreover

$$E_+ = \left\{ x = \sum_{n=1}^{\infty} \lambda_n e_n \mid \lambda_n \geqslant 0 \text{ for each } n \right\},$$

we say that $\{e_n\}$ is **a positive basis** of E.

A positive basis is unique in the sense that if $\{b_n\}$ is an other positive basis of E, then each element of $\{b_n\}$ is a positive multiple of an element of $\{e_n\}$. If $\{e_n\}$ is a positive basis of E, then the following statements are equivalent, see in [17], Theorem 16.3.

1. The basis $\{e_n\}$ is unconditional.
2. The cone E_+ is generating and normal.

Remind that cone E_+ is generating if $E = E_+ - E_+$ and E_+ is normal (or self-allied) is there exists $c \in \mathbb{R}_+$ such that $0 \leq x \leq y$ implies $\|x\| \leq c\|y\|$. In the case where X is an n-dimensional subspace of E and the positive cone X_+ of X is generating the following statements are equivalent:

(i) X is a lattice-subspace of E.

(ii) X has a positive basis.

(iii) X is order-isomorphic to $\mathbb{R}^n$.[1]

The equivalence of (i) and (ii) is the most useful criterion for finite-dimensional lattice-subspaces of E.

For notation and terminology not defined here we refer to [8, 5, 10].

3. Subspaces of $\mathbb{R}^\Omega$ with positive bases

In this article we shall denote by Ω a nonempty set, by $\mathbb{R}^\Omega$ the space of real valued functions defined in Ω and by $\mathbb{R}^\Omega_+ = \{x \in \mathbb{R}^\Omega \mid x(t) \geqslant 0, \quad \text{for each } t \in \Omega\}$, the positive cone of $\mathbb{R}^\Omega$.

Suppose that $\{b_r\}$ is a (finite or infinite) sequence of $\mathbb{R}^\Omega$. If t is a point of Ω and there exists $m \in \mathbb{N}$ such that $b_m(t) \neq 0$ and $b_r(t) = 0$ for each $r \neq m$, then we shall say that t is an m-**node** (or simply a **node**) of the sequence $\{b_r\}$. If for each r there exists an r-node t_r of $\{b_r\}$, we shall say that $\{b_r\}$ is a **sequence** of $\mathbb{R}^\Omega$ **with nodes** and also that $\{t_r\}$ is a **sequence of nodes** of $\{b_r\}$.

THEOREM 1. *Let E be an ordered subspace of $\mathbb{R}^\Omega$ and suppose that $\{b_r\}$ is a sequence of E consisting of positive functions.*

[1] I.e., there exists a linear operator T of Y onto $\mathbb{R}^n$ with the property: $y \in Y_+$ if and only if $T(y) \in \mathbb{R}^n_+$.

(i) *If* $\{b_r\}$ *is a positive basis of* E, *then*[2] *for each* m *there exists a sequence* $\{\omega_\nu\}$ *of* Ω *(depending on* m*) such that for each* $k \in \mathbb{N}$ *we have*

$$0 \leq \sum_{i=1, i \neq m}^{k} \frac{b_i(\omega_k)}{b_m(\omega_k)} < \frac{1}{k},$$

therefore $\lim_{\nu\to\infty} b_i(\omega_\nu)/b_m(\omega_\nu) = 0$ *for each* $i \neq m$.

(ii) *If* E *is an* n*-dimensional subspace of* $\mathbb{R}^\Omega$ *and the sequence* $\{b_r\}$ *is consisting of* n *vectors* $b_1, b_2, ..., b_n$, *the converse of* (i) *is also true, i.e. if for each* $1 \leq m \leq n$ *there exists a sequence* $\{\omega_\nu\}$ *of* Ω *(depending on* m*) satisfying*

$$\lim_{\nu\to\infty} \frac{b_i(\omega_\nu)}{b_m(\omega_\nu)} = 0 \ \textit{for each } i \neq m,$$

then $\{b_1, \ldots, b_n\}$ *is a positive basis of* E.

Proof. Suppose that $\{b_r\}$ is a positive basis of E. Then for each k we put $z_k = -\frac{1}{k} b_m + \sum_{i=1, i\neq m}^{k} b_i$. Since $\{b_r\}$ is a positive basis we have that $z_k \notin E_+$, therefore there exists $\omega_k \in \Omega$ such that $z_k(\omega_k) < 0$, or

$$0 \leq \sum_{i=1, i \neq m}^{k} \frac{b_i(\omega_k)}{b_m(\omega_k)} < \frac{1}{k}, \text{ for each } k.$$

For each fixed $i \neq m$ we have $0 \leqslant b_i(\omega_k)/b_m(\omega_k) < 1/k$, for each $k \geqslant i$. By letting $k \to \infty$ we have that (i) is true.

To prove the converse assume that $b_1, b_2, \ldots, b_n$ satisfy the assumptions of (ii). To show that $\{b_1, \ldots, b_n\}$ is a positive basis assume that $x = \sum_{i=1}^{n} \lambda_i b_i \in E_+$. Then we have that

$$0 \leqslant \frac{x(\omega_\nu)}{b_m(\omega_\nu)} = \sum_{i=1}^{n} \lambda_i \frac{b_i(\omega_\nu)}{b_m(\omega_\nu)},$$

and taking limits as $\nu \to \infty$ we have $\lambda_m = \lim_{\nu\to\infty} x(\omega_\nu)/b_i(\omega_\nu) \geq 0$ for each m. Also if we suppose that $x = 0$, similarly we have that $\lambda_m = 0$ for each m, therefore the vectors b_i are also linearly independent. Hence $\{b_1, b_2, \ldots, b_n\}$ is a positive basis of E. □

PROPOSITION 2. *Let* E *be an ordered subspace of* $\mathbb{R}^\Omega$ *and let* $\{b_1, \ldots, b_n\}$ *be a positive basis of* E. *For each* $x = \sum_{i=1}^{n} \lambda_i b_i \in E$, *we have:*

(i) *If a point* t_i *is an* i*-node of the basis, then* $\lambda_i = x(t_i)/b_i(t_i)$.

