Financial Engineering with Python

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Contents

Preface

The title Financial Engineering with Python was deliberately chosen to convey the book's central aim: to present techniques from financial mathematics that are not only rigorous in theory but also readily applicable in practice.
Below we summarize the key findings of this book.

- By separating predictive modeling from portfolio construction, we can design portfolios that achieve the desired payoff across specific market scenarios. This is accomplished by strategically combining existing call and put options, providing a natural extension of Markowitz's mean-variance framework.

  Decoupling the prediction stage allows the development of more advanced forecasting techniques beyond simple means and variances. It is clear that future price behavior depends on past, recent and forthcoming events rather than solely on historical numerical data. Our goal, therefore, is to use a given prediction as the basis for constructing the optimal portfolio tailored to that scenario.

- In the context of option pricing, we introduce a model-free concept of fair value, one that does not rely on a volatility parameter, and demonstrate that it is the unique fair price that remains arbitrage-free with respect to the currently listed call and put options. Clearly, for both parties to reach an agreement, any effective method for determining a fair price must be model-free. By contrast, the Black-Scholes price,
  - depends on an assumed volatility (and is therefore `non-unique`)
  - because it does not incorporate the prevailing option quotes, need `not be arbitrage-free` relative to them.
  - is `not fair` because nobody can construct the proposed replicating portfolio.

- Finally, we discuss path-dependent options and present a pricing and hedging method based on a systematic sell high, buy low trading strategy.

This book correspond to two semester-long courses. In the first semester, the following can be taught: Chapters 1-7 and 9.3 and 9.4. The rest in the second semester after dedicating enough time to probability theory and stochastic analysis.
You can download some Python codes related to the content of these notes.

- Python1
- Python2

In these compressed files, you will also find two PDF files containing Python and Latex lessons entirely prepared by AI. The latex word processor appears particularly useful for communicating with AI when dealing with complex mathematical problems.

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Chapter 1

Basic Definitions

The prices of shares and the contracts tied to them are governed by supply and demand. Just as crucial, however, is the notion of `arbitrage`*—the opportunity to earn a risk-free profit.*

1.1 Shares

`Definition` **1** *A share is the ownership of one unit of the share capital of a company.*

The price of a share is determined after daily trading. Shareholders wishing to sell make a corresponding declaration to the stock exchange (possibly offering a lower acceptable sale price) and interested buyers purchase. If interest in buying shares exceeds the number offered for sale, then obviously the price rises, otherwise it falls. The last sale price on a given day is the price of the share on that day.
It is obvious that no one can predict with certainty the price of the share the next day, but one can guess that perhaps tomorrow there will be high interest in buying because the company is about to expand its activities, etc. Nevertheless, no one can predict exactly how much it will rise, if that indeed happens.

`Definition` **2 (Market Makers)** *A market maker or liquidity provider is a company or an individual that quotes both a buy and a sell price in a tradable asset held in inventory, hoping to make a profit on the difference, which is called the bid-ask spread or turn.*

`Definition` **3 (Bid-Ask Spread)** *The bid-ask spread (also bid-offer or bid/ask and buy/sell in the case of a market maker) is the difference between the prices quoted (either by a single market maker or in a limit order book) for an immediate sale (ask) and an immediate purchase (bid) for stocks, futures contracts, options, or currency pairs in some auction scenario. The size of the bid-ask spread in a security is one measure of the liquidity of the market and of the size of the transaction cost. If the spread is 0 then it is a frictionless asset.*

Dynamics of the Bid–Ask Spread The *bid–ask spread* of a stock is not constant; it fluctuates significantly over time due to market conditions and trading activity. Several key factors explain its variation:

- **Liquidity:** Highly liquid stocks, such as large-cap companies with high daily trading volume, usually have very tight spreads (only a few cents). In contrast, illiquid stocks often exhibit wider spreads because fewer buy and sell orders are available.

- **Time of Day:** The spread is typically wider at the opening and closing of the market session, when order books are thinner and uncertainty is higher. During midday, as liquidity improves, the spread usually narrows.

- **Uncertainty:** When volatility increases—for instance, during earnings announcements or unexpected news—market makers widen spreads to protect themselves from adverse selection and higher risk.

- **Market Conditions:** In times of market stress or crisis (e.g., financial crashes or sudden macroeconomic shocks), spreads tend to widen even for highly liquid stocks. Conversely, in calm and stable markets, spreads remain narrow.

In summary, the bid–ask spread is a dynamic measure that reflects both the *liquidity* and the *risk perception* of the market at any given moment. It is therefore an important cost consideration for traders and investors, especially when trading less liquid securities or during periods of heightened uncertainty.

Order Book and Liquidity Example Consider the following simplified order book:

Ask (Sell Orders)	Bid (Buy Orders)
\$101.00 × 200	\$99.00 × 150
\$100.50 × 300	\$98.50 × 250
\$100.20 × 100	\$98.00 × 300

Suppose a trader wants to *buy 500 shares at market price.*

- The first 100 shares are bought at \$100.20.

- The next 300 shares are bought at \$100.50.

- Only 100 shares remain to be filled; these are taken at \$101.00.

Thus, although the *best ask* is \$100.20, the trader cannot buy all 500 shares at that price, because only 100 shares were available there. The *average purchase price* will be higher, reflecting the limited liquidity of the market.

This illustrates why liquidity matters: it determines not only whether you can trade immediately, but also the price impact of large trades.

Lab 4 **(Downloading Prices)** *You can use the Python code for Colab* `DownLoadPastData.ipynb` *with which you can download past numerical data from Yahoo Finance for any share and for any time period of your choice. If you are not sure how to use this code, you can upload it to an AI and ask anything you wish.*

Of course, one can find other financial products besides shares. We will study the so-called derivatives (options) which are (written) on a share.

1.2 Options

Options have a long history: as early as the 6th century BCE, Thales of Miletus is described by Aristotle as using "options" on olive presses to lock in future use; by the 17th century, rights on shares were traded in the Dutch Republic during the tulip era, with Joseph de la Vega documenting them in his 1688 *Confusion of Confusions*. In the 18th–19th centuries they flourished in London and Amsterdam, mostly over the counter, while in 19th-century United States they were handled by curbstone brokers. A pivotal scholarly milestone came in 1900 when Louis Bachelier's doctoral thesis, *Théorie de la spéculation*, introduced a stochastic model of prices (arithmetic Brownian motion) and an early option-valuation formula—foreshadowing modern theory. The watershed in practice arrived in 1973 with the launch of the Chicago Board Options Exchange, the creation of the Options Clearing Corporation, and the publication of the Black–Scholes–Merton model, enabling standardization and mass pricing. Innovations followed—index options (S&P 100, 1983), options on currencies and interest rates, VIX options (2006), the spread of electronic trading, and since the 2010s, weekly/mini contracts and a surge in retail participation—cementing options as core tools for hedging, pricing, and speculation.

`Definition` **5** *A call or put option is a contract that grants the holder the right, but not the obligation, to buy or sell a specific quantity of an underlying asset at a predetermined strike price on or before a predetermined date, depending on the type of option.*

We will now describe these two options in more detail. Let us now suppose a share with current price S_0, an amount K which is called the strike price, and a future point in time (maturity) T, for example 3 months. The payoffs for these options are

$$\begin{aligned} (S_T - K)^+ &= \max(S_T - K, 0) \quad \text{call option} \\ (K - S_T)^+ &= \max(K - S_T, 0) \quad \text{put option} \end{aligned}$$

`Remark` **6 (How payoffs arise)** *In practice, the writer of the call option is obliged to deliver to the buyer one share at the price K, if $S_T > K$, while in the case of a put option, the writer is obliged to buy one share at the price K, if $S_T < K$. In the first case, the buyer can sell the share at the current price and therefore earn the amount $(S_T - K)^+$. In the second case, the buyer can buy one share at the price $S_T < K$ and sell it at the price K to the option writer, thus earning the amount $(K - S_T)^+$. Note that in the above case, the option writer has the obligation while the buyer has the choice to make the corresponding move.*

For example, in the case of buying a call option with strike price $K = 220$ for $Y = 30$, then the profit for the holder of the option at time T will be equal to

$$(S_T - K)^+ - Y = (S_T - 220)^+ - 30$$

The price S_T is unknown and may be a number greater than or equal to zero. The profit function will be

$$\Pi(x) = (x - 220)^+ - 30.$$

The graph of this function can be easily represented with the appropriate software. It then becomes obvious that it can also take negative values, which means that there is a possibility of loss.

When buying an option, one may hold it until its maturity or sell it before maturity. The selling price is determined according to the law of supply and demand. Therefore, a profit may also arise if we sell it earlier than its expiration date, thus receiving the corresponding difference.

1.3 Formation of the values of call put options

The values of shares and their corresponding contracts arise according to the law of supply and demand. However, there is a very useful concept that also plays an important role, and that is the concept of `arbitrage`, i.e., the opportunity for risk-free profit. This concept has a significant contribution to price formation because when there is an opportunity for risk-free profit in the market, investors rush to exploit it, with the result that prices change in such a way that it disappears. To take advantage of a risk-free profit opportunity, you must buy something cheap and sell something expensive.

Symbols and assumption

Let:

- S: current price of the underlying asset (the underlying).
- K: strike price.
- r: continuously compounded risk-free interest rate.
- T: time to maturity.
- C: price of a European call.
- P: price of a European put.

We assume that the underlying asset does not pay dividends and there are no transaction costs. The inequalities arise from the absence of arbitrage.

1.3.1 Possible values for the European call and put (no dividends)

Call: Lower bound

$$C \geq \max\big(0,\, S - Ke^{-rT}\big)\,.$$

Arbitrage argument: If $C < S - Ke^{-rT}$, form the portfolio:

1. Buy one call (cost C).
2. Invest Ke^{-rT} at the risk-free rate (so you have K at T).

At maturity:

- If $S_T > K$: Exercise the call, receive $S_T - K$, and the deposit yields K; total S_T.
- If $S_T \leq K$: The call expires worthless, the deposit yields $K \geq S_T$.

Thus the portfolio payoff is at least S_T in all states. Since holding the asset costs S and pays S_T, no-arbitrage implies

$$C + Ke^{-rT} \geq S \quad \Rightarrow \quad C \geq S - Ke^{-rT}.$$

Trivially $C \geq 0$, hence $C \geq \max(0, S - Ke^{-rT})$.

Call: Upper bound

$$C \leq S.$$

A call cannot be worth more than immediate ownership of the asset. If $C > S$, short the call (receive C) and buy the asset (pay S) for a riskless profit, contradicting no-arbitrage.

Put: Lower bound

$$P \geq \max\bigl(0,\, Ke^{-rT} - S\bigr)\,.$$

Arbitrage argument (protective put): Consider one put plus one unit of the asset. At maturity,

$$S_T + (K - S_T)^+ \;\geq\; K \quad \text{for all } S_T.$$

Therefore the present value satisfies $S + P \geq Ke^{-rT}$, i.e.

$$P \geq Ke^{-rT} - S.$$

Since $P \geq 0$, we obtain $P \geq \max(0, Ke^{-rT} - S)$.

Put: Upper bound

$$P \leq Ke^{-rT}.$$

The put's payoff is at most K at T, so today's price cannot exceed its discounted maximum.

1.3.2 Put-Call Parity

$$C - P = S - Ke^{-rT}.$$

Arbitrage argument: Compare the portfolios

$$\text{A: 1 call} + Ke^{-rT} \quad \text{and} \quad \text{B: 1 put} + \text{1 share.}$$

At T:

- If $S_T > K$: A pays $(S_T - K) + K = S_T$; B pays S_T.
- If $S_T \leq K$: A pays K; B exercises the put, sells the share for K, thus pays K.

The payoffs coincide in all states, hence $C + Ke^{-rT} = P + S$, i.e. $C - P = S - Ke^{-rT}$.

In summary

European call:	$\max(0,\, S - Ke^{-rT}) \leq C \leq S,$
European put:	$\max(0,\, Ke^{-rT} - S) \leq P \leq Ke^{-rT},$
Put-call parity:	$C - P = S - Ke^{-rT}.$

If any of these inequalities were violated, there would be an arbitrage possibility with suitable combinations of buying/selling the underlying, the options, and investing at the risk-free rate.

