Statistical Process Control

MSc: Statistics and Actuarial-Financial Mathematics

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Nonparametric control charts for unknown distributions

- All the charts we have studied so far (Shewhart, CUSUM, EWMA) assume the process distribution is **normal**, or at least belongs to a known parametric family
- When this assumption is violated, the actual IC ARL can be **dramatically different** from the nominal value, leading to too many false alarms or dangerously slow detection
- **Nonparametric** (distribution-free) charts do not require a parametric form for the IC distribution
  - Their IC properties hold regardless of the true process distribution
  - They rely on ranks, signs, or data categorization instead of raw values

Why the normality assumption matters

- Consider the standard CUSUM chart with $k = 0.5$, designed for $\text{ARL}_0 = 500$ under the assumption that the IC distribution is $N(0,1)$
- If the true IC distribution is a standardized χ^2 or t, the actual IC ARL can be far from 500
- For example, a standardized χ^2_3 gives an actual ARL_0 around 50, meaning the process is stopped **ten times more often** than intended
- The problem is not unique to the CUSUM. Any parametric chart (Shewhart, EWMA) has the same vulnerability
- The EWMA chart with small λ is sometimes called “robust,” but the required λ depends on how far the true distribution is from normal, which we typically do not know

The Nonparametric Signed-Rank (NSR) Shewhart chart

- Assume batch data of size m at each time point, with the IC distribution **symmetric** about its median $\tilde{\mu}_0$
- At time i, compute the Wilcoxon signed-rank statistic[1]

$$\psi_i = \sum_{j=1}^{m} \text{sign}(X_{ij} - \tilde{\mu}_0)\, R_{ij}^{+}$$

  where R_{ij}^{+} is the rank of $|X_{ij} - \tilde{\mu}_0|$ among $\{|X_{i1} - \tilde{\mu}_0|, \ldots, |X_{im} - \tilde{\mu}_0|\}$
- Under IC, positive and negative signed-ranks cancel out on average, so ψ_i is centred at zero
- The upward NSR chart signals when $\psi_i \geq U$, where $U > 0$ is chosen from tables (Table 8.1 in Qiu) to achieve a target ARL_0
- Since ψ_i takes only finitely many values, only a **discrete set** of ARL_0 levels is achievable

[1] Bakir, S. T. (2004). A distribution-free Shewhart quality control chart based on signed-ranks. *Qual. Eng.*, 16, 613–623.

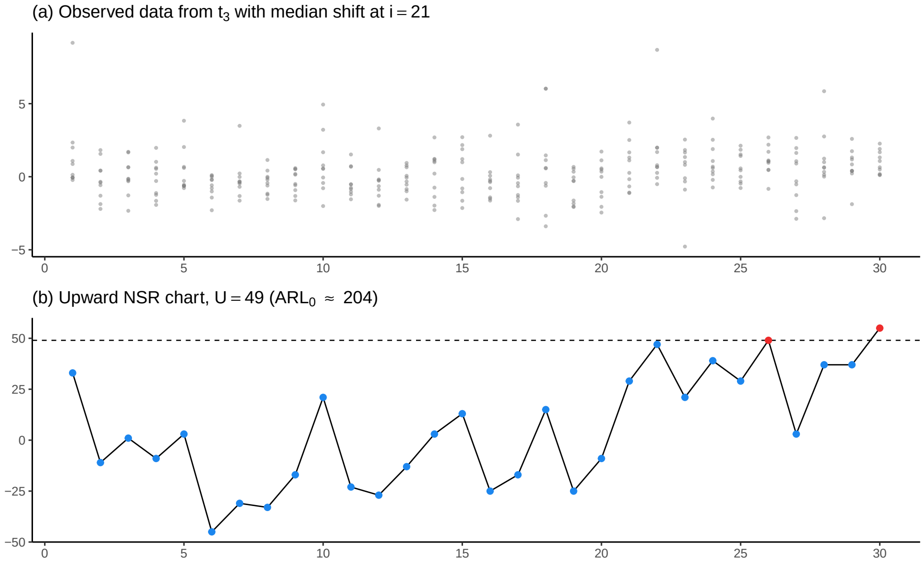
Example 8.1 Setup and data generation

- We simulate 30 batches of size $m = 10$ from a heavy-tailed symmetric distribution (t_3), with known IC median $\tilde{\mu}_0 = 0$
- The first 20 batches are IC; from batch 21 onward, a location shift of $\delta = 1$ is introduced

$$X_{ij} \sim \begin{cases} t_3 & i = 1, \dots, 20 \\ t_3 + 1 & i = 21, \dots, 30 \end{cases}$$