(ii) *If* $\{\omega_\nu\}$ *is a sequence of* Ω *such that* $\lim_{\nu\to\infty} b_j(\omega_\nu)/b_i(\omega_\nu) = 0$ *for each* $j \neq i$, *then* $\lambda_i = \lim_{\nu\to\infty} x(\omega_\nu)/b_i(\omega_\nu)$.

[2] We suppose also that E is a Banach space with respect to some norm which is defined on E.

Proof. If the point t_i is an i-node, then $x(t_i) = \lambda_i b_i(t_i)$. Statement (ii) is also true because $\lim_{\nu\to\infty} x(\omega_\nu)/b_i(\omega_\nu) = \lim_{\nu\to\infty} \sum_{j=1}^{n} \lambda_j b_j(\omega_\nu)/b_i(\omega_\nu) = \lambda_i$. □

4. Finite-dimensional lattice-subspaces of $\mathbb{R}^\Omega$

In this section we suppose that $x_1, x_2, \dots, x_n$ are fixed, linearly independent positive elements of $\mathbb{R}^\Omega$, $z = x_1 + x_2 + ... + x_n$ and we suppose that

$$X = [x_1, x_2, \dots, x_n],$$

is the subspace of $\mathbb{R}^\Omega$ generated by the functions x_i.

DEFINITION 3. Let $y_1, y_2, \dots, y_m \in \mathbb{R}^\Omega_+$. The **basic function (curve)** of $y_1, y_2, \dots, y_n$ is the function

$$\gamma(t) = \left(\frac{y_1(t)}{w(t)}, \frac{y_2(t)}{w(t)}, \dots, \frac{y_m(t)}{w(t)}\right), \quad t \in \Omega \text{ with } w(t) > 0,$$

where w is the sum of the functions y_i[1]

In this paper we will denote by β the basic function of $x_1, x_2, \dots, x_n$, i.e.

$$\beta(t) = \left(\frac{x_1(t)}{z(t)}, \frac{x_2(t)}{z(t)}, \dots, \frac{x_n(t)}{z(t)}\right), \quad t \in \Omega \text{ with } z(t) > 0,$$

where z is the sum of the functions x_i. Also we will denote by $D(\beta)$ the domain of β, by $R(\beta)$ the range of β and by K the convex hull of the closure of the range of β, i.e.,

$$K = co\overline{R(\beta)}.$$

Recall that for any subset C of a linear topological space, denote by $\overline{C}$ the closure of C, by co C the convex hull of C and by $\overline{co}C$ the closure of co C. A subset C of $\mathbb{R}^l$ is a **polytope** if C is the convex hull of a finite subset of $\mathbb{R}^l$ and C is an **r-simplex** if it is the convex hull of $r+1$ affinely independent vectors of $\mathbb{R}^l$. In both cases the extreme points of C are referred as **vertices** of C. Also for any matrix A we denote by A^T the transpose and by A^{-1} the inverse matrix of A.

Finite dimensional lattice-subspaces of the space of the continuous real valued functions $C(\Omega)$ defined on a compact, Hausdorff, topological space Ω, have been studied in [14] and [15]. In this section we generalize the results of these articles in $\mathbb{R}^\Omega$, where Ω is a set. The proofs of this section (except that of Theorem 11) are the same with the proofs of the corresponding results of [14] and [15]. So we give only the proof of the next Theorem and we omit the others.

[1] If Ω is an interval of the real line, then γ defines a curve in Δ_n. For this reason γ is referred also as the basic curve of the functions x_i.

THEOREM 4. *The following statements are equivalent.*
(i) *X is a lattice-subspace*
(ii) *K is an (n-1)-simplex.*

Suppose that statement (ii) is true and that $P_1, P_2, \dots, P_n$ are the vertices of K. Then for each $i = 1, 2, \dots, n$ there exists a sequence $\{\omega_{i\nu}\}$ of Ω such that

$$P_i = \lim_{\nu\to\infty} \beta(\omega_{i\nu}).$$

Suppose also that A is the $n \times n$ matrix whose, for each $i = 1, 2, \dots, n$, the i^{th} column is the vector P_i and $b_1, b_2, \dots, b_n$ are the functions defined by the formula

$$(b_1(t), b_2(t), \dots, b_n(t))^T = A^{-1}(x_1(t), x_2(t), \dots, x_n(t))^T. \tag{4.1}$$

Then the set $\{b_1, b_2, \dots, b_n\}$ is a positive basis of X and

$$\lim_{\nu\to\infty} \left(\frac{b_j}{b_i}\right)(\omega_{i\nu}) = 0 \text{ for each } j \neq i.$$

If $P_k = \beta(t_k)$, then t_k is a k-node of the basis $\{b_1, \dots, b_n\}$.