1.3.3 Arbitrage-Free Interval

Meaning of the bounds. In practice, these bounds tell us where a rational price should fall. In a market with only a risk-free bank account and the underlying asset, a trader considering a European call with strike K and maturity T should expect the price to lie between the previously derived lower and upper bounds. Under no-arbitrage and rational behavior, supply and demand drive the price into this interval; any quote outside it would enable arbitrage. In this one-stock/one-bank-account setting, the interval exactly coincides with the set of arbitrage-free prices for the call.

Why the arbitrage-free interval shrinks when more options trade. If the market also lists options at nearby strikes $K_1 < K_2 < \cdots < K_n$ (and possibly the corresponding puts), cross-strike no-arbitrage constraints-such as monotonicity, convexity in K, and put-call parity-apply simultaneously. These additional constraints tighten the feasible set for $C(K)$, so the arbitrage-free interval around the target strike becomes narrower.

Computing the interval. For any option written on the same stock, its admissible price lies within an arbitrage-free interval that is computed using all existing option quotes on that stock. Determining this interval reduces to solving two linear programs; see section 9.4.

Market-consistent valuation (Option Pricing) Any derivative pricing method that aims to produce arbitrage-free values must, at a minimum, incorporate the observed market prices of the available call and put options (on the same underlying and maturity). This ensures that the proposed prices for a new contract lie within the arbitrage-free interval implied by the traded options; otherwise, the method risks generating inconsistent quotes that admit arbitrage.
Initial quotes for a derivative may lie outside the arbitrage-free interval. Arbitrageurs correct such mispricings by constructing portfolios that lock in riskless profits. Sooner or later, prices typically converge into this interval.
The task of derivative valuation is not to predict the contract's future value. Prediction would require additional considerations (e.g., behavioral finance and other market effects). The objective is to compute a fair and arbitrage-free price for the contract and, equally important, to provide investors with hedging methods, i.e., ways to offset risk by buying and/or selling the contract against other traded instruments.
Therefore, if you want to design a mathematical based valuation method, i.e., to compute a fair and arbitrage-free price, do not force it to match the prices that will be observed in the market. Investors neither know nor think according to your method (or model)!
We will study the option pricing problem at the chapter 9.

`Remark` **7 (Speculating using options)** *In many cases the prices of the call and put options are much cheaper than the asset price itself. Therefore, if the investor predicts that the value of the share will increase significantly, it may be more profitable to buy (many) call options instead of shares.*

`Lab` **8 (Downloading Call Put Options prices from Yahoo Finance)** *One can use the Python code `Download-Call-Put.ipynb` to download the prices of these contracts for any share you are interested in and with any available expiration date. Again, AI can help you with exactly how to use it.*

1.4 Over the Counter (OTC) Market

Of course, there are also other contracts that differ mainly in the payoff function. Also, the way we exercise them is different: if the option is to be exercised only at time T (maturity), then it

is called European type. If the option can be exercised at any time up to its maturity, then it is called American type, while if it can be exercised only at specific times it is called Bermudan.
Outside the stock exchange market, there is also the (over-the-counter (OTC) market) where one can buy more complex options compared to the standard call and put options. For example, an investor may not be sure if they want a call or a put, so they can buy an option with payoff

$$f(x) = \max\{(x - K_1)^+, (K_2 - x)^+\}$$

that is, they will receive the greater of the two amounts above. The price of such an option will be determined according to the law of supply and demand and of course the notion of arbitrage will play a significant role.
So far, payoffs depend on the final price of the share. There are also other types of options whose payoff depends on all (or some) of the share prices in the interval $[0, T]$. For example, the payoff function may have the form

$$f(S_{t_1}, \cdots, S_{t_n}) = \left(\frac{\sum_{i=1}^{n} S_{t_i}}{n} - K \right)^+$$

where $t_i \in [0, T]$. This type of options is called (path dependent).
Contracts with payoffs that depend on two or more shares are called (multi-asset options). One example is the option with the following payoff function:

$$f(x, y) = \max\{x, y, K\}$$

where x refers to the price of share S_1 at time T and y refers to the price of share S_2 at time T.

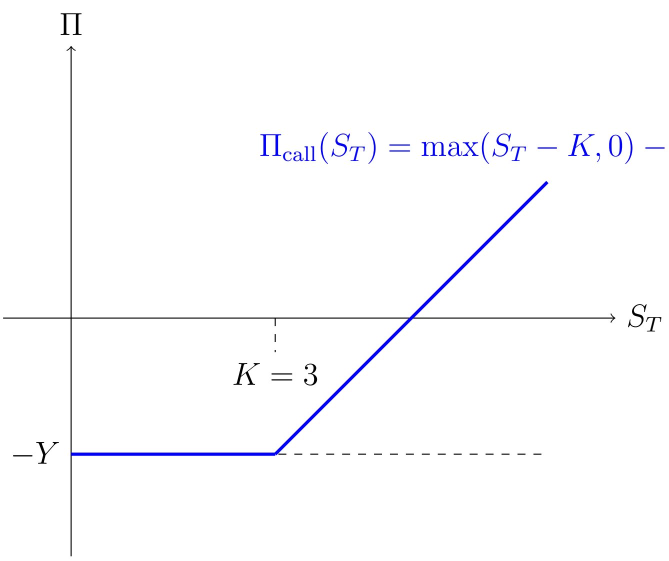


Figure 1.1: **Payoff of a Call Option** with strike price $K = 3$ and premium $Y = 2$.

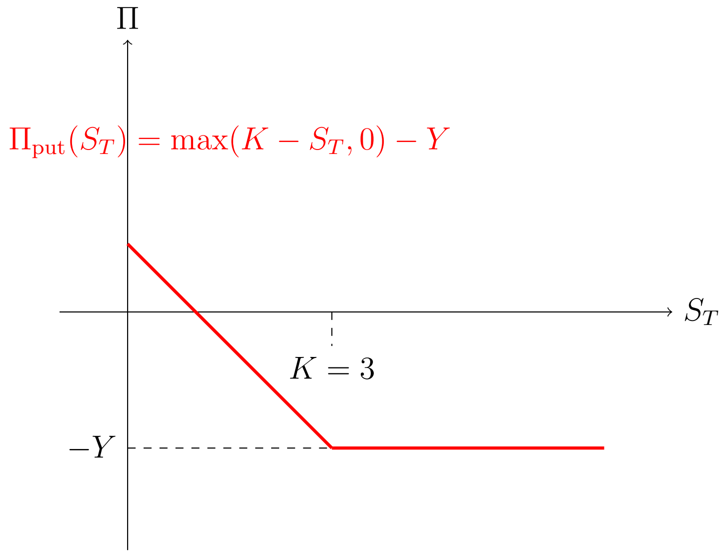


Figure 1.2: **Payoff of a Put Option** with strike price $K = 3$ and premium $Y = 2$.

1.5 Forwards and Futures

Definition 9 **(Forward Contract)** *A forward contract is a non-standardized agreement between two parties, in which the buyer agrees to purchase and the seller agrees to sell an underlying asset at a predetermined price K at a future time T. The net payoff at time T for the long position is*

$$P_{forward} = S_T - K,$$

where S_T is the price of the underlying asset at T.

Definition 10 **(Futures Contract)** *A futures contract is a standardized agreement in the exchange market, obligating both parties to buy/sell the underlying asset at a future date. In contrast to the forward, futures are subject to marking to market: profits and losses are valued and settled daily via margin accounts, which affects financing and may lead to deviations when interest rates are stochastic or correlated with the price of the underlying.*

Remark 11 **(Differences)** *The main differences between forward and futures are:*

- *forwards are non-standardized and bear counterparty risk, whereas futures are standardized with central clearing,*
- *futures have daily settlement through marking to market and require margin, resulting in dynamically adjusted exposure, and*
- *pricing may deviate when financing parameters (such as interest rates) are correlated with the underlying. These contracts are used for hedging and speculation.*

1.6 What is a Long Position

A long position means that the investor has a "buy" exposure to the underlying asset, i.e., benefits if its price rises. Examples:

- **Stock:** An investor who buys a stock has a long position in it. If S_0 is the purchase price and S_T the price at time T, then the profit is

$$\Pi_{long\ stock} = S_T - S_0.$$

- **Forward/Futures:** The long position corresponds to an agreement for the investor to buy the underlying at a future date T at a predetermined price K. The payoff at T is

$$\Pi_{long\ forward} = S_T - K.$$

- **Call option:** The long call gives the right (but not the obligation) to the holder to buy the underlying at the strike price K. The payoff is

$$\Pi_{long\ call} = \max(S_T - K, 0).$$

The long position has *positive* exposure to the rise of the underlying's price and *negative* exposure to its fall.

1.7 What is a Short Position

A short position means that the investor benefits if the price falls. Examples:

- **Stock:** An investor with a short position in a stock has payoff

$$\Pi_{short\ stock} = S_0 - S_T,$$

  i.e., gains when $S_T < S_0$.

- **Forward/Futures:** The short position agrees to sell the underlying at a future date T at price K. The payoff at T is

$$\Pi_{short\ forward} = K - S_T.$$

- **Put option:** The long put gives the right to sell at the strike price K, so the short put has payoff

$$\Pi_{short\ put} = -\max(K - S_T, 0).$$

The short position has *positive* exposure to the fall of the underlying's price and *negative* to its rise.
The long and short positions are terminology used in the stock market but have no mathematical value and therefore we will not deal with them again.

Comparison Long vs Short

Instrument	**Position**	**Payoff at T**	**Directional exposure**
Stock	Long	$S_T - S_0$	Benefits from rise
Stock	Short	$S_0 - S_T$	Benefits from fall
Forward/Futures	Long	$S_T - K$	Benefits from rise
Forward/Futures	Short	$K - S_T$	Benefits from fall
Call option	Long	$\max(S_T - K, 0)$	Position in rise (non-linear)
Call option	Short	$-\max(S_T - K, 0)$	Exposed to rise
Put option	Long	$\max(K - S_T, 0)$	Benefits from fall
Put option	Short	$-\max(K - S_T, 0)$	Exposed to fall

1.8 Exercises

True or False

1. **Statement:** A call option gives its holder the right, but not the obligation, to sell a stock at a specified strike price.
   True / False: ______ **Justification:** ______________________________

2. **Statement:** The long position in a forward contract gains when the price of the underlying asset increases.
   True / False: ______ **Justification:** ______________________________

3. **Statement:** The payoff of a put option is zero when the stock price is greater than the strike price.
   True / False: ______ **Justification:** ______________________________

4. **Statement:** Futures contracts are non-standardized and are not subject to daily settlement (marking to market).
   True / False: ______ **Justification:** ______________________________

5. **Statement:** The short position in a stock gains when the stock price decreases.
   True / False: ______ **Justification:** ______________________________

Multiple Choice

1. The payoff of a call option with strike price K and stock price S_T is:

   (A) $\max(S_T - K, 0)$

   (B) $\max(K - S_T, 0)$

   (C) $S_T - K$

   (D) $K - S_T$

   Answer: ______

2. Which of the following is not a characteristic of futures contracts?

   (A) Daily settlement (marking to market)

   (B) Central clearing

   (C) Non-standardized contracts

   (D) Execution obligation

   Answer: ______

3. The short position in a forward contract has payoff:

   (A) $S_T - K$

   (B) $K - S_T$

(C) $\max(S_T - K, 0)$

(D) $\max(K - S_T, 0)$

Answer: ________

4. Which of the following is a derivative product?

(A) Stock

(B) Option

(C) Gold

(D) Bank deposit

Answer: ________

5. If an investor buys a put option, then they expect:

(A) The stock price to rise

(B) The stock price to fall

(C) The stock price to remain stable

(D) To secure a stable return

Answer: ________

Matching

Match each description with the correct choice. Write the letter next to the number.

1. The right to buy a stock	A. Put Option
2. The obligation to sell a stock	B. Call Option
3. The obligation to buy a stock	C. Short Forward
4. The right to sell a stock	D. Long Forward

Answers: 1._____ 2._____ 3._____ 4._____

Fill in the Blank

1. A _______________ is a contract that gives its holder the right, but not the obligation, to buy or sell an asset at a predetermined price.

2. The _______________ position in a stock gains when the stock price decreases.

3. _______________ contracts are standardized and involve daily settlement.

4. The payoff of a put option is _______________, when the stock price is greater than the strike price.

5. The strike price in an option is called _______________.

Theoretical and Computational Exercises

1. **Payoff Calculation:** An investor buys a call option with strike price $K = 100$. If the stock price at maturity is $S_T = 120$, calculate the payoff of the option.

2. **Profit from Short Position:** An investor sells a stock at price $S_0 = 50$ and buys the same stock at price $S_T = 40$. Calculate the investor's profit.