- The upward NSR chart uses $U = 49$ as control limit, which gives $\text{ARL}_0^+ \approx 204$ (Table 8.1)
- At each batch, the Wilcoxon signed-rank statistic $\psi_i = \sum_{j=1}^{m} \text{sign}(X_{ij})\, R_{ij}^+$ is computed and compared to U
- The chart should detect the shift shortly after batch 20, despite the heavy tails of the t_3 distribution

Example 8.1 NSR chart detecting a shift in t_3 data



Nonparametric CUSUM charts based on signed-ranks

- Bakir and Reynolds[2] combined the Wilcoxon signed-rank statistic ψ_n with the CUSUM recursion to detect small persistent shifts

$$C_n^+ = \max\left(0,\ C_{n-1}^+ + \psi_n - k\right), \qquad C_0^+ = 0$$

  and signal upward shift when $C_n^+ > h$

- Both k and h are chosen as integers (since ψ_n is integer-valued). Table 8.4 in Qiu gives approximate optimal k for different shift sizes and batch sizes
- The chart inherits the CUSUM's ability to accumulate evidence while remaining distribution-free under the symmetry assumption
- The downward and two-sided versions are defined analogously
- Practical advantage: the chart is robust to scattered large observations that would trigger a Shewhart chart

[2] Bakir, S. T. and Reynolds, M. R., Jr. (1979). A nonparametric procedure for process control based on within group ranking. *Technometrics*, 21, 175–183.

Example 8.3 Setup and data generation

- We simulate 30 batches of size $m = 6$ from the t_3 distribution (symmetric, heavy-tailed), with known IC median $\tilde{\mu}_0 = 0$
- The first 20 batches are IC; from batch 21 onward, a location shift of $\delta = 1$ is introduced

$$X_{nj} \sim \begin{cases} t_3 & n = 1, \ldots, 20 \\ t_3 + 1 & n = 21, \ldots, 30 \end{cases}$$

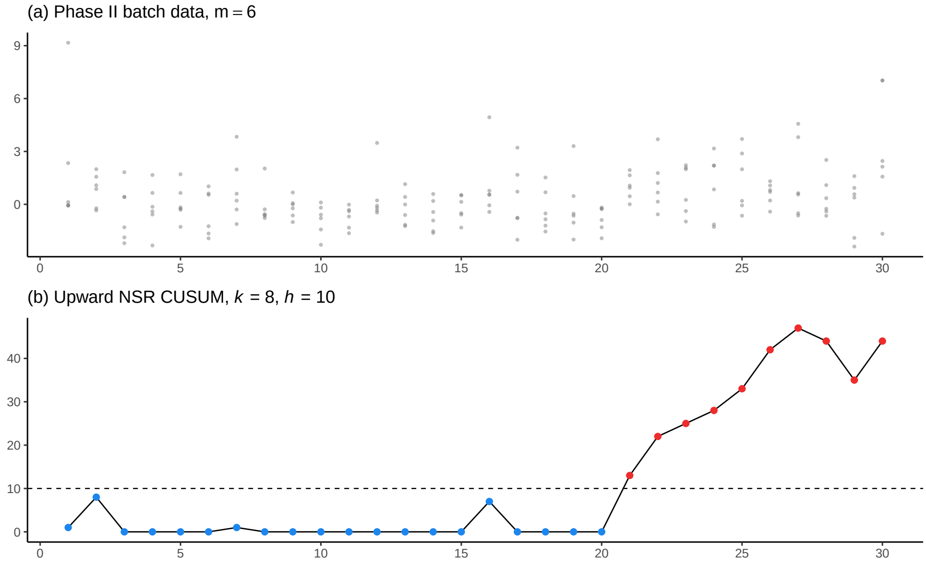
- The upward NSR CUSUM uses $k = 8$ and $h = 10$, tuned for detecting a 1σ shift (Table 8.4). The corresponding ARL_0 lies between 97 and 155 (Table 8.5)
- The CUSUM recursion accumulates the signed-rank statistics

$$C_n^+ = \max\left(0,\ C_{n-1}^+ + \psi_n - 8\right), \qquad C_0^+ = 0$$

and signals when $C_n^+ > 10$

- The chart should detect the shift quickly while remaining robust to the scattered large observations typical of t_3 data