Proof. Suppose that statement (ii) is true. Since the vectors P_i are extreme points of K we have that P_i is not a convex combination of elements of $\overline{R(\beta)}$, therefore we have that $P_i \in \overline{R(\beta)}$, hence there exists a sequence $\{\omega_{i\nu}\}$ such that $P_i = \lim_{\nu\to\infty} \beta(\omega_{i\nu})$. Suppose that the functions b_i are defined as in the Theorem. Since $\{x_1, x_2, \dots, x_n\}$ is a basis of X and the vectors P_i are linearly independent we have that $\{b_1, b_2, \dots, b_n\}$ is a basis of X. Let

$$P_i = (a_{i1}, a_{i2}, \dots, a_{in}), \quad i = 1, 2, \dots, n.$$

Since the vectors P_i belong to the simplex $\Delta_n = \left\{\xi \in \mathbb{R}^n_+ \mid \sum_{i=1}^n \xi_i = 1\right\}$, it follows that $\sigma_i = \sum_{j=1}^n a_{ij} = 1$ for each i. By (4.1) we have that $x_i = \sum_{j=1}^n a_{ji} b_j$, therefore

$$z = \sum_{i=1}^n x_i = \sum_{i=1}^n \sigma_i b_i = \sum_{i=1}^n b_i.$$

Let $\beta(t) = \sum_{i=1}^n \xi_i(t) P_i$ be the expansion of $\beta(t)$ relative to the basis $\{P_1, P_2, \dots, P_n\}$. Then

$$\frac{1}{z(t)} (x_1(t), x_2(t), \dots, x_n(t))^T = A\,(\xi_1(t), \xi_2(t), \dots, \xi_n(t))^T$$

and in view of (4.1) we get

$$(\xi_1(t), \xi_2(t), \dots, \xi_n(t))^T = \frac{1}{z(t)} (b_1(t), b_2(t), \dots, b_n(t))^T.$$

Since $\beta(t)$ is a convex combination of the vectors P_i, we have that $\xi_i(t) \in \mathbb{R}_+$, therefore $b_i(t) \in \mathbb{R}_+$ for each i. Thus, $b_i \in X_+$ for each i. From (4.1) we have

$$(\beta(t))^T = A\left(\frac{b_1(t)}{z(t)}, \frac{b_2(t)}{z(t)}, ..., \frac{b_n(t)}{z(t)}\right)^T.$$

Since $0 \leqslant b_j(t)/z(t) \leqslant 1$, there exists a subsequence of $\{\omega_{i\nu}\}$ which we will denote again by $\{\omega_{i\nu}\}$ such that $b_j(\omega_{i\nu})/z(\omega_{i\nu})$ is convergent for each j and suppose that $\eta_{ij} = \lim_{\nu\to\infty} b_j(\omega_{i\nu})/z(\omega_{i\nu})$.

Replacing t by $\omega_{i\nu}$ and taking limits, we get

$$(a_{i1}, a_{i2}, \dots, a_{in})^T = A(\eta_{i1}, \eta_{i2}, \dots, \eta_{in})^T. \tag{4.2}$$

We remark that the columns of A are the vectors P_k, therefore $\eta_{ii} = 1$ and $\eta_{ij} = 0$ for each $i \neq j$. Since $\omega_{i\nu}$ belong to the domain of β we have that $z(\omega_{i\nu}) > 0$, therefore $b_j(\omega_{i\nu}) > 0$ for at least one j. Since $\eta_{ii} = 1$, we have $b_i(\omega_{i\nu}) > 0$ for each ν. Therefore,

$$\lim_{\nu\to\infty}\left(\frac{b_i}{z}\right)(\omega_{i\nu}) = \lim_{\nu\to\infty}\frac{1}{1+\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{b_j}{b_i}\right)(\omega_{i\nu})} = 1.$$

Hence

$$\lim_{\nu\to\infty}\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{b_j}{b_i}\right)(\omega_{i\nu}) = 0.$$

Since

$$0 \leqslant \frac{b_j}{b_i} \leqslant \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{b_j}{b_i},$$

we have that

$$\lim_{\nu\to\infty}\left(\frac{b_j}{b_i}\right)(\omega_{i\nu}) = 0 \text{ for each } j \neq i, \tag{4.3}$$

and by Theorem 1, $\{b_1, \dots, b_n\}$ is a positive basis of X, therefore X is a lattice-subspace. Hence (ii) implies (i). To prove the last assertion of (ii) we suppose now that $P_k = \beta(t_k)$. Then we may suppose that $\omega_{k\nu} = t_k$ for each ν. As we have remarked before, $b_k(\omega_{k\nu}) > 0$ for each ν and

$$\lim_{\nu\to\infty}\left(\frac{b_j}{b_k}\right)(\omega_{k\nu}) = 0 \quad \text{for each} \quad j \neq k.$$

Therefore we have that $b_k(t_k) > 0$ and $b_j(t_k)/b_k(t_k) = 0$, for each $j \neq k$, hence $b_j(t_k) = 0$ for each $j \neq k$, therefore t_k is a k-node for the basis $\{b_1, b_2, \ldots, b_n\}$. The proof that (i) implies (ii) is the same with the corresponding proof of [14], Theorem 3.6. □

4.1. SUBLATTICES

LEMMA 5 ([15], Lemma 3.4). *The functions $y_i \in \mathbb{R}_+^{\Omega}, i = 1, 2, \ldots, m$ are linearly independent if and only if the space generated by the range of the basic function of $y_1, y_2, \ldots, y_m$ is $\mathbb{R}^m$.*

THEOREM 6 ([15], Theorem 3.6). *The following statements are equivalent:*
(i) *X is a sublattice of $\mathbb{R}^{\Omega}$.*
(ii) $R(\beta) = \{P_1, P_2, \ldots, P_n\}$.
If statement (ii) is true, a positive basis $\{b_1, b_2, \ldots, b_n\}$ of X is given by the formula:

$$(b_1(t), b_2(t), \ldots, b_n(t))^T = A^{-1}(x_1(t), x_2(t), \ldots, x_n(t))^T, \tag{4.4}$$

where A is the $n \times n$ matrix whose the i^{th} column is the vector P_i, for each $i = 1, 2, \ldots, n$.