3. **Forward Contract:** An investor enters into a forward contract to buy a stock at price $K = 80$. If the stock price at maturity is $S_T = 90$, calculate the payoff of the long position.

4. **Put Option Payoff:** An investor buys a put option with strike price $K = 60$. If the stock price at maturity is $S_T = 55$, calculate the payoff of the option.

5. **Hedging with Futures:** An investor wants to protect against a fall in the price of their stock. Propose a hedging strategy using futures contracts.

Python-Based Exercises

1. **Payoff Graph:** Use Python to create the payoff graph of a call option with strike price $K = 100$ for stock prices $S_T \in [80, 120]$.

2. **Forward Payoff:** Use Python to calculate and plot the payoff of a long position in a forward contract with strike price $K = 50$ for stock prices $S_T \in [40, 60]$.

3. **Put Option Hedging:** Use Python to simulate a hedging strategy that uses a put option to protect the price of a stock.

Chapter 2

Portfolios

Call options multiply profit when the stock price rises sufficiently above the strike; put options provide insurance by paying when the price falls below it.

2.1 Profit Function or Payoff Function

Portfolios containing stocks and call/put options are of particular mathematical interest. We will study the behavior of portfolios containing a stock and the corresponding call and put options. Later, we will generalize our conclusions to portfolios containing d stocks and the corresponding contracts.

`Example` **12** *Suppose that the current stock price is* 406 *Euros and that there exists in the market a call option with strike price* $K = 400$ *and* $C(400) = 41$ *Euros. Consider the following two strategies: in the first strategy, you buy one stock at the price of* 406*, and in the second strategy, you buy* $\frac{406}{41}$ *call options.*
The profit functions at time T *(the option expiration date) are as follows:*

- $\Pi_1(x) = x - 406$
- $\Pi_2(x) = \frac{406}{41}(x - 400)^+ - 406$

We can easily see that if the stock price becomes greater than 445 *at time* T*, then the second strategy will yield greater profit, while otherwise the first strategy will be more profitable. Of course, one can combine the above strategies, ending up with an infinite number of combinations! Taking into account all existing call and put options, the investor can decide exactly how to invest their money. For this purpose, they must mathematically study the profit functions and their behavior.*
Note that usually we can only buy/sell an integer number of contracts. At first, we do not take this into account in order to extract basic conclusions about the impact of these contracts on portfolios.

`Lab` **13 (Designing the Profit Function of a Portfolio)** *With the* ***`Profit-Function.ipynb`*** *code, you can experiment by creating portfolios with one stock and 5 call and put options. You must provide the stock value, the values of the contracts with strike prices* $K_1, \ldots, K_5$*, and the number you will buy/sell of each. Then you plot the profit function, which gives you the profit/loss at each possible stock price. By changing the parameters, we see that we can create portfolios that are profitable in any scenario we want.*

`Remark` **14 (Properties of Call and Put Options)** *After similar experiments with real data, you will arrive at the following conclusion: call options multiply the profit in the case where the stock price takes a sufficiently high value, but in the case where the stock price falls below the strike price, the money is lost. Similarly, put options act as a hedging factor, in the sense that if the stock price falls below the strike price, we receive the corresponding amount.*

`Remark` **15 (Scenario Selection)** *Suppose you have decided to invest the amount V in a company's stocks. As we saw in the previous example, you can invest it entirely in stock purchases, but you can also follow an infinite number of different strategies. Each strategy is advantageous in a different scenario. Therefore, in order to choose the one that is right for you, you must essentially choose which scenario you want to bet your money on.* □

`Question` **16** *If you experiment enough, questions will arise:*

- *Ultimately, which portfolio should I construct? In other words, in which scenario do I want to make a profit? The answer to this question will not be given with mathematical precision by making some calculations because it is a prediction! An experienced investor makes predictions by evaluating various factors and events that will affect the stock price. Therefore, we will not deal with prediction techniques, but with the problem of constructing a portfolio given a prediction.*
- *Another question is: how many stocks/contracts can I buy/sell at a given price at the same time? Even if you have infinite money, you cannot buy as many shares as you want at a given price because there may not be enough sellers offering all the desired shares at the same price. Some will have placed sell orders for N_1 shares at price Y_1, some N_2 shares at price Y_2, etc. So you will buy the first N_1 at price Y_1, the next ones at price Y_2, etc. This problem is important because you must know in advance how many stocks/contracts you can buy/sell at once at a given price in order to make your calculations. This indicator is called liquidity. We will not deal with this problem either! Such detailed market data, including full order book depth and bid–ask sizes, is typically available only through professional trading platforms or paid data providers. Free services usually provide only top-of-book quotes, while access to Level II market depth requires a subscription.*

2.2 Exercises

True or False

1. **Statement:** A portfolio containing only stocks is always less profitable than a portfolio containing stocks and options.
   True / False: _______ **Justification:** ____

2. **Statement:** The profit function of a portfolio containing only call options is always non-negative.
   True / False: _______ **Justification:** ____

3. **Statement:** Put options can be used to create a portfolio that protects against declines in stock price.
   True / False: _______ **Justification:** ____

4. **Statement:** The profit function of a portfolio containing stocks and call options is linear.
   True / False: _______ **Justification:** ____

5. **Statement:** A portfolio containing only put options is always profitable when the stock price falls below the strike price.
   True / False: _______ **Justification:** ____

Multiple Choice

1. Which of the following profit functions corresponds to a portfolio containing one stock and one call option?

   (A) $\Pi(x) = x - S_0 + \max(x - K, 0) - C$

   (B) $\Pi(x) = x - S_0 - C$

   (C) $\Pi(x) = \max(x - K, 0) - C$

   (D) $\Pi(x) = x - S_0 + C$

   Answer: _______

2. What is the optimal strategy for an investor predicting a significant increase in a stock's price?

   (A) Buy stocks

   (B) Buy call options

   (C) Buy put options

   (D) Sell stocks

   Answer: _______

3. Which of the following is an advantage of options compared to stocks?

   (A) Higher liquidity

   (B) Higher risk

   (C) Greater potential for profit amplification

   (D) Lower entry cost

   Answer: _______

4. What is the profit function of a portfolio containing only put options?

   (A) $\Pi(x) = \max(K - x, 0) - P$

   (B) $\Pi(x) = x - K$

   (C) $\Pi(x) = \max(x - K, 0)$

   (D) $\Pi(x) = K - x$

   Answer: _______

5. Which of the following strategies offers the best protection against a decline in stock price?

   (A) Buy stocks

(B) Buy call options

(C) Buy put options

(D) Sell stocks

Answer: _______

Matching

Match each description with the correct choice:

1. Strategy that amplifies profit when the stock price increases	A. Put Option
2. Strategy that protects against a stock price drop	B. Call Option
3. Strategy that offers linear profit	C. Long Forward
4. Strategy used for hedging	D. Short Forward

Answers: 1._____ 2._____ 3._____ 4._____

Fill in the Blank

1. A ______________ is a financial derivative that gives its holder the right to buy a stock at a predetermined price.
2. The profit function of a portfolio containing only stocks is ______________.
3. The ______________ of a call option is zero when the stock price is below the strike price.
4. The strike price of an option is called ______________.
5. A portfolio containing stocks and put options is used for ______________.

Theoretical and Computational Exercises

1. **Profit Calculation:** An investor buys one stock at price $S_0 = 400$ and a call option with strike price $K = 400$ and cost $C = 40$. If the stock price at expiration is $S_T = 450$, calculate the portfolio profit.
2. **Profit Function:** Give the profit function of a portfolio containing: - 2 stocks, - 3 call options with strike price $K = 400$, - 1 put option with strike price $K = 380$.
3. **Optimal Strategy:** An investor predicts that a stock's price will fall below $K = 380$. Propose a strategy that will allow the investor to profit from this movement.
4. **Profit Function Simulation:** Using Python, plot the profit function of a portfolio containing: - 1 stock, - 2 call options with $K = 400$, - 1 put option with $K = 380$.
5. **Hedging Portfolio:** An investor holds 100 shares of a company with current price $S_0 = 400$. Propose a hedging strategy using put options to protect the portfolio value.

Chapter 3

Linear Programming Problems

> IN ECONOMICS, OPTIMIZATION IS NOT OPTIONAL – IT IS THE ESSENCE.

In this chapter, we will describe a type of optimization problem called a *Linear Programming (LP)* problem.
Find x_1, x_2, x_3 such that the quantity

$$5x_1 + 4x_2 + 3x_3$$

is maximized subject to the following constraints:

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &\leq 5, \\ 4x_1 + x_2 + 2x_3 &\leq 11, \\ 3x_1 + 4x_2 + 2x_3 &\leq 8, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

The above problem is usually written as:

$$\begin{aligned} \text{max} \quad & 5x_1 + 4x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 + x_3 \leq 5, \\ & 4x_1 + x_2 + 2x_3 \leq 11, \\ & 3x_1 + 4x_2 + 2x_3 \leq 8, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

We define a matrix A and vectors x, b, c as follows. The matrix A is constructed by taking the coefficients of x_j in each constraint:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix}, \quad c = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}.$$

Thus, the inequality $Ax \leq b$ encodes the three constraints:

$$\begin{cases} 2x_1 + 3x_2 + x_3 \leq 5, \\ 4x_1 + x_2 + 2x_3 \leq 11, \\ 3x_1 + 4x_2 + 2x_3 \leq 8, \end{cases}$$

and we add the non-negativity constraint $x \geq 0$.
Hence, the original problem can now be written as:

$$\max\ c^T x \quad \text{such that} \quad Ax \leq b, \quad x \geq 0.$$

Such a problem may have one solution (or even infinitely many solutions) or none at all. Applying the appropriate solving procedure yields that the maximum is attained at $x_1 = 2, x_2 = 0, x_3 = 1$ and equals 13.

3.1 Linear Programming: Basic Theory

`Definition` **17 (Linear program)** *A (finite-dimensional) linear program (LP) is an optimization problem of the form*

$$\max\{\, c^\top x : Ax \leq b,\ x \geq 0 \,\},$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. Variants include minimization, equality or "$\geq$" constraints, and sign–free variables; all are equivalent up to simple transformations.

`Definition` **18 (Feasibility, optimality)** *The* feasible region *is $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$. A point $x^\star \in P$ is* optimal *if $c^\top x^\star \geq c^\top x$ for all $x \in P$. An LP is* infeasible *if $P = \varnothing$ and* unbounded *if for every M there exists $x \in P$ with $c^\top x \geq M$.*

Standard forms and slack variables

A common "standard form" for maximization is

$$\max\ c^\top x \quad \text{s.t.} \quad Ax \leq b,\ x \geq 0.$$

A $\leq$-constraint $a_i^\top x \leq b_i$ can be turned into an equality by adding a *slack* $s_i \geq 0$: $a_i^\top x + s_i = b_i$. For a $\geq$-constraint, multiply by -1. A free variable $u \in \mathbb{R}$ can be written as $u = u^+ - u^-$ with $u^+, u^- \geq 0$.

Geometry and basic feasible solutions

The feasible set P is a convex polyhedron. A *basic feasible solution* (BFS) is a feasible point obtained by activating (setting tight) n linearly independent constraints.

`Theorem` **19 (Extreme-point optimality)** *If an LP in standard form has an optimal solution and its feasible region is nonempty and bounded, then there exists an optimal BFS; equivalently, an optimal* vertex *of P.*

Duality

Given the primal in standard max form

$$\max\ c^\top x \quad \text{s.t.} \quad Ax \leq b,\ x \geq 0, \tag{P}$$

its *dual* is

$$\min\ b^\top y \quad \text{s.t.} \quad A^\top y \geq c,\ y \geq 0. \tag{D}$$

Sign/shape rules (quick reference). For a maximization primal:

Primal row i type	$\Rightarrow$	Dual variable y_i sign
$\leq$		$y_i \geq 0$
$=$		y_i free
$\geq$		$y_i \leq 0$
Primal variable x_j sign	$\Rightarrow$	**Dual column/constraint j**
$x_j \geq 0$		$(A^\top y)_j \geq c_j$
x_j free		$(A^\top y)_j = c_j$
$x_j \leq 0$		$(A^\top y)_j \leq c_j$

Theorem **20 (Weak duality)** *If x is feasible for* (P) *and y is feasible for* (D)*, then*

$$c^\top x \;\leq\; b^\top y.$$

Proof. Since $A^\top y \geq c$ and $x \geq 0$, we have $c^\top x \leq y^\top Ax$. Also $Ax \leq b$ and $y \geq 0$ imply $y^\top Ax \leq y^\top b$. Combine the two.