Example 8.3 NSR CUSUM with $m = 6$



Nonparametric CUSUM based on the Wilcoxon rank-sum

- When a reference sample $(Y_1, \ldots, Y_M)$ is available, each Phase II batch is pooled with it and ranked; the Wilcoxon rank-sum statistic is $W_n = \sum_{j=1}^{m} R_{nj}$, where R_{nj} is the rank of X_{nj} among the $M + m$ pooled observations
- The NRS CUSUM chart[3] applies the CUSUM recursion to the centred rank-sum

$$C_n^+ = \max\left(0,\ C_{n-1}^+ + \left(W_n - \frac{m(m+M+1)}{2}\right) - k\right)$$

  and signals when $C_n^+ > h$
- A common choice is $k = 0.5\,\sigma_{W_n}$, where $\sigma_{W_n} = \sqrt{mM(m+M+1)/12}$
- The two-sided version adds a downward chart C_n^- and signals when either chart crosses its limit
- Unlike the NSR CUSUM, this chart does **not** require the IC distribution to be symmetric

[3] Li, S. Y., Tang, L. C., and Ng, S. H. (2010). Nonparametric CUSUM and EWMA control charts for detecting mean shifts. *J. Qual. Technol.*, 42, 209–226.

Nonparametric EWMA charts

- Any rank-based charting statistic can be embedded in an EWMA recursion for smoother detection of small shifts
- The NSR EWMA chart[4] uses the Wilcoxon signed-rank statistic ψ_n from a batch of size m

$$E_n = \lambda \psi_n + (1-\lambda) E_{n-1}, \qquad E_0 = 0$$

- The steady-state control limits are

$$U = \rho \sqrt{\frac{m(m+1)(2m+1)}{6} \cdot \frac{\lambda}{2-\lambda}}, \qquad L = -U$$

  where ρ is chosen from Table 8.6 in Qiu for a target ARL_0

- A companion NRS EWMA chart (Li et al., 2010) replaces ψ_n by the centred Wilcoxon rank-sum $W_n - \mu_{W_n}$ for cases with a reference sample

[4] Graham, M. A., Chakraborti, S., and Human, S. W. (2011). A nonparametric exponentially weighted moving average signed-rank chart for monitoring location. *Comput. Statist. Data Anal.*, 55, 2490–2503.

Nonparametric change-point detection (NCPD) chart

- Hawkins and Deng[5] proposed a distribution-free chart based on the Mann-Whitney statistic applied sequentially
- At time n, for each candidate change-point $1 \leq j \leq n-1$, compute the standardised Mann-Whitney statistic

$$T_{jn} = \frac{U_{jn}}{\sqrt{j(n-j)(n+1)/3}}$$

  where $U_{jn} = \sum_{i=1}^{j} \sum_{i'=j+1}^{n} \text{sign}(X_i - X_{i'})$
- The chart signals when $T_{\max,n} = \max_{1 \leq j \leq n-1} |T_{jn}| > h_n$
- When a signal is given at time n, the estimated change-point is $\hat{r} = \arg\max_j |T_{jn}|$, so we get both detection and **location estimation** simultaneously
- Requires $n_0 \geq 14$ initial IC observations; control limits h_n are tabulated (Table 8.8 in Qiu)

[5] Hawkins, D. M. and Deng, Q. (2010). A nonparametric change-point control chart. *J. Qual. Technol.*, 42, 165–173.

Nonparametric Shewhart chart for scale (Mood statistic)

- All rank-based charts seen so far target **location** shifts. To monitor **scale** (variance) without normality assumptions, we can use the Mood statistic[6]
- Given a reference sample of size M and a Phase II batch of size m, pool all $N = M + m$ observations and rank them. The Mood statistic for the Phase II batch is

$$\mathcal{M}_n = \sum_{j=1}^{m} \left(R_{nj} - \frac{N+1}{2} \right)^2$$

  where R_{nj} is the rank of X_{nj} in the combined sample
- Under IC, $E[\mathcal{M}_n] = m(N^2 - 1)/12$. When the variance increases, extreme ranks become more frequent and $\mathcal{M}_n$ grows
- Signal an increase in variance when $\mathcal{M}_n > U_{\mathcal{M}}$, where $U_{\mathcal{M}}$ is determined by simulation for a target ARL_0
- This is a **Shewhart-type** chart, suitable for large variance shifts. For small persistent scale shifts, a memory-type chart is preferred

[6] Murakami, H. and Matsuki, T. (2010). A nonparametric control chart based on the Mood statistic for dispersion. *Int. J. Adv. Manuf. Technol.*, 49, 757–763.