THEOREM 7 (**Construction of the sublattice generated by** X**).** *Let Z be the sublattice of $\mathbb{R}^{\Omega}$ generated by $x_1, x_2, \ldots, x_n$ and let $m \in \mathbb{N}$. Then statements (i) and (ii) are equivalent:*
(i) $\dim(Z) = m$.
(ii) $R(\beta) = \{P_1, P_2, \ldots, P_m\}$.

If statement (ii) is true then Z is constructed as follows:
(a) Enumerate $R(\beta)$ so that its n first vectors to be linearly independent. (A such enumeration exists by Lemma 5). Denote again by P_i, $i = 1, 2, \ldots, m$ the new enumeration and let $I_i = \beta^{-1}(P_i)$, $i = 1, 2, \ldots, m$.
(b) Define the functions

$$x_{n+k}(t) = a_k(t)\, z(t), \quad t \in \Omega, \quad k = 1, 2, \ldots, m-n$$

where a_k is the characteristic function of I_{n+k}.
(c) $Z = [x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_m]$.

The proof of the above theorem is the same with the proof of Theorem 3.7, in [15].

4.2. MINIMAL LATTICE-SUBSPACES

Suppose that $L(X)$ is the set of lattice-subspaces of $\mathbb{R}^{\Omega}$ which contain X. (Recall that X is the subspace of $\mathbb{R}^{\Omega}$ generated by the functions $x_1, x_2, \ldots, x_n$). If $Y \in L(X)$ and for any proper subspace Z of Y we have that $Z \notin L(X)$, then we say that Y is a **minimal lattice-subspace** of $\mathbb{R}^{\Omega}$ containing $x_1, x_2, \ldots, x_n$. We study the problem: **Does a finite-dimensional minimal lattice-subspace of $\mathbb{R}^{\Omega}$ containing $x_1, x_2, \ldots, x_n$ exist?**

THEOREM 8 ([15], Theorem 3.8). *Let Y be an l-dimensional lattice-subspace of $\mathbb{R}^{\Omega}$ containing $x_1, x_2, \ldots, x_n$. Suppose that $\{b_1, b_2, \ldots, b_l\}$ is a positive basis of Y,*

$$x_i = \sum_{j=1}^{l} \lambda_{ij} b_j, \quad i = 1, 2, \ldots, n,$$

$$\sigma_i = \sum_{j=1}^{n} \lambda_{ji}, \quad i = 1, 2, \ldots, l,$$

$$\Phi = \{i \in \{1, 2, \ldots, l\} \mid \sigma_i \neq 0\},$$

$$P_i = \frac{1}{\sigma_i}(\lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{ni}), \quad i \in \Phi$$

and K is the convex hull of $\overline{R(\beta)}$. Then

(i) *$P_i \in \overline{R(\beta)}$ for each $i \in \Phi$.*

(ii) *K is a polytope with vertices $P_{i_1}, P_{i_2}, \ldots, P_{i_m}$ where $n \leqslant m \leqslant l$ and $i_\nu \in \Phi$ for each $\nu = 1, 2, \ldots, m$.*

THEOREM 9 (**Construction of a minimal lattice-subspace).** *Let the set $K = \mathrm{co}\,\overline{R(\beta)}$ be a polytope with vertices $P_1, P_2, \ldots, P_m$. Suppose that the n first vertices $P_1, P_2, \ldots, P_n$ of K are linearly independent.*[1] *Suppose also that ξ_i, $i = 1, 2, \ldots, m$ are positive, real-valued functions defined on $D(\beta)$ such that $\sum_{i=1}^{m} \xi_i(t) = 1$ and $\beta(t) = \sum_{i=1}^{m} \xi_i(t) P_i$, for each $t \in D(\beta)$. Let x_{n+i}, $i = 1, 2, \ldots, m-n$ be the functions $x_{n+i}(t) = \xi_{n+i}(t)\, z(t)$ for each $t \in D(\beta)$ and $x_{n+i}(t) = 0$ if $t \notin D(\beta)$. Then*

$$Y = [x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_m]$$

is a minimal lattice-subspace of $\mathbb{R}^{\Omega}$ containing $x_1, x_2, \ldots, x_n$ and $\dim Y = m$.

---

[1] Such an enumeration of the vertices of K exists by Lemma 5.

A positive basis $\{b_1, b_2, \dots, b_m\}$ *of* Y *is given by the formula*

$$(b_1, b_2, \dots, b_m)^T = D^{-1} (x_1, x_2, \dots, x_m)^T, \tag{4.5}$$

where D *is the* $m \times m$ *matrix with columns the vectors*

$$R_i = \frac{M_i}{\|M_i\|_1}, i = 1, 2, \dots, m, \tag{4.6}$$

with $M_i = (P_i, 0)$ *for* $i = 1, 2, \dots, n$ *and* $M_i = (P_{n+i}, e_i)$ *for* $i = 1, 2, \dots, m-n$.[1]

Proof. The proof is the same with the proof of Theorem 3.10 of [15]. □

The next result says that all the finite-dimensional minimal lattice-subspaces containing X are of the same dimension.

THEOREM 10 ([15], Theorem 3.20). *Let* $K = \operatorname{co}\overline{R(\beta)}$ *and let* L *be the set of finite-dimensional minimal lattice-subspaces of* $\mathbb{R}^\Omega$ *containing* $x_1, x_2, \dots, x_n$. *Then the following statements are equivalent:*
(i) K *is a polytope with* m *vertices.*
(ii) $L \neq \emptyset$ *and* $\dim Y = m$, *for each* $Y \in L$.
(iii) $L \neq \emptyset$.