Theorem **21 (Strong duality)** *If either* (P) *or* (D) *has an optimal solution and the other is feasible, then both have optimal solutions and their optimal values coincide:* $\max(P) = \min(D)$.

Corollary **22 (Complementary slackness)** *Let $x^\star, y^\star$ be feasible for* (P) *and* (D)*. They are both optimal iff*

$$y_i^\star\big(b_i - (Ax^\star)_i\big) = 0 \quad \forall i, \qquad x_j^\star\big((A^\top y^\star)_j - c_j\big) = 0 \quad \forall j.$$

Equivalently, with reduced costs $r = c - A^\top y^\star$, one has $r \leq 0$ and $x_j^\star > 0 \Rightarrow r_j = 0$.

Optimality/KKT conditions for LP

For LPs, the Karush–Kuhn–Tucker conditions reduce to: *primal feasibility* $Ax \leq b,\ x \geq 0$, *dual feasibility* $A^\top y \geq c,\ y \geq 0$, and *complementary slackness* as in Theorem 22. These are necessary and sufficient.

Farkas' Lemma (in a common form)

Exactly one of the following systems has a solution:

$$\text{(i) } Ax = b,\ x \geq 0 \qquad \text{or} \qquad \text{(ii) } A^\top y \geq 0,\ b^\top y < 0.$$

This is a fundamental alternative underpinning infeasibility certificates and duality.

Reduced costs and sensitivity (shadow prices)

Given an optimal dual $y^\star$, the *reduced cost* of variable x_j is

$$r_j = c_j - (A^\top y^\star)_j \leq 0.$$

If $r_j < 0$, increasing x_j from zero would decrease the objective (for a max LP), so $x_j^\star = 0$. If $x_j^\star > 0$, then $r_j = 0$. The dual components $y_i^\star$ are *shadow prices*: for small perturbations Δb, the optimal value changes by approximately $(y^\star)^\top \Delta b$.

Algorithmic notes (one-paragraph overview)

The simplex method moves along vertices (BFSs) of P, improving the objective until no non-positive reduced costs remain (optimality). Degeneracy can cause cycling; anti-cycling rules (e.g., Bland's rule) guarantee termination. Interior-point methods follow a central path inside P and achieve polynomial-time complexity; both approaches return primal/dual solutions and certificates.

Lab **23 (Solving Linear Programming Problems)** *You can use the provided Python code (`Linear-programming-problems.ipynb`) to solve* $\max 5x_1+4x_2+3x_3$ *subject to* $Ax \le b,\ x \ge 0$*:*

Remark *24* *The scipy routine linprog solves problems in* minimization *form:*

$$\min_{x\ge 0} f(x) \quad \textit{subject to} \quad A_{ub}x \le b_{ub}.$$

To solve the problem

$$\max c^T x \quad \textit{subject to} \quad Ax \le b, \quad x \ge 0,$$

we use the identity $\max c^T x = -\min(-c)^T x$ *and pass the vector* $-c$ *to linprog as the objective coefficients. The call is then:*

$$\textit{linprog}(-c,\ A_{ub} = A,\ b_{ub} = b,\ \textit{bounds} = (0,\infty))$$

and the returned solution x^* *is the optimum for the original maximization problem, with* $c^T x^* = -$*(value returned by linprog).*
If we solve the dual in the form

$$\min b^T y \quad \textit{subject to} \quad A^T y \ge c, \quad y \ge 0,$$

we must also transform the inequality to match the form $A_{ub}y \le b_{ub}$*:*

$$A^T y \ge c \quad \Longleftrightarrow \quad -A^T y \le -c,$$

so in linprog we set $A_{ub} = -A^T$ *and* $b_{ub} = -c$*.*

```
import numpy as np from scipy.optimize import linprog

def solve_lp(c, A, b):
    # Solves max c^T x  s.t. A x <= b, x >= 0
    res = linprog(-np.array(c), A_ub=np.array(A), b_ub=np.array(b),
                  bounds=[(0, None)] * len(c), method="highs")
    if not res.success:
        raise RuntimeError("Error: " + res.message)
    x = res.x
    value = float(np.array(c) @ x)
    return x, value

# Example x, val = solve_lp([5,4,3],
                  [[2,3,1],
                   [4,1,2],
                   [3,4,2]],
                  [5,11,8])
print("x =", x) print("max =", val)
```

Listing 3.1: Simple Python Code

3.2 Exercises

1. Recall the example:

   - Write explicitly the matrix A, and vectors b and c for $\max 5x_1 + 4x_2 + 3x_3$ with the constraints:
$$\begin{cases} 2x_1 + 3x_2 + x_3 \le 5, \\ 4x_1 + x_2 + 2x_3 \le 11, \\ 3x_1 + 4x_2 + 2x_3 \le 8, \\ x_1, x_2, x_3 \ge 0. \end{cases}$$
   - Implement the solution using the provided Python routine and verify that $x = (2, 0, 1)$ gives a maximum of 13.

2. Change the objective to max $7x_1 + 2x_2 + 5x_3$ with the same constraints:

   - Construct A, b, c.
   - Find the optimal solution and the maximum.

3. Add an extra constraint $x_1 + x_2 \le 3$ to the original problem:

   - Update matrix A and vector b.
   - Solve the new problem and compare the solution with the previous one.
   - What happens if you remove the constraint $x_3 \ge 0$? Is it still feasible? Explain.
   - What should you change if you require the solution to have integer values?

4. Construct the dual of the original problem:
$$\max\ c^T x \quad \text{such that} \quad Ax \le b, \quad x \ge 0$$
   - Write the corresponding $\min b^T y$ with $A^T y \ge c,\ y \ge 0$.
   - Solve the dual using Python.

5. Sensitivity cases:

   - Slightly change c (e.g. from $[5, 4, 3]$ to $[5.1, 4, 3]$ or $[5, 3.9, 3]$) and repeat. How does the solution change?
   - Change the right-hand side b (e.g. $b = (5, 11, 8) \to (5, 11, 9)$) and discuss how the maximum and active constraints are affected.

3.3 Exercises

True or False

1. **Statement:** Every linear program with a nonempty and bounded feasible region attains its optimum.
   True / False: _______ **Justification:** ____

2. **Statement:** If the primal (in max–$\le$–$x \ge 0$ form) is unbounded, then its dual is infeasible.
   True / False: _______ **Justification:** ____

3. **Statement:** If an LP has two distinct optimal feasible points, then every convex combination of them is also optimal.
   True / False: _______ **Justification:** ____

4. **Statement:** Adding a redundant constraint (one implied by the others) can change the optimal value of an LP.
   True / False: _______ **Justification:** ____

5. **Statement:** In the simplex method, cycling (revisiting the same basis) is impossible.
   True / False: _______ **Justification:** ____

Multiple Choice

1. The dual of $\max\{\, c^\top x : \; Ax \le b, \; x \ge 0 \,\}$ is:

   (A) $\min\{\, b^\top y : \; A^\top y \ge c, \; y \ge 0 \,\}$
   (B) $\min\{\, c^\top y : \; Ay \ge b, \; y \ge 0 \,\}$
   (C) $\max\{\, b^\top y : \; A^\top y \le c, \; y \ge 0 \,\}$
   (D) $\min\{\, b^\top y : \; Ay \le c, \; y \ge 0 \,\}$

   Answer: _______

2. Which statement is a correct form of complementary slackness for primal max and dual min above?

   (A) $x_j\,((A^\top y)_j - c_j) = 0$ for all j
   (B) $y_i\,(b_i - (Ax)_i) = 0$ for all i
   (C) Both (A) and (B)
   (D) Neither (A) nor (B)

   Answer: _______

3. Consider the chapter's example with $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix}$, $b = (5, 11, 8)^\top$, $c = (5, 4, 3)^\top$. Which point is feasible?

   (A) $(x_1, x_2, x_3) = (2, 0, 1)$
   (B) $(1, 2, 2)$
   (C) $(3, 0, 0)$
   (D) $(0, 3, 2)$

   Answer: _______

4. In a maximization LP solved by simplex, an entering variable typically has (with reduced costs $r = c - A^\top y$):

   (A) $r_j < 0$
   (B) $r_j = 0$

(C) $r_j > 0$
(D) No condition on r_j

Answer: _______

5. If the primal is feasible and the dual is infeasible, then:

(A) The primal has multiple optima
(B) The primal is infeasible
(C) The primal is unbounded
(D) No conclusion can be made

Answer: _______

Matching

Match each description with the correct choice:

1. Variable added to convert $\leq$ into equality	A. Shadow price
2. Dual optimal multiplier interpreting marginal value	B. Slack variable
3. Vertex solution obtained by activating constraints	C. Basic feasible solution
4. Phenomenon causing stalling in simplex pivots	D. Degeneracy

Answers: 1._____ 2._____ 3._____ 4._____

Fill in the Blank

1. The feasible region $P = \{x : Ax \leq b,\ x \geq 0\}$ is a convex ______________.
2. For the dual of the standard max form, the constraint reads $A^\top y \geq$ __________.
3. At optimality (for max form), reduced costs satisfy $r = c - A^\top y^\star \leq 0$.
4. Complementary slackness gives $y_i^\star\big(b_i - (Ax^\star)_i\big) = 0$ for all i.
5. For a small RHS perturbation Δb, the change in the optimal value is approximately $(y^*)^\top \Delta b$.

Theoretical and Computational Exercises

1. **Solve and interpret (chapter LP).** Solve

$$\max\ 5x_1 + 4x_2 + 3x_3 \text{ s.t. } Ax \leq b,\ x \geq 0,$$

with A, b as in the chapter. Report an optimal x^*, the optimal value, and which constraints are binding at x^*.

2. **Dual formation and verification.** Form the dual of the LP in (1). Find a feasible y achieving the same objective value and verify strong duality and complementary slackness explicitly.

3. **Geometry and BFS.** Consider

$$\max\ 3x_1 + 2x_2 \quad \text{s.t.} \quad \begin{cases} x_1 + 2x_2 \leq 6, \\ 2x_1 + x_2 \leq 6, \\ x_1, x_2 \geq 0. \end{cases}$$

(i) Sketch the feasible region. (ii) List all vertices (BFS). (iii) Evaluate the objective at each vertex and identify an optimal solution.

4. **Sensitivity (shadow prices).** For the chapter LP, let $b(\varepsilon) = (5 + \varepsilon,\ 11,\ 8)^\top$.
   (i) Using the dual optimal y^*, predict the first-order change of the optimal value as a function of ε.
   (ii) Solve the perturbed primal for $\varepsilon = \pm 0.5$ and compare the actual optimal values with your prediction.

5. **Python implementation.** Using `scipy.optimize.linprog`, implement a function that solves $\max\ c^\top x$ s.t. $Ax \le b,\ x \ge 0$ by minimizing $-c^\top x$.
   Run it on: (a) the chapter LP; (b) the 2-variable LP in Exercise 3; and print x^*, the optimal value, and which constraints are binding in each case.

Chapter 4

Optimal Portfolios

We will not deal with prediction techniques, but with the problem of constructing a portfolio given a prediction.

Let $S_0 = 238$, $C(230) = 65$, $P(200) = 30$, where S_0 is the current stock price, $C(230)$ is the price of the call option with strike 230, and $P(200)$ is the price of the put option with strike 200. Suppose you want to invest 1,000 Euros in the above. For example, a portfolio can be constructed with profit function

$$\Pi(x) = \frac{200}{238}x + \frac{300}{65}(x - 230)^+ + \frac{500}{30}(200 - x)^+ - 1000.$$

If we plot the profit function, we will observe that the maximum possible loss across all scenarios (i.e., for all $x \geq 0$) is about 832 Euros.