EWMA-Lepage chart for joint location and scale monitoring

- Mukherjee and Chakraborti[7] combined Wilcoxon rank-sum and Ansari-Bradley statistics into a **Lepage-type**[8] chart
- Pool the reference sample and the Phase II batch; if R_{nj} is the rank of X_{nj}, then $W_n = \sum_{j=1}^{m} R_{nj}$ monitors location and $A_n = \sum_{j=1}^{m} a(R_{nj})$, with $a(r) = \min\{r, M + m + 1 - r\}$, monitors scale
- The Lepage statistic is

$$L_n = \left(\frac{W_n - \mu_W}{\sigma_W}\right)^2 + \left(\frac{A_n - \mu_A}{\sigma_A}\right)^2$$

- The EWMA-Lepage chart uses

$$E_n = \lambda L_n + (1 - \lambda) E_{n-1}, \qquad E_0 = 0,$$

  and signals when $E_n > h_L$
- Under IC, $E[L_n] \approx 2$; location, scale, or joint shifts inflate L_n, so one chart monitors both effects simultaneously

[7] Mukherjee, A. and Chakraborti, S. (2012). A distribution-free control chart for the joint monitoring of location and scale. *Qual. Reliab. Eng. Int.*, 28, 335–352.

[8] Lepage, Y. (1971). A combination of Wilcoxon's and Ansari-Bradley's statistics. *Biometrika*, 58, 213–217.

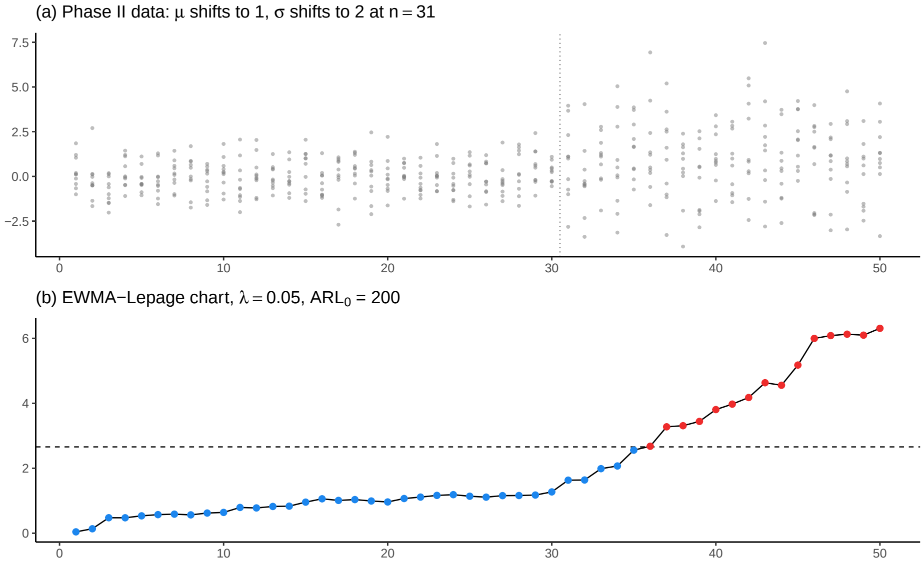
EWMA-Lepage example   Setup and data generation

- A reference sample of $M = 100$ observations is drawn from $N(0, 1)$
- 50 Phase II batches of size $m = 10$ are monitored. The first 30 batches are IC; from batch 31 onward, a **pure variance shift** is introduced (no change in mean)

$$X_{nj} \sim \begin{cases} N(0,\ 1) & n = 1, \ldots, 30 \\ N(1,\ 4) & n = 31, \ldots, 50 \end{cases}$$

- The Lepage statistic combines the standardised Wilcoxon rank-sum Z_W^2 (location) and the standardised Ansari-Bradley statistic Z_A^2 (scale) at each batch
- The EWMA recursion with $\lambda = 0.05$ smooths the Lepage statistics over time; the control limit h_L is calibrated by simulation for $\text{ARL}_0 = 200$
- This example demonstrates that the chart can detect a **scale-only** shift, even though no mean shift is present

EWMA-Lepage chart detecting a pure variance shift



Multivariate nonparametric charts

- The normality problem is even worse in the multivariate case, since **joint** normality requires all marginals and all subsets to be normal
- Two main approaches have been proposed
  - **Longitudinal ranking**: extend the sign and signed-rank statistics component-wise, then combine into a quadratic form[9]
  - **Cross-component ranking**: use the antirank of the observation vector to detect shifts in individual components[10]
- In both cases, the goal is to replace the raw multivariate observations by rank-based information whose IC behaviour does not depend on multivariate normality
- Compared with the univariate case, multivariate nonparametric charts are more difficult to calibrate and interpret, but they provide a useful alternative when the normality assumption is questionable

[9] Boone, J. M. and Chakraborti, S. (2012). Two simple Shewhart-type multivariate nonparametric control charts. *Appl. Stoch. Models Bus. Ind.*, 28, 130–140.

[10] Qiu, P. and Hawkins, D. M. (2001). A rank based multivariate CUSUM procedure. *Technometrics*, 43, 120–132.