4.3. THE DERIVATIVE CRITERION

In general it is difficult to study if K is a polytope or not and if it is it is also difficult to determine its vertices. The derivative criterion says that if K a polytope and $\beta(t_0)$ is a vertex of K, then the derivative of the restriction of β at any curve of Ω having t_0 as an interior point is equal to zero, whenever the derivative exists. We start with a more general criterion for the case where the functions are not differentiable.

THEOREM 11. *Let* K *be a polytope and let* $\beta(t_0)$ *be a vertex of* K. *Suppose that* $\{a_r\}$ *is a sequence of real numbers convergent to zero with* $a_{2r} > 0$ *and* $a_{2r+1} < 0$ *for each* r *and suppose also that* $\{t_r\}$ *is a sequence of* Ω. *If* $\lim_{r\to\infty} \frac{\beta(t_r)-\beta(t_0)}{a_r} = \ell$, $\ell \in \mathbb{R}^n$, *then* $\ell = 0$.

Proof. Let $\lim_{r\to\infty} \frac{\beta(t_r)-\beta(t_0)}{a_r} = \ell \neq 0$. Then there exists r_0 such that $\beta(t_r) \neq \beta(t_0)$ for each $r > r_0$. Hence

$$\lim_{r\to\infty} \frac{\beta(t_{2r}) - \beta(t_0)}{\|\beta(t_{2r}) - \beta(t_0)\|} = \lim_{r\to\infty} \frac{\beta(t_{2r}) - \beta(t_0)}{a_{2r}} \cdot \lim_{r\to\infty} \frac{1}{\left\| \frac{\beta(t_{2r})-\beta(t_0)}{a_{2r}} \right\|} = \frac{\ell}{\|\ell\|},$$

[1] $(P_i, 0)$, (P_{n+i}, e_i) are the vectors of $\mathbb{R}^m$ whose the n first coordinates are the corresponding coordinates of P_i and the other are the coordinates of zero, of e_i respectively, where $\{e_k\}$ is the usual basis of $\mathbb{R}^{m-n}$. Also $\|M_i\|_1$ is the ℓ_1-norm of M_i.

and similarly

$$\lim_{r\to\infty} \frac{\beta(t_{2r+1}) - \beta(t_0)}{\|\beta(t_{2r+1}) - \beta(t_0)\|} = -\frac{\ell}{\|\ell\|},$$

because

$$\|\frac{\beta(t_{2r+1}) - \beta(t_0)}{a_{2r+1}}\| = -\frac{\|\beta(t_{2r+1}) - \beta(t_0)\|}{a_{2r+1}}.$$

Since $\beta(t_0)$ is a vertex of K it is easy to show that there exists a real number $\rho > 0$ such that

$$\beta(t_0) + \rho\frac{\xi - \beta(t_0)}{\|\xi - \beta(t_0)\|} \in K, \quad \text{for each } \xi \in K, \xi \neq \beta(t_0),$$

therefore

$$\lim_{r\to\infty}\left(\beta(t_0) + \rho\frac{\beta(t_{2r}) - \beta(t_0)}{\|\beta(t_{2r}) - \beta(t_0)\|}\right) = \beta(t_0) + \rho\frac{\ell}{\|\ell\|} = z_1 \in K$$

and

$$\lim_{r\to\infty}\left(\beta(t_0) + \rho\frac{\beta(t_{2r+1}) - \beta(t_0)}{\|\beta(t_{2r+1}) - \beta(t_0)\|}\right) = \beta(t_0) - \rho\frac{\ell}{\|\ell\|} = z_2 \in K.$$

Hence $\beta(t_0) = \frac{1}{2}(z_1 + z_2)$, contradiction. Therefore $\ell = 0$. □

COROLLARY 1 (**The derivative criterion).** *Let K be a polytope and let $\beta(t_0)$ be a vertex of K. Suppose that σ is a function defined on the real interval $(-\epsilon, \epsilon)$ with values in Ω with $\sigma(0) = t_0$ and suppose that $\varphi = \beta o \sigma$ is the composition of σ, β. Then*

$$\varphi'(0) = 0,$$

whenever the derivative $\varphi'(0)$ of φ at the point 0 exists.

REMARK 12. Suppose that the assumptions of the derivative criterion are satisfied and that the range of β is closed. Then $K = coR(\beta)$, therefore the extreme points of K are images of elements of Ω. The following remarks show the way we study if K is a polytope or not.

In the simplest case where Ω is the real interval $[a, b]$ we proceed as follows: suppose that K is a polytope and $\beta(t_0)$ is a vertex of K. Then $\beta'(t_0) = 0$ or t_0 is not an interior point of $[a, b]$ therefore the vertices of K belong to the set

$$G = \{\beta(t) \mid t \quad \text{is a root of the equation } \beta'(t) = 0 \text{ or } t = a \text{ or } t = b\}.$$

So in order to study if K is a polytope or not we solve the equation $\beta'(t) = 0$ and determine G. In the sequel we study if a finite and minimal subset Φ of G exists such that each $\beta(t)$ is a convex combination of elements of Φ. If a such set Φ exists, then K is a polytope and the elements of Φ are the vertices of K, therefore a minimal lattice-subspace is determined by Theorem 9.

If Ω is a convex subset of $\mathbb{R}^l$ the situation is analogous but more complicated. So if we suppose that K is a polytope and $\beta(t_0)$ is a vertex of K we have: If t_0 is an interior point of Ω, then the partial derivatives of β at the point t_0 are equal to zero and if t_0 belongs to the boundary $\vartheta(\Omega)$ of Ω, then the derivative at t_0 of the restriction of β at any differentiable curve of $\vartheta(\Omega)$ having t_0 as an interior point is equal to zero. Hence the points t_0 of Ω whose the images $\beta(t_0)$ are vertices of K can be obtained as solutions of a system of equations or they are extreme points of Ω which cannot be interior points of a differentiable curve of Ω.