`Question 1` *Is there a different allocation of the 1,000 Euros such that the maximum possible loss is smaller? If yes, what is the allocation that leads to the portfolio with the smallest possible loss?*

The above question can be transformed into a linear programming problem. Let us first state the following observation. Consider a linear function $f(x)$ that satisfies $f(x) \geq -D$ for every $x \geq 0$. For this reason, it is enough to require $f(0) \geq -D$ and $f'(0+) \geq 0$. This conclusion generalizes to a piecewise linear function with finitely many segments.
The profit function $\Pi(x)$ described above is piecewise linear; therefore, the following equivalence holds

$$\Big(\Pi(x) \geq -D \text{ for every } x \geq 0\Big) \Longleftrightarrow \begin{cases} \Pi(0) \geq -D, \\ \Pi(200) \geq -D, \\ \Pi(230) \geq -D, \\ \Pi'(230+) \geq 0, \end{cases}$$

where

$$\Pi(x) = ax + b(x - 230)^+ + c(200 - x)^+ - 1000$$

for some D, or equivalently

$$\Big(\Pi(x) \geq -D \text{ for every } x \geq 0\Big) \Longleftrightarrow \begin{cases} 200c - 1000 \geq -D, \\ 200a - 1000 \geq -D, \\ 230a - 1000 \geq -D, \\ a + b \geq 0. \end{cases}$$

Hence, portfolios with parameters a, b, c that satisfy the above inequalities and the budget equality $238a + 65b + 30c = 1000$ have a maximum possible loss of order D.

`Question 2` *What is the smallest value of D for which there exists a portfolio with maximum possible loss D?*

To find this value, it suffices to solve the following linear programming problem:

$$\begin{array}{ll} \min & D \\ \text{subject to} & 200c - 1000 + D \geq 0, \\ & 200a - 1000 + D \geq 0, \\ & 230a - 1000 + D \geq 0, \\ & a + b \geq 0, \\ & 238a + 65b + 30c = 1000. \end{array}$$

Let $x_1 = a$, $x_2 = b$, $x_3 = c$, $x_4 = D$. We obtain the following linear programming problem:

$$\begin{array}{ll} \max & -x_4 \\ \text{subject to} & 200x_3 - 1000 + x_4 \geq 0, \\ & 200x_1 - 1000 + x_4 \geq 0, \\ & 230x_1 - 1000 + x_4 \geq 0, \\ & x_1 + x_2 \geq 0, \\ & 238x_1 + 65x_2 + 30x_3 = 1000. \end{array}$$

The solution of the above system is approximately $a = 4.9$, $b = -4.9$, $c = 5$, and the maximum possible loss is about 20 Euros. Negative signs correspond to short positions in the underlying. If all positions are required to be long (nonnegative), the problem has a solution approximately $a = 3.5$, $b = 0$, $c = 5$, with corresponding maximum possible loss about 286 Euros.
The above described how an investor can construct a portfolio with the smallest possible loss. However, the investor may believe that the stock price will rise.
One way to construct the corresponding profit-seeking portfolio is to require the slope of the line in the last segment to be sufficiently large. That is, $x_1 + x_2 \geq M$ for some $M > 0$. The larger M is, the greater the profit will be in the event of a stock price increase. However, the maximum possible loss will also increase. Therefore, the investor must decide which portfolio suits them. For example, choosing $M = 6.5$ yields $a = 2.84$, $b = 3.65$, $c = 2$, while the maximum possible loss now becomes 432 Euros.
Another way to construct the portfolio depends on the investor's prediction for the stock price at time T. For example, suppose the investor believes the stock price will be in the interval $(220, 250)$. Then they want to construct a portfolio that is profitable if indeed the price lies in this interval. For this purpose, we must have $\Pi(x) \geq 0$ for every $x \in (220, 250)$.
Specifically, the following linear program can be solved:

$$\begin{array}{ll} \max & x_4 + x_5 + x_6 \\ \text{subject to} & 200x_3 - 500 \geq 0, \\ & 200x_1 - 500 \geq 0, \\ & 220x_1 - 1000 - x_4 \geq 0, \\ & 230x_1 - 1000 - x_5 \geq 0, \\ & 250x_1 + 20x_2 - 1000 - x_6 \geq 0, \\ & x_1 + x_2 \geq 0, \\ & 238x_1 + 65x_2 + 30x_3 = 1000. \end{array}$$

Solving this linear program yields a portfolio whose loss is no greater than 500 Euros and, at the same time, achieves the largest possible profit if the stock price lies in the above interval. The above inequalities follow from the following reasoning: since we want a profitable portfolio on $(220, 250)$, we should require $\Pi(220) \geq D_1$, $\Pi(230) \geq D_2$, $\Pi(250) \geq D_3$, with D_1, D_2, D_3 as large as possible.
On the other hand, to ensure that the portfolio's loss is no more than 500 Euros, we should require $\Pi(x) \geq -500$ for every $x \geq 0$. Thus, we require $\Pi(0) \geq -500$, $\Pi(200) \geq -500$, and $\Pi'(230+) \geq 0$.
Next we will discuss the following equivalences,

$$\Big(\Pi(x) \geq -D \text{ for every } x \geq 0\Big) \iff \begin{cases} \Pi(0) \geq -D, \\ \Pi(K_1) \geq -D, \\ \vdots \\ \Pi(K_n) \geq -D, \\ \Pi'(K_n+) \geq 0. \end{cases}$$

$$\Big(\Pi(x) \geq 0 \text{ for every } x \in (a,b)\Big) \iff \begin{cases} \Pi(K_i) \geq D_i \text{ for every } K_i \in (a,b), \\ \Pi(a) \geq D_a, \\ \Pi(b) \geq D_b \text{ if } b < +\infty, \\ \Pi'(K_n+) \geq M \text{ if } b = +\infty, \\ D_i, D_a, D_b, M \geq 0. \end{cases}$$

where D_a, D_b, D_i, M must be nonnegative.
We will give rigorous proofs for the case with one asset and for the general case with d-assets as well.

`Theorem` **25 (Static superhedging on a finite grid (one-asset case))** *Fix strikes $\{K_1, \dots, K_m\}$ for calls and $\{K'_1, \dots, K'_{\tilde{m}}\}$ for puts on a single stock. Consider a static portfolio (in discounted units) with profit function*

$$\Pi(x) \;=\; b \;+\; \phi(x) - V, \qquad x \in \mathbb{R}_+,$$

where

$$\phi(x) \;=\; a\,x \;+\; \sum_{j=1}^{m} c_j\,(x - K_j)^+ \;+\; \sum_{k=1}^{\tilde{m}} d_k\,(K'_k - x)^+,$$

for real coefficients b, a, c_j, d_k (long > 0, short < 0). Let $V \in \mathbb{R}$ be the (discounted) initial setup cost and fix $D \in \mathbb{R}$. Define the node set

$$S \;:=\; \{0\} \cup \{K_1, \dots, K_m, K'_1, \dots, K'_{\tilde{m}}\}.$$

Then the following are equivalent:

(i) $\Pi(x) \;\geq\; D$ *for all* $x \in \mathbb{R}_+$.

(ii) The following finite family of linear inequalities holds:

$$a + \sum_{j=1}^{m} c_j \;\geq\; 0, \tag{4.1}$$

$$(b - V) + \phi(s) \;\geq\; D \quad \textit{for every } s \in S. \tag{4.2}$$

Proof. Write the profit function as $\Pi(x) = (b - V) + \phi(x)$. We have

$$\inf_{x \geq 0} \big(\Pi(x)\big) \;=\; (b - V) + \inf_{x \geq 0} \phi(x). \tag{4.3}$$

Step 1: tail slope and boundedness. On $[0, \infty)$ the function ϕ is piecewise affine with nodes at S, and its right slope for $x \to \infty$ equals $a + \sum_j c_j$. If this slope is negative then $\phi(x) \to -\infty$ as $x \to \infty$, so no uniform lower bound D can hold; hence (4.1) is necessary. Assuming (4.1), the function ϕ is bounded below on $[0, \infty)$.

Step 2: minimum occurs at a node (or at 0). Between consecutive points of S, the function ϕ is affine; thus its minimum over any closed interval is attained at an endpoint. Under (4.1), the global minimum over $[0, \infty)$ is attained at some $s \in S$, i.e.

$$\inf_{x \geq 0} \phi(x) = \min_{s \in S} \phi(s). \tag{4.4}$$

Step 3: equivalence. Combining (4.3) and (4.4) yields

$$\inf_{x \geq 0} \big(\Pi(x)\big) \;=\; \min_{s \in S} \big[(b - V) + \phi(s)\big].$$

Therefore, $\Pi(x) \geq D$ for all $x \geq 0$ holds if and only if (4.2) holds for every $s \in S$, together with the necessary tail condition (4.1) to preclude $-\infty$ along the ray $x \to \infty$. This proves the equivalence of (i) and (ii).

Theorem **26 (Finite-point certification on a subset of $\mathbb{R}_+$)** *Fix finite strike sets $\{K_1, \ldots, K_m\}$ for calls and $\{K'_1, \ldots, K'_{\tilde{m}}\}$ for puts on a single stock. Consider a static portfolio (in discounted units) with profit function*

$$\Pi(x) = b - V + \phi(x), \qquad \phi(x) = a\,x + \sum_{j=1}^{m} c_j\,(x - K_j)^+ \;+\; \sum_{k=1}^{\tilde{m}} d_k\,(K'_k - x)^+,$$

for real coefficients b, a, c_j, d_k (long > 0, short < 0). Let $V \in \mathbb{R}$ be the (discounted) setup cost. Let $G \subseteq \mathbb{R}_+$ be a set that is a finite union of intervals and/or singletons (hence with finite boundary). Define the node set $S := \{0\} \cup \{K_1, \ldots, K_m, K'_1, \ldots, K'_{\tilde{m}}\}$ and

$$E_G \;:=\; \big(S \cap \overline{G}\big) \;\cup\; \partial G,$$

where $\overline{G}$ is the closure and ∂G the topological boundary of G in $\mathbb{R}_+$. Then the following are equivalent:

(i) $\Pi(x) \;\geq\; 0$ *for all* $x \in G$.

(ii) The following finite family of linear conditions holds:

$$\textit{(Tail, only if } G \textit{ is unbounded above)} \qquad a + \sum_{j=1}^{m} c_j \;\geq\; 0, \tag{4.5}$$

$$\textit{(Grid on } G\textit{)} \qquad (b - V) + \phi(s) \;\geq\; 0 \quad \textit{for every } s \in E_G. \tag{4.6}$$

Equivalently, there exist points $x_i \in E_G$ and numbers $D_i \geq 0$ (one for each x_i) such that

$$(b - V) + \phi(x_i) \;\geq\; D_i \qquad \textit{for all } i,$$

and in particular one may take $D_i = 0$ for all i.

`Proof.` On $[0,\infty)$ the function ϕ is piecewise affine with nodes at S.

Step 1: tail boundedness. If G is unbounded above (i.e. $\sup G = +\infty$) and $a + \sum_j c_j < 0$, then $\phi(x) \to -\infty$ along G, so (i) fails. Thus (4.5) is necessary whenever G is unbounded above. Under (4.5) (or if G is bounded), ϕ is bounded below on G.

Step 2: where the minimum over G can occur. Between consecutive nodes in S, ϕ is affine. On any connected component I of G lying inside such an affine segment, the minimum of the continuous affine function $(b-V)+\phi$ over I is attained at an endpoint of I (possibly an endpoint not belonging to G if I is half-open). Hence the global minimum of $(b-V)+\phi$ on G is attained at a point in $(S \cap \overline{G}) \cup \partial G = E_G$.

Step 3: equivalence. Therefore,

$$\inf_{x\in G}\big(\Pi(x)\big) = \min_{s\in E_G}\big((b-V)+\phi(s)\big),$$

provided the tail is nonnegative when G is unbounded. Consequently, $\Pi(x) \geq 0$ for all $x \in G$ holds if and only if $(b-V)+\phi(s) \geq 0$ for all $s \in E_G$, together with the necessary tail condition when $\sup G = +\infty$. This is precisely (4.6)-(4.5). Finally, since E_G is finite, (4.6) is equivalent to the existence of nonnegative margins $D_i \geq 0$ at finitely many points $x_i \in E_G$ with $(b-V)+\phi(x_i) \geq D_i$ (e.g. take $D_i = 0$).