If for example Ω is the square $[0, 1] \times [0, 1]$ of $\mathbb{R}^2$, and $\beta(t_0)$ is a vertex of K, then t_0 is:

(i) a root of the system of equations $D_1\beta(t) = 0$, $D_2\beta(t) = 0$[1](if t_0 is an interior point of Ω), or

(ii) a root of an equation $D_i\beta(t) = 0$ (if t_0 is an interior point of an edge of Ω), or

(iii) t_0 is a vertex of the square.

If Ω is the circle of $\mathbb{R}^2$ with center 0 and radius 1, the restriction of β on the boundary of Ω is $\sigma(\vartheta) = \beta(cos\vartheta, sin\vartheta)$, therefore the vertices of K are of the form $\beta(t_0)$, where t_0 is a root of the system $D_1\beta(t) = 0$, $D_2\beta(t) = 0$, or $t_0 = \sigma(\vartheta_0)$ with $\sigma'(\vartheta_0) = 0$.

5. Linear functions

In this section we suppose also that $x_1, x_2, \dots, x_n$ are linearly independent positive elements of $\mathbb{R}^\Omega$, z is the sum of x_i and X is the subspace of $\mathbb{R}^\Omega$ generated by the functions x_i but we add the assumption that Ω is a convex set and the functions x_i are **linear**, i.e.,

$$x_i\left(\sum_{k=1}^{m} \lambda_k t_k\right) = \sum_{k=1}^{m} \lambda_k x_i(t_k),$$

for each convex combination $\sum_{k=1}^{m} \lambda_k t_k$ of Ω and $x_i(\lambda t) = \lambda x_i(t)$, for each positive real number λ with $t, \lambda t \in \Omega$.

We denote also by β the basic function of the elements x_i and by K the convex hull of the closure of the range of β. The domain $D(\beta)$ of β is convex because $D(\beta) = \{t \in \Omega \quad \text{with } z(t) > 0\}$ and the function z is linear.

THEOREM 13. *Suppose that $x_1, x_2, \dots, x_n$ are linear functions. Then*

(i) *the basic function β is homogeneous of degree zero in the sense that $\beta(\lambda t) = \beta(t)$ for each positive real number λ with $t, \lambda t \in D(\beta)$.*

[1] D_i is the operator of the i^{th} partial derivative.

(ii) *for each positive, linear combination* $t = \sum_{k=1}^{m} \lambda_k t_k \in D(\beta)$ *of elements of* $D(\beta)$ *we have*

$$\beta(t) = \sum_{k=1}^{m} \frac{\lambda_k z(t_k)}{z(t)} \beta(t_k),$$

therefore $\beta(t)$ *is a convex combination of the vectors* $\beta(t_i)$, $i = 1, 2, \dots, m$.

(iii) *If* $t = t_1 + t_2$ *with* $t_1, t_2 \in \Omega$, $z(t) > 0$ *and* $z(t_2) = 0$, *then* $\beta(t) = \beta(t_1)$.

(iv) *Let* τ *be a topology on* Ω *and suppose that* $T = \{t_i, i \in I\}$ *and* A *are subsets of the domain* $D(\beta)$ *of* β.

(a) *If* A *is contained in the positive linear span of*[1] T, *then the image* $\beta(A)$ *of* A *is contained in the convex hull of the set* $\{\beta(t_i) \mid i \in I\}$.

(b) *If* A *is contained in the* τ*-closure* Ψ *of the positive linear span of* T *and the basic function* β *is* τ*-continuous on* Ψ, *then the image* $\beta(A)$ *of* A *is contained in the closed convex hull of the set* $\{\beta(t_i), i \in I\}$.

Proof. Statement (i) is an easy consequence of the linearity of x_i. To prove (ii) suppose that $r(\omega) = (x_1(\omega), x_2(\omega), \dots, x_n(\omega))$, $\omega \in \Omega$. Then we have:

$$\beta(t) = \frac{r(t)}{z(t)} = \frac{1}{z(t)} \sum_{k=1}^{m} \lambda_k r(t_k) = \frac{1}{z(t)} \sum_{k=1}^{m} \lambda_k z(t_k) \beta(t_k) = \sum_{k=1}^{m} \frac{\lambda_k z(t_k)}{z(t)} \beta(t_k).$$

Statement (iii) is also true because if we suppose that $z(t_2) = 0$, then by the fact that the functions x_i are positive we have $x_i(t_2) = 0$ for each i, therefore $z(t) = z(t_1)$ and $x_i(t) = x_i(t_1)$, for each i. Hence $\beta(t) = \beta(t_1)$. Statement (iv) is proved as follows: Suppose that each $t \in A$ is a positive linear combination of elements of T. Then by (ii), $\beta(t)$ is a convex combination of $\beta(t_i)$, therefore $\beta(A) \subseteq co\{\beta(t_i) | i \in I\}$. In the case (b), each element $t \in A$ is the limit of a net $\{\omega_\alpha\}$ where ω_α is a positive linear combination of t_i, therefore $\beta(\omega_\alpha)$ is a convex combination of elements of $\beta(T)$ and by the continuity of β on Ψ we have that $\beta(t)$ belongs to the closed convex hull of the set $\{\beta(t_i) \mid i \in I\}$, therefore (b) is also true. □

COROLLARY 2. *Suppose that* $x_1, x_2, \dots, x_n$ *are linear functions and that* $\beta(t_0)$ *is an extreme point of* K. *Then* t_0 *is an extreme point of* $D(\beta)$ *or* t_0 *has the property: If* c *is a line segment of* $D(\beta)$, *having* t_0 *as an interior point, then the function* β *is constant on* c *and also*

$$\frac{x_j(t)}{x_i(t)} = \frac{x_j(t_0)}{x_i(t_0)}, \quad \textit{for each } t \in c \textit{ and each } j, i \textit{ with } x_i(t_0) \neq 0.$$

Proof. Suppose that t_0 as an interior point of a line segment c and that $t_1 \in c$ with $t_1 \neq t_0$. Then t_0 is a convex combination $t_0 = \lambda_1 t_1 + \lambda_2 t_2$ with $t_2 \in c$, therefore

[1] The positive linear span of T is the set of positive linear combinations of elements of T

$\beta(t_0) = \lambda_1 \frac{z(t_1)}{z(t_0)} \beta(t_1) + \lambda_2 \frac{z(t_2)}{z(t_0)} \beta(t_2)$, hence $\beta(t_1) = \beta(t_2) = \beta(t_0)$ because $\beta(t_0)$ is an extreme point of K. Therefore $\beta(t) = \beta(t_0)$, for each $t \in c$. So we have that $x_k(t)/z(t) = x_k(t_0)/z(t_0)$, for each $t \in c$ and for each k, therefore $x_j(t)/x_i(t) = x_j(t_0)/x_i(t_0)$. □

Suppose that Y is an ordered space. A set S is an **absorbing subset of** Y_+ if $S \subseteq Y_+$ and for each $y \in Y_+$, $y \neq 0$, there exists a real number $\lambda > 0$ such that $\lambda y \in S$. The positive cone Y_+, and also each base for the cone Y_+ [1] of Y are absorbing subsets of Y_+. If Y is an ordered normed space the positive part $U_Y^+ = \{y \in Y_+ \mid \|y\| \leqslant 1\}$ of the closed unit ball of Y is an absorbing subset of Y_+ and if e is an order unite of Y, the order interval $[0, e]$ of Y is also an absorbing subset of Y_+.

DEFINITION 14. A dual system $\langle Y, G \rangle$ is[2] an **ordered dual system** if Y, G are ordered spaces with

$$Y_+ = \{y \in Y \mid \langle y, g \rangle \geqslant 0 \text{ for each } g \in G_+\} \textit{ and}$$

$$G_+ = \{g \in G \mid \langle y, g \rangle \geqslant 0 \text{ for each } y \in Y_+\}.$$

Suppose $\langle Y, G \rangle$ is an ordered dual system and suppose that Ω is a convex and absorbing subset of Y_+ .

Any element of G can be considered as an element of $\mathbb{R}^\Omega$ but the equality in G and equality in $\mathbb{R}^\Omega$ are not the same. If for example we suppose that $g_1, g_2 \in G$, then $g_1 = g_2$ in G, if and only if $\langle y, g_1 \rangle = \langle y, g_2 \rangle$ for each $y \in Y$ and $g_1 = g_2$ in $\mathbb{R}^\Omega$, if and only if $g_1(t) = g_2(t)$ for each $t \in \Omega$. It is clear that the equality in G implies equality in $\mathbb{R}^\Omega$ but the converse is not true in general. If the cone Y_+ is generating (i.e. $Y = Y_+ - Y_+$) it is easy to show that the two equalities are equivalent, therefore then we can identify algebraically G with a subspace of $\mathbb{R}^\Omega$. If for example Y is a Banach lattice, then the cone Y_+ is generating therefore we may suppose that G is a subspace of $\mathbb{R}^\Omega$. For the space Y hold also similar results.

Suppose that $x_1, x_2, \dots, x_n$ are linearly independent positive elements of G and that X is the subspace of $\mathbb{R}^\Omega$ generated by these functions. As before, the basic function of the elements x_i is

$$\beta(t) = \left(\frac{x_1(t)}{z(t)}, \frac{x_2(t)}{z(t)}, \dots, \frac{x_n(t)}{z(t)} \right), \ t \in \Omega \text{ with } z(t) > 0,$$

where z is the sum of the elements x_i.

Also we denote by $D(\beta)$ the domain, by $R(\beta)$ the range of β and by K the convex hull of the closure of $R(\beta)$.

[1] Any subset $B = \{y \in Y_+ \mid f(y) = 1\}$, where f is a strictly positive linear functional of Y is a base for the cone Y_+.

[2] Remind that in any dual system $\langle Y, G \rangle$, G separates the points of Y and conversely.

THEOREM 15. *Let $\langle Y, G\rangle$ be an ordered dual system. Suppose that G is an ordered normed space and Y the topological dual of G.*

(i) *If $\Omega = U_Y^+$, then $K \subseteq \overline{co}\{\beta(t) \mid t \in ep(U_Y^+)\}$.*

(ii) *If e is an order unite of Y, the cone Y_+ is normal and $\Omega = [0, e]$, then $K \subseteq \overline{co}\{\beta(t) \mid t \in L_e\}$, where L_e is the Boolean algebra of components of*[3] *e.*

Proof. Suppose that τ is the $\sigma(Y, G)$ topology of Y.

(i) The positive cone Y_+ of Y is τ-closed, therefore the positive part U_Y^+ of the closed unit ball of Y is τ-compact, hence U_Y^+ is the τ-closed convex hull of its extreme points. Therefore by Theorem 13, $K \subseteq \overline{co}\{\beta(t) \mid t \in ep(U_Y^+)\}$.