`Remark` **27** *If G is bounded, the tail condition (4.5) is not needed. If $G = [\alpha,\infty)$, then $E_G = (S \cap [\alpha,\infty)) \cup \{\alpha\}$ and the single tail inequality $a + \sum_j c_j \geq 0$ together with $(b-V)+\phi(s) \geq 0$ for $s \in E_G$ is necessary and sufficient.*

`Theorem` **28 (Static superhedging on a finite grid (d–asset case))** *Fix $d \in \mathbb{N}$ and, for each asset $i = 1,\dots,d$, finite strike sets $\{K_{i,1},\dots,K_{i,m_i}\}$ for calls and $\{K'_{i,1},\dots,K'_{i,\tilde m_i}\}$ for puts. Consider a static portfolio (in discounted units) with profit function*

$$\Pi(x) \;=\; b - V \;+\; \sum_{i=1}^{d} \phi_i(x_i), \qquad x = (x_1,\dots,x_d) \in \mathbb{R}^d_+,$$

where, for each i,

$$\phi_i(u) \;=\; a_i\,u \;+\; \sum_{j=1}^{m_i} c_{i,j}\,(u - K_{i,j})^+ \;+\; \sum_{k=1}^{\tilde m_i} d_{i,k}\,(K'_{i,k} - u)^+,$$

with real coefficients $b, a_i, c_{i,j}, d_{i,k}$ (long > 0, short < 0). Let $V \in \mathbb{R}$ be the (discounted) initial setup cost and fix $D \in \mathbb{R}$. For each i define the node set

$$S_i \;:=\; \{0\} \,\cup\, \{K_{i,1},\dots,K_{i,m_i},K'_{i,1},\dots,K'_{i,\tilde m_i}\}.$$

Then the following are equivalent:

(i) $\Pi(x) \;\geq\; D$ *for all* $x \in \mathbb{R}^d_+$.

(ii) The following finite *system of linear inequalities holds:*

$$a_i + \sum_{j=1}^{m_i} c_{i,j} \;\geq\; 0 \qquad \text{for each } i = 1,\dots,d, \tag{4.7}$$

$$(b-V) + \sum_{i=1}^{d} \phi_i(s_i) \;\geq\; D \qquad \text{for every } (s_1,\dots,s_d) \in S_1 \times \cdots \times S_d. \tag{4.8}$$

`Proof.` By separability,

$$\inf_{x\in\mathbb{R}^d_+}\big(\Pi(x)\big) \;=\; (b-V)+\sum_{i=1}^{d}\inf_{u\geq 0}\phi_i(u). \tag{4.9}$$

Step 1: tail slopes and boundedness. For each i, the function ϕ_i is piecewise affine on $[0,\infty)$ with nodes at S_i. Its right slope as $u\to\infty$ equals $a_i+\sum_j c_{i,j}$. If this slope is negative for some i, then $\phi_i(u)\to-\infty$ and no uniform lower bound D for $\Pi(x)$ can hold. Hence the tail conditions (4.7) are necessary. Under (4.7), each ϕ_i is bounded below on $[0,\infty)$.
Step 2: minima occur at nodes (or at 0). Between consecutive points of S_i, the function ϕ_i is affine; therefore its minimum over any closed interval is attained at an endpoint. Using (4.7), the global minimum over $[0,\infty)$ satisfies

$$\inf_{u\geq 0}\phi_i(u) \;=\; \min_{s_i\in S_i}\phi_i(s_i), \qquad i=1,\dots,d. \tag{4.10}$$

Step 3: equivalence. Combining (4.9) and (4.10) yields

$$\inf_{x\in\mathbb{R}^d_+}\big(\Pi(x)\big)=\min_{(s_1,\dots,s_d)\in S_1\times\cdots\times S_d}\Big[(b-V)+\sum_{i=1}^{d}\phi_i(s_i)\Big].$$

Thus $\Pi(x)\geq D$ for all $x\in\mathbb{R}^d_+$ holds if and only if (4.8) holds for every grid point, together with the necessary tail conditions (4.7) to preclude $-\infty$ along the rays $x_i\to\infty$. This proves the equivalence of (i) and (ii).

`Theorem` **29 (Finite-point certification on a subset of $\mathbb{R}^d_+$)** *Fix $d\in\mathbb{N}$ and, for each asset $i=1,\dots,d$, finite strike sets $\{K_{i,1},\dots,K_{i,m_i}\}$ for calls and $\{K'_{i,1},\dots,K'_{i,\tilde m_i}\}$ for puts. Consider a static portfolio (in discounted units) with profit function*

$$\Pi(x) \;=\; b-V \;+\; \sum_{i=1}^{d}\phi_i(x_i), \qquad x=(x_1,\dots,x_d)\in\mathbb{R}^d_+,$$

where

$$\phi_i(u) \;=\; a_i\,u \;+\; \sum_{j=1}^{m_i}c_{i,j}\,(u-K_{i,j})^+ \;+\; \sum_{k=1}^{\tilde m_i}d_{i,k}\,(K'_{i,k}-u)^+,$$

with real coefficients $b,a_i,c_{i,j},d_{i,k}$ (long >0, short <0). Let $V\in\mathbb{R}$ be the (discounted) setup cost.
Let $G\subseteq\mathbb{R}^d_+$ be a finite union of axis-aligned boxes*, i.e.*

$$G=\bigcup_{\ell=1}^{L}\prod_{i=1}^{d}I_{i,\ell},$$

where each $I_{i,\ell}$ is one of $[\alpha,\beta]$, $[\alpha,\infty)$, or the singleton $\{\alpha\}$ with $0\leq\alpha\leq\beta<\infty$. For each i, define

$$S_i:=\{0\}\cup\{K_{i,1},\dots,K_{i,m_i},K'_{i,1},\dots,K'_{i,\tilde m_i}\}, \qquad B_i:=\{\textit{all finite endpoints appearing among the } I_{i,\ell}\},$$

and set

$$E_i \;:=\; \big(S_i\cap\overline{\pi_i(G)}\big)\;\cup\;B_i, \qquad E:=E_1\times\cdots\times E_d,$$

where $\pi_i(G)$ is the projection of G onto the i-th axis and $\bar{\cdot}$ denotes closure. Let $I_\infty:=\{\,i:\sup\pi_i(G)=+\infty\,\}$ be the set of coordinates along which G is unbounded above.
Then the following are equivalent:

(i) $\Pi(x) \geq 0$ *for all* $x \in G$.

(ii) The following finite system of linear conditions holds:

$$a_i + \sum_{j=1}^{m_i} c_{i,j} \geq 0 \qquad \text{for every } i \in I_\infty, \tag{4.11}$$

$$(b - V) + \sum_{i=1}^{d} \phi_i(s_i) \geq 0 \qquad \text{for every } s = (s_1, \ldots, s_d) \in E \cap \overline{G}. \tag{4.12}$$

Equivalently, there exist finitely many points $x^{(1)}, \ldots, x^{(N)} \in E \cap \overline{G}$ *and numbers* $D_1, \ldots, D_N \geq 0$ *(e.g. one can take all* $D_\ell = 0$*) such that*

$$\Pi\big(x^{(\ell)}\big) \geq D_\ell, \qquad \ell = 1, \ldots, N,$$

and (4.11) *holds.*

`Proof.` Each ϕ_i is piecewise affine on $[0, \infty)$ with nodes at S_i.
Step 1: tail boundedness. For $i \in I_\infty$, the right slope of ϕ_i as $u \to \infty$ equals $a_i + \sum_j c_{i,j}$. If this slope is negative for some $i \in I_\infty$, then along G we can send $x_i \to \infty$ keeping the other coordinates fixed, and obtain $\phi_i(x_i) \to -\infty$; hence (i) fails. Thus (4.11) is necessary. When (4.11) holds (or if $i \notin I_\infty$), each ϕ_i is bounded below on $\pi_i(G)$, so $\Pi(\cdot)$ is bounded below on G.
Step 2: reduction to a finite grid. For each i, order the finite set E_i increasingly and use it to partition $[0, \infty)$ into intervals with endpoints in E_i. On the resulting *grid* of products of such intervals, each ϕ_i is affine on every interval factor, hence the sum $(b - V) + \sum_i \phi_i$ is affine on each grid cell. Intersecting this finite grid with G (a finite union of boxes) yields a finite union of convex polytopes, on each of which $(b - V) + \sum_i \phi_i$ is affine. Therefore the minimum of an affine function over each such polytope is attained at one of its vertices. Every such vertex has coordinates in E_i, hence belongs to $E \cap \overline{G}$. Consequently,

$$\inf_{x \in G} \big(\Pi(x)\big) = \min_{s \in E \cap \overline{G}} \Big[(b - V) + \sum_{i=1}^{d} \phi_i(s_i)\Big],$$

provided (4.11) holds on I_∞.
Step 3: equivalence. From Step 2, (i) holds iff the right-hand side above is ≥ 0, i.e. iff (4.12) holds for all $s \in E \cap \overline{G}$, together with the necessary tail conditions (4.11). Since $E \cap \overline{G}$ is finite, this is equivalent to the existence of finitely many points $x^{(\ell)}$ in $E \cap \overline{G}$ and nonnegative margins $D_\ell \geq 0$ with $\Pi(x^{(\ell)}) - V \geq D_\ell$ (for instance $D_\ell = 0$), completing the proof.

`Remark` **30** *If G is bounded, no tail condition is required (i.e. $I_\infty = \varnothing$). When $G = \prod_{i=1}^{d}[\alpha_i, \infty)$, one has $E_i = (S_i \cap [\alpha_i, \infty)) \cup \{\alpha_i\}$ and the single family of inequalities* (4.11)–(4.12) *is necessary and sufficient.*

`Problem` **31 (Constructing an Optimal Portfolio)** *Suppose an investor wants to invest an amount Y in a stock and the corresponding calls and puts. Suppose they predict that the stock price will lie in the interval (c, v) at time T. Then they can allocate Y after solving the following linear program.*
Given an acceptable loss D, find the coefficients $a, b, \gamma_i, \delta_i, D_c, D_v, D_i, M$ such that

$$\begin{array}{ll} \max & w_c D_c + w_1 D_1 + \cdots + w_l D_l + w_v D_v + w_M M \\ \textit{subject to} & \Pi(x) \geq -D \textit{ for every } x \geq 0 \\ & \Pi(x) \geq 0 \textit{ for every } x \in (c, v) \\ & ax + be^{rT} + \sum_{i=1}^{n} \gamma_i C(K_i) + \delta_i P(K_i) = Y, \end{array} \tag{4.13}$$

where w_c, w_v, w_i, w_M are investor-chosen weights.
If the above problem is unbounded, one may impose bounds on the coefficients, for example $\gamma_i, \delta_i \in [-H, H]$ for some investor-chosen H.

`Lab` **32 (Solving Problem 4.13 in Python)** *The Python notebook `PCUP.ipynb` computes the optimal portfolio given a prediction by solving Problem 4.13. It uses real market data and, in fact, the bid-ask spread. Upload the code to an AI assistant and ask how to use it. While you're at it, also ask what the bid-ask spread is!* □

`Remark` **33 (Low Risk - Low Return!)** *Note that the larger the prediction set, the more strikes will fall inside it. This adds more constraints to the optimization problem, resulting in fewer portfolios that satisfy them. In turn, this leads to a solution with less profit. Hence, low risk - low return, high risk - high return, which is intuitively clear.* □

`Example` **34 (Butterfly Strategy)** *Suppose there is an asset with current price $S_0 = 248$ and the following calls: $C(150) = 110$, $C(200) = 63.4$, $C(300) = 12.7$. We can use these three calls to construct a butterfly portfolio by buying one call with strike 150, buying one call with strike 300, and selling two calls with strike 200. Constructing this portfolio leaves 4.1 Euros in our pocket, while the maximum possible loss is 45.9 Euros.*
Can we recover the same portfolio using the above theory? Yes, by solving the following linear program:

$$\begin{aligned} \max \quad & D \\ \textit{subject to} \quad & \Pi(x) \geq -45.9 \textit{ for every } x \geq 0, \\ & \Pi(x) \geq D \textit{ for every } x \in (150, 200), \\ & 110a + 63.4b + 12.7c = -4.1, \\ & D \geq 0, \\ & a, b, c \in [-2, 2], \end{aligned}$$

where the profit function is

$$\Pi(x) = a(x-150)^+ + b(x-200)^+ + c(x-300)^+ + 4.1.$$

Now, suppose we also allow buying stock. Then we solve the following linear program:

$$\begin{aligned} \max \quad & D \\ \textit{subject to} \quad & \Pi(x) \geq -45.9 \textit{ for every } x \geq 0, \\ & \Pi(x) \geq D \textit{ for every } x \in (150, 200), \\ & 248a + 110b + 63.4c + 12.7d = -4.1, \\ & D \geq 0, \\ & a, b, c, d \in [-2, 2], \end{aligned}$$

where the profit function now is

$$\Pi(x) = ax + b(x-150)^+ + c(x-200)^+ + d(x-300)^+ + 4.1.$$

In real trading, we can usually buy/sell an integer number of contracts. In such a case, we must use a solver that supports integrality constraints. □

The above construction can be applied so that the resulting portfolio is profitable on any subset of $\mathbb{R}_+$, not necessarily a single interval. In fact, the smaller the subset, the greater the profit when the investor's prediction is confirmed. Therefore, in constructing a profitable portfolio, the investor's prediction and its accuracy play a crucial role. Simple constructions based on the above methodology include bull spread, bear spread, butterfly, and straddle strategies, among others.