(ii) Since e is an order unit, the order interval $[0, e]$ is an absorbing subset of Y_+. Also $[0, e]$ is norm-bounded therefore it is τ-compact, hence $K \subseteq \overline{co}\{\beta(t) \mid t \in L_e\}$. □

THEOREM 16. *Let $\langle Y, G\rangle$ be an ordered dual system. Suppose that Y is a Banach lattice with a positive basis $\{d_i\}$ with $\|d_i\| = 1$, for each i and suppose also that $\Omega = U_Y^+$. If $x_1, x_2, \ldots, x_n$ are linearly independent positive elements of G and β is the basic function of the vectors x_i, then*

$$R(\beta) \subseteq \overline{co}\{\beta(d_i) \mid i \in \Phi\},$$

where $\Phi = \{i \in \mathbb{N} \mid d_i \in D(\beta)\}$.

If Y is an m-dimensional space with a positive basis $\{b_1, b_2, \ldots, b_m\}$, then

$$K = co\{\beta(d_i) \mid i \in \Phi\},$$

where $\Phi = \{i \in \{1, 2, \ldots, m\} \mid d_i \in D(\beta)\}$, therefore K is a polytope and a finite dimensional minimal lattice subspace of $\mathbb{R}^\Omega$ which contains the elements x_i exists and is determined by Theorem 9.

Proof. The positive cone Y_+ of Y is generating and normal, therefore the basis $\{b_i\}$ is unconditional, [17], Theorem 16.3. Let Y_1 be the closed linear span of the vectors $d_i, i \in \Phi$. Then Y_1 is a closed sublattice of Y and each element y of Y has a unique expression $y = y_1 + y_2$, with $y_1 \in Y_1$ and $y_2 \in Y_2$, where Y_2 is the closed linear span of the vectors $d_i,\ i \notin \Phi$. It is clear that $z(y_2) = 0$, therefore $\beta(y) = \beta(y_1)$ by statement (iii) of Theorem 13, hence $R(\beta) = \beta(U_Y^+ \cap (Y_1^+ \setminus \{0\}))$. Since each positive linear functional of a Banach lattice is continuous, [5], Theorem 12.3, the x_i are continuous linear functionals of Y. Since each element of Y_1^+ is limit of a sequence of positive linear combinations of elements of the basis $\{b_i \mid i \in \Phi\}$, by statement (iv) of the Theorem 13 we have that

$$R(\beta) \subseteq \overline{co}\{\beta(d_i) \mid i \in \Phi\},$$

therefore the first part of the Theorem is true.

Suppose now that $\dim Y = m$ and $\{b_1, b_2, \ldots, b_m\}$ is a positive basis of Y. As we have shown above $R(\beta) \subseteq \overline{co}\{\beta(d_i) \mid i \in \Phi\}$. Since the set Φ is finite,

[3] I.e. L_e is the set of extreme points of $[0, e]$.

$co\{\beta(d_i) \mid i \in \Phi\}$ is closed therefore $\overline{R(\beta)}$ is contained in the set $co\{\beta(d_i) \mid i \in \Phi\}$, hence $K = co\{\beta(d_i) \mid i \in \Phi\}$. Therefore K is a polytope and the Theorem is true. □

EXAMPLE 17. Suppose that X is the subspace of ℓ_∞ generated by the vectors x_1, x_2, x_3, where

$$x_1(3n) = \frac{n^2+n+2}{n^2}, x_1(3n+1) = \frac{n^2+n+2}{n^2}, x_1(3n+2) = \frac{n^2+n+2}{n^2},$$

$$x_2(3n) = \frac{n^2+3}{n^2}, x_2(3n+1) = \frac{2n+1}{n^2}, x_2(3n+2) = \frac{2n^2+n+2}{n^2},$$

$$x_3(3n) = \frac{2n^2+n+2}{n^2}, x_3(3n+1) = \frac{n^2+3}{n^2}, x_3(3n+2) = \frac{2n+1}{n^2}.$$

Consider the dual system $\langle \ell_1, \ell_\infty \rangle$ and suppose that $\{e_i\}$ is the usual basis of ℓ_1. By Theorem 16 we have that $K \subseteq \overline{co}\{\beta(e_i)\}$. It is easy to see that

$$\beta(e_{3n}) = \frac{1}{4n^2+2n+7}(n^2+n+2, n^2+3, 2n^2+n+2),$$

$$\beta(e_{3n+1}) = \frac{1}{2n^2+3n+6}(n^2+n+2, 2n+1, n^2+3) \text{ and}$$

$$\beta(e_{3n+2}) = \frac{1}{3n^2+4n+5}(n^2+n+2, 2n^2+n+2, 2n+1).$$

The vectors $\lim_{n\longrightarrow\infty} \beta(e_{3n}) = \frac{1}{4}(1,1,2)$, $\lim_{n\longrightarrow\infty} \beta(e_{3n+1}) = \frac{1}{2}(1,0,1)$ and $\lim_{n\longrightarrow\infty} \beta(e_{3n+2}) = \frac{1}{3}(1,2,0)$, belong to $\overline{R(\beta)}$. We remark that the second and third coordinate of the vectors $\beta(e_i)$ are greater than zero and also that the first coordinate of these vectors is greater than $\frac{1}{4}$. After these remarks we show that each $\beta(b_i)$ is a convex combination of the vectors $\frac{1}{4}(1,1,2)$, $\frac{1}{2}(1,0,1)$, $\frac{1}{3}(1,2,0)$, therefore K is a simplex and X a lattice-subspace. A positive basis of X is given by the formula $(b_1, b_2, b_3)^T = A^{-1}(x_1, x_2, x_3)^T$, where A is the matrix with columns the vectors $\frac{1}{4}(1,1,2)$, $\frac{1}{2}(1,0,1)$, $\frac{1}{3}(1,2,0)$. So we find that $b_1 = \frac{4}{3}(-2x_1 + x_2 + 2x_3)$, $b_2 = \frac{2}{3}(4x_1 - 2x_2 - x_3)$ and $b_3 = x_1 + x_2 - x_3$ is a positive basis of X.

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