Remark **35 (Deposit Rate $\neq$ Borrowing Rate!)** *If in a mathematical finance problem the interest rate r appears both as a borrowing rate and as a risk-free rate, we should think of it as the rate of our personal bank account. If we deposit money, we earn interest r; if we withdraw and later replace money, it is as if we borrow at rate r again.*

Remark **36 (Financial Mathematics)** *Note that we will not discuss prediction techniques here, despite their importance. Instead, given a prediction, our goal is to construct an optimal portfolio that is as profitable as possible in the predicted scenario, while ensuring the loss does not exceed a fixed amount in the bad scenario. In prediction, probability theory, statistics, behavioral finance, stochastic analysis (in particular stochastic differential equations), machine learning, etc., play a significant role.*

Markowitz-type Portfolios. Solving optimization problems also yields other types of portfolios. For example, an investor may be interested in constructing a portfolio where the maximum possible loss is no more than 500 Euros. As discussed, such portfolios exist and, in fact, there are many. Among them, one may choose the one with the smallest variance, or some similar property. For this purpose, one should assume S_T follows a known distribution, hence a prediction for the future behavior of the stock price must be made. Here, the prediction model differs from the previous case. Therefore, the following optimization problem can be solved:

$$\begin{aligned}
\text{min} \quad & w_1 \operatorname{Var}\big(\Pi(S_T)\big) \;-\; w_2\,\mathbb{E}(\Pi(S_T) \mid S_T \in G) \\
\text{subject to} \quad & 200c - 1000 + 500 \geq 0, \\
& 200a - 1000 + 500 \geq 0, \\
& 230a - 1000 + 500 \geq 0, \\
& a + b \geq 0, \\
& 238a + 65b + 30c = 1000,
\end{aligned}$$

where $G \subseteq \mathbb{R}_+$ is a set in which the investor predicts the stock price will lie at time T, and w_1, w_2 are investor-chosen weights.
The resulting portfolio will have maximum possible loss no greater than 500 Euros in all scenarios. This construction generalizes the classical Markowitz construction, as it accounts for available put contracts.

4.1 Exercises

True or False

1. **Statement:** The maximum possible loss of a piecewise linear payoff can be controlled only by checking the value at specific knots and the right slope at the last strike.
   True / False: ______ **Justification:** ________________________________

2. **Statement:** Adding the constraint $a + b \geq M$ in a portfolio with calls and puts changes the slope of the last segment and makes it more bullish.
   True / False: ______ **Justification:** ________________________________

3. **Statement:** A butterfly portfolio can be obtained as the solution of a linear program with profit constraints on an interval and a maximum-loss constraint.
   True / False: ______ **Justification:** ________________________________

4. **Statement:** A short position in a stock profits when the stock price rises.
   True / False: ______ **Justification:** ________________________________

Multiple Choice

1. In a portfolio with payoff $\Pi(x) = ax + b(x - K_1)^+ + c(K_2 - x)^+ - Y$, the constraint that the *maximum possible loss* is D translates to:

   (A) $\Pi(K_1) \geq -D,\ \Pi(K_2) \geq -D,\ a + b \geq 0$
   (B) $\Pi(0) \geq -D,\ \Pi(K_1) \geq -D,\ \Pi(K_2) \geq -D,\ a + b \geq 0$
   (C) $\Pi(0) \leq D,\ \Pi(K_1) \leq D,\ \Pi(K_2) \leq D$
   (D) $\Pi(x) \geq D$ for every $x \in (K_1, K_2)$

   Answer: ______

2. What does it mean that a linear program is unbounded?

   (A) There is no solution satisfying the constraints.
   (B) The objective can increase without bound via feasible moves-there is potential for infinite profit.
   (C) The loss is always zero.
   (D) The portfolio is unique.

   Answer: ______

3. In constructing a butterfly, which description matches?

   (A) Buy two calls at the middle strike and sell one at each wing.
   (B) Sell one call at the low strike, buy two at the middle, and sell one at the high.
   (C) Buy one call at the low and one at the high strike, sell two at the middle.
   (D) Buy one call and sell one put with the same strike.

   Answer: ______

4. Which constraint adds an upward tilt to the payoff?

   (A) $\Pi(K_i) \geq D_i$ for K_i inside the prediction interval.
   (B) $a + b \geq M$, with $M > 0$.
   (C) $\Pi(0) \geq -D$.
   (D) $c \geq 0$ where c multiplies $(K - x)^+$.

   Answer: ______

5. In the general problem with prediction interval (c, v), what do D_c, D_v, D_i represent?

   (A) Strike prices.
   (B) Amounts invested in the underlying.
   (C) Minimum acceptable profits inside and at the edges of the prediction interval.
   (D) Slopes at the last strike.

   Answer: ______

Matching

Match each description with the correct concept or strategy. Write the letter next to the number.

1. Buy one call at the low strike, sell two at the middle, buy one at the high	A. Butterfly
2. Constraint requiring the payoff to be at least zero on a prediction interval	B. Prediction-constrained portfolio
3. The smallest D such that $\Pi(x) \geq -D$ for every $x \geq 0$	C. Minimum worst-case loss
4. Unbounded profit due to unconstrained positions	D. Unbounded arbitrage

Answers: 1._____ 2._____ 3._____ 4._____

Fill in the Blank

1. The property that for a piecewise linear payoff it suffices to check values at the ______________ and the slope at the last point to bound the maximum loss.

2. Adding the constraint $a + b \geq M$ introduces an ______________ toward an increase in the underlying's price.

3. When a linear program has no upper bound on the objective, it is ______________.

4. In the butterfly example, we sell two calls at the ______________ strike.

5. The strategy that uses put options to protect a stock position is called hedging with ______________.

Theoretical and Computational Exercises

1. **Verify the solution:** Given $S_0 = 238$, $C(230) = 65$, $P(200) = 30$, and investment $Y = 1000$. Consider the proposed portfolio with $a = 4.9, b = -4.9, c = 5$. Compute $\Pi(0), \Pi(200), \Pi(230)$ and show that the maximum possible loss is about 20 Euros.

2. **Upward tilt:** Starting from the previous portfolio, impose the constraint $a + b \geq 6.5$. Solve numerically (or outline the method) to find new a, b, c and compute the new maximum loss.

3. **Butterfly as LP:** Provide the corresponding linear program that yields a butterfly strategy with call strikes 150, 200, 300, under the constraints that the maximum loss is 45.9 and that the payoff is positive on $(150, 200)$.

4. **Prediction on an interval:** An investor predicts the price will lie in $(220, 250)$ and wants $\Pi(x) \geq 0$ on this interval while the maximum loss does not exceed 500. Formulate the full LP (with node variables D_1, D_2, D_3) that maximizes profits inside the interval under this constraint.

Python-Based Exercises

1. **Payoff Visualization:** Write Python code that plots the payoff $\Pi(x) = ax + b(x - 230)^+ + c(200 - x)^+ - 1000$ for given a, b, c. Test $a = 4.9, b = -4.9, c = 5$ and $a = 2.84, b = 3.65, c = 2$ and compare the maximum possible loss graphically.

2. **Solving the LP:** Implement in Python (e.g., with `scipy.optimize.linprog`) the linear program for minimizing D as in the first example and return the optimal a, b, c, D.

3. **Profit vs Risk:** In Python, study the payoff obtained by varying M in the constraint $a + b \geq M$: for each M, compute the minimal D and the expected profit on a prediction interval, and plot the trade-off between M (or profit) and D.

Chapter 5

Arbitrage

Is there such a thing as arbitrage? Absolutely–that's what arbitrageurs are for!

In this chapter, we will study in more detail the concept of a riskless profit opportunity (arbitrage).

`Definition` **37 (Arbitrage)** *Let there be a stock and the corresponding call and put options that define a market. We say that there exists a riskless profit opportunity in the market (Arbitrage) if a portfolio with time horizon T and payoff function can be constructed such that*

$$\begin{aligned} \Pi(x) &\geq 0 \quad \textit{for every } x \geq 0, \\ \Pi(x) &> 0 \quad \textit{for some } x \geq 0 \end{aligned}$$

where

$$\Pi(x) = ax + be^{rT} + \sum \gamma_i (x - K_i)^+ + \delta_i (K_i - x)^+ - Y$$

and Y is the cost of constructing this portfolio. Note that Y may be negative if a, γ_i, δ_i can also be negative.

`Problem` **38 (Detecting a riskless profit opportunity)** *Find the parameters $a, b, \gamma_i, \delta_i, D$ such that the following hold:*

$$\begin{aligned} & \min D \\ \textit{subject to} \quad & aS_0 + b + \sum_{i=1}^{n} (\gamma_i C(K_i) + \delta_i P(K_i)) = Y \\ & \Pi(x) \geq -D \quad \textit{for every } x \geq 0 \\ & \gamma_i, \delta_i \in [-N, N] \end{aligned} \tag{5.1}$$

where $C(K_i), P(K_i)$ are the current prices of the options. If $D < 0$ then we have identified a riskless profit opportunity.

`Lab` **39 (Detecting a riskless profit opportunity with Python)** *The Python code* ***`FindingArbitrage.ipynb`*** *solves problem 5.1 with real data and using the bid-ask spread.*

5.1 Put-Call Parity and riskless profit opportunity

Suppose there is a stock in the market with price S_0, a call option with price $C(K)$, a put option with price $P(K)$, and a bank account with interest rate r. These contracts have the same underlying asset and the same expiration date.
If the following relation (put-call parity) is not true

$$C(K) + Ke^{-rT} = S_0 + P(K),$$

there exists a riskless profit opportunity.

`Remark` **40 (Put-Call Parity and Arbitrage)** *How is the put-call parity formula related to the solution of the linear programming problem 5.1? This linear programming problem is a generalization of the put-call parity, since it will detect the portfolio with the greatest arbitrage opportunity taking into account all available option contracts.*

In practice, if one attempts to repeat this process infinitely many times, this will lead to price changes and consequently to the disappearance of the arbitrage opportunity. Moreover, in practice, transaction costs, bid-ask spread, dividends, etc., must also be included in the analysis.

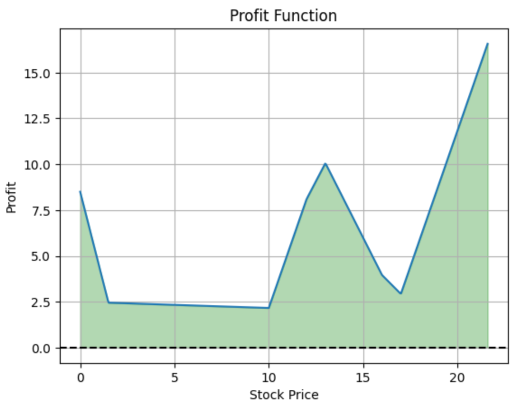


Figure 5.1: The graph of the profit function of a portfolio that exploits a risk-free profit opportunity (Arbitrage).

5.2 Exercises

True or False

1. **Statement:** If the put-call parity is violated, then there exists arbitrage.
   True / False: ________ **Justification:** ____

2. **Statement:** The existence of an unbounded linear programming problem for minimizing the maximum loss means that there is no possibility of arbitrage.
   True / False: ________ **Justification:** ____

3. **Statement:** Restricting the positions in calls and puts to the interval $[-N, N]$ makes the corresponding linear programming problem bounded and may result in a guaranteed profit.
   True / False: ________ **Justification:** ____

4. **Statement:** Constructing an arbitrage portfolio always requires knowing the actual future stock price S_T.
   True / False: ______ **Justification:** ____________________

Multiple Choice

1. The put-call parity is given by $C(K) + Ke^{-rT} = S_0 + P(K)$. If we find that $C(K) + Ke^{-rT} > S_0 + P(K)$, which arbitrage strategy can we apply?

   (A) Buy a call, sell a put, and borrow Ke^{-rT}.

   (B) Buy a put and the stock, sell a call, and invest/borrow the rest in a bank account.

   (C) Sell a call and a put simultaneously.

   (D) Buy a call and sell the stock.

   Answer: ______

2. In the opposite case, if $C(K) + Ke^{-rT} < S_0 + P(K)$, what is the appropriate arbitrage construction?

   (A) Buy a call, sell a put, and sell the stock.

   (B) Buy a put, sell a call, and sell the stock.

   (C) Buy a put and the stock, sell a call.

   (D) Sell a call and buy a put.

   Answer: ______

3. What does it mean that the arbitrage problem is unbounded?

   (A) There is no solution satisfying the constraints.

   (B) The objective (e.g., profit) can be increased indefinitely with feasible positions, indicating infinite profit potential.

   (C) The loss is always zero.

   (D) Option prices are unrealistic.

   Answer: ______

4. In the arbitrage problem with constraints $\gamma_i, \delta_i \in [-N, N]$, what do these constraints represent?

   (A) The maximum amount we can borrow.

   (B) The limits on exposure to each option (how many of each are bought/sold).

   (C) The strike price.

   (D) The interest rate.

   Answer: ______

5. Which of the following theoretically reduces an arbitrage opportunity in the market when attempted repeatedly?

   (A) Increasing the number of strikes.

(B) Imposing $[-N, N]$ position limits.

(C) Price adjustments due to transactions (price impact) and transaction costs such as bid-ask spread.

(D) Excluding the put-call parity.

Answer: _______

Matching Terms

Match each description in column A with the correct English term in column B.

1.	Simultaneous buying and selling of an asset for a profit without risk.	A. Arbitrage
2.	Right to buy the underlying asset at a predetermined price (strike).	B. Call option
3.	Right to sell the underlying asset at a predetermined price (strike).	C. Put option
4.	Relationship $C(K) + Ke^{-rT} = S_0 + P(K)$.	D. Put-call parity
5.	Difference between the lowest asking price (ask) and the highest bidding price (bid).	E. Bid-ask spread
6.	Periodic payment made by a company to its shareholders.	F. Dividend

Matching Symbols

Match each symbol in column A with its correct financial interpretation in column B.

1.	Units of the underlying asset S_0 held at time 0.	A. Units of the underlying asset
2.	Units of the zero-coupon bond held at time 0.	B. Units of the zero-coupon bond
3.	Position (number of contracts) in call options $C(K_i)$.	C. Position in call options
4.	Position (number of contracts) in put options $P(K_i)$.	D. Position in put options
5.	Maximum possible loss (worst-case loss) D.	E. Minimum worst-case loss

Fill in the Blank

1. The riskless profit opportunity in the market is called ____________________.

2. The relationship $C(K) + Ke^{-rT} = S_0 + P(K)$ is known as ____________________.

3. If $C(K) + Ke^{-rT} > S_0 + P(K)$, we buy ____________________ and ____________________, and sell ____________________ to exploit arbitrage.

4. The constraints $\gamma_i, \delta_i \in [-N, N]$ set ____________________ on each option.

5. Adding bid-ask spread and transaction costs to the analysis reduces the ____________________ of the arbitrage opportunity.

Theoretical and Computational Exercises

1. **Arbitrage detection:** Given $S_0 = 100$, $C(50) = 55$, $P(50) = 2$, $r = 0.05$, $T = 1$. Check if the put-call parity is violated and construct the corresponding arbitrage strategy (buy/sell what, what to do with the amount in the bank account) and calculate the riskless profit.

2. **LP Formulation:** Formulate the linear program for the minimum worst-case loss D from eq. (5.1) with constraint $\gamma_i, \delta_i \in [-N, N]$ and with invested amount $Y = 100$. Describe what each term represents and how it ensures $\Pi(x) \geq -D$.

3. **Effect of Bound:** Compare theoretically what happens to the arbitrage problem if we remove the constraints $\gamma_i, \delta_i \in [-1, 1]$ versus when we impose them. What does this mean for boundedness and the possibility of infinite profit?

4. **Put-Call Parity with zero interest rate:** Simplify the put-call parity when $r = 0$ and explain with what hedging we can detect if there is arbitrage.

5. **Application to real data:** Describe how you would extend problem (5.1) to account for bid-ask spread, transaction costs, and dividends. What changes would you make to the formulation and constraints?

6. **Strategy payoff:** Suppose the put-call parity is violated and we construct the corresponding arbitrage portfolio. Show why the final payoff is always non-negative and positive for some x, i.e., it satisfies the definition of arbitrage.

Python-Based Exercises

1. **Put-call parity check:** Write Python code that takes as input $S_0, C(K), P(K), r, T$ and checks if the put-call parity is violated. If there is a violation, suggest which arbitrage strategy should be followed and compute the immediate profit.

2. **Solving LP Arbitrage:** Implement in Python (e.g., using `scipy.optimize.linprog`) the linear programming problem (5.1) with a small set of available calls and puts (e.g., 2 strikes) and constraints $\gamma_i, \delta_i \in [-N, N]$. Show the optimal allocation and the value of D.

3. **Effect of transaction cost:** Add to the previous problem a simple transaction cost term (e.g., $c|\gamma_i| + c|\delta_i|$) and repeat the solution. Compare the final D with and without cost.

Chapter 6

Markowitz Choice Theory and Value at Risk

Diversification is the only free lunch in finance.

— Harry Markowitz

6.1 Elements of Probability Theory

To develop this theory, we first present the basic mathematical tools that arise from probability theory.
We begin with the concept of the covariance of two random variables X, Y, which describes how the change in the value of one random variable affects the other. It is defined as

$$\operatorname{Cov}(X, Y) = \mathbb{E}(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))$$

It can be easily shown that

$$\operatorname{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

The following properties hold for covariance:

- $\operatorname{Cov}(X, X) = \operatorname{VaR}(X)$,
- $\operatorname{Cov}(X, Y) = \operatorname{Cov}(Y, X)$,
- $\operatorname{Cov}(X + Y, Z) = \operatorname{Cov}(X, Z) + \operatorname{Cov}(Y, Z)$,
- $\operatorname{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{m} Y_i\right) = \sum_{i=1}^{n}\sum_{i=1}^{m} Cov(X_i, Y_i)$,
- $\operatorname{Cov}(aX, bY) = ab\operatorname{Cov}(X, Y)$,
- $\operatorname{VaR}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \operatorname{Cov}(X_i, X_j)$.

6.2 The Problem

Suppose that the amount V can be invested in n companies. That is, the goal is to buy k_i shares of company S_i for $i = 1, \cdots, n$. We denote

$$w_i = \frac{k_i S_i^0}{\displaystyle\sum_{j=1}^{n} k_j S_j^0}$$

where clearly

$$\sum_{i=1}^{n} w_i = 1$$

meaning that the amount $w_i V$ will be used to buy k_i shares of S_i. We focus on the values of the weights w_i so that the profit at time T is maximized with the smallest possible risk.
The return of each share μ_i over the interval $[0, T]$ is a random variable such that

$$\frac{S_i^T}{S_i^0} = 1 + \mu_i$$

where S_i^0 and S_i^T are the share prices at times 0 and T, respectively. We assume that the expected value m_i of the return μ_i, the variance σ_i of the return, and the covariance σ_{ij} of the returns of shares of companies S_i and S_j are estimated from historical data. We also assume that these values remain valid for the time interval $[0, T]$. Our goal is to construct a portfolio at time 0 such that at time T it has the maximum expected return m and the minimum possible variance.
The (total) return μ of the portfolio will be such that

$$\frac{V^T}{V} = 1 + \mu$$

thus

$$\mu = \frac{\sum_{i=1}^{n} k_i (S_i^T - S_i^0)}{V} = \frac{\sum_{i=1}^{n} \mu_i k_i S_i^0}{V} = \sum_{i=1}^{n} \mu_i w_i$$

Therefore, the expected return of the portfolio will be

$$m = \mathbb{E}(\mu) = \sum_{i=1}^{n} w_i \mathbb{E}(\mu_i)$$

The variance of the portfolio return will be

$$\sigma = \mathrm{VaR}(\mu) = \mathrm{VaR}\left(\sum_{i=1}^{n} w_i \mu_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

Setting $\mathbf{w}^t = (w_1, \cdots, w_n)$, $\boldsymbol{\Sigma} = [\sigma_{ij}]$ and $\mathbf{m}^t = (m_1, \cdots, m_n)$, we have

$$\begin{aligned} \mathbf{1}^t \mathbf{w} &= 1, \\ \mathbf{w}^t \mathbf{m} &= m, \\ \mathbf{w}^t \boldsymbol{\Sigma} \mathbf{w} &= \sigma \end{aligned}$$

where $\mathbf{1}^t = (1, \cdots, 1)$ and $\mathbf{w}^t$ is the transpose of $\mathbf{w}$. Note that, since the portfolio variance is always a positive number, the covariance matrix Σ is positive definite.

`Problem` **41 (Construction of a Markowitz Portfolio)** *The problem is reduced to a constrained minimization problem. Let m_0 be a required mean return of the portfolio. The goal is to compute the w_i so as to achieve the minimum possible variance of the portfolio, i.e., to solve the problem*

$$\min \quad \mathbf{w}^t \boldsymbol{\Sigma} \mathbf{w}$$

subject to the conditions

$$\begin{aligned} \mathbf{1}^t \mathbf{w} &= 1, \\ \mathbf{w}^t \mathbf{m} &= m_0 \end{aligned} \tag{6.1}$$

6.3 Theoretical Solution of the Markowitz Problem

Using the method of Lagrange multipliers we obtain the following equations:

$$\begin{aligned} 2\boldsymbol{\Sigma}\mathbf{w} &= \lambda_1 \mathbf{1} + \lambda_2 \mathbf{m}, \\ \mathbf{1}^t \mathbf{w} &= 1, \\ \mathbf{w}^t \mathbf{m} &= m_0 \end{aligned}$$

If the matrix $\boldsymbol{\Sigma}$ is invertible, solving the first equation for $\mathbf{w}$ yields

$$\mathbf{w} = \frac{1}{2}\boldsymbol{\Sigma}^{-1}(\lambda_1 \mathbf{1} + \lambda_2 \mathbf{m}) = \frac{1}{2}\Sigma^{-1}[\mathbf{m} \quad \mathbf{1}] \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{6.2}$$

The equations $\mathbf{1}^t \mathbf{w} = 1$ and $\mathbf{w}^t \mathbf{m} = m_0$ can also be written as

$$[\mathbf{m} \quad \mathbf{1}]^t \mathbf{w} = \begin{pmatrix} m_0 \\ 1 \end{pmatrix}$$

Multiplying 6.2 by $[\mathbf{m} \quad \mathbf{1}]^t$ gives

$$[\mathbf{m} \quad \mathbf{1}]^t \mathbf{w} = \frac{1}{2}[\mathbf{m} \quad \mathbf{1}]^t \boldsymbol{\Sigma}^{-1} [\mathbf{m} \quad \mathbf{1}] \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} m_0 \\ 1 \end{pmatrix}$$

Let $\mathbf{A} = [\mathbf{m} \quad \mathbf{1}]^t \boldsymbol{\Sigma}^{-1} [\mathbf{m} \quad \mathbf{1}]$ and show that it is positive definite. Indeed,

$$[y_1 \quad y_2]\mathbf{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = [y_1 \mathbf{m} + y_2 \mathbf{1}]^t \boldsymbol{\Sigma}^{-1} [y_1 \mathbf{m} + y_2 \mathbf{1}]^t$$

therefore the conclusion follows since the matrix $\boldsymbol{\Sigma}^{-1}$ is positive definite.
Note that $\boldsymbol{\Sigma}$ is invertible if $\mathbf{w}^t \boldsymbol{\Sigma} \mathbf{w} = \sigma > 0$, since it follows that all eigenvalues of $\boldsymbol{\Sigma}$ are strictly positive and from the properties of eigenvalues it follows that it is invertible. Furthermore, its inverse will have $\frac{1}{\lambda_i}$ as eigenvalues, which will also be all positive, hence it will also be positive definite. Substituting $\mathbf{A}$ we get

$$\frac{1}{2}\mathbf{A} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} m_0 \\ 1 \end{pmatrix}$$

By inverting $\mathbf{A}$, we compute the vector $\mathbf{w}$ and obtain

$$\mathbf{w} = \boldsymbol{\Sigma}^{-1} [\mathbf{m} \quad \mathbf{1}] \mathbf{A}^{-1} \begin{pmatrix} m_0 \\ 1 \end{pmatrix} \tag{6.3}$$

The variance can be computed using the vector $\mathbf{w}$:

$$\sigma^2 = \mathbf{w}^t \mathbf{\Sigma} \mathbf{w} = \frac{A_{11} - 2A_{12}m_0 + A_{14}m_0^2}{|A|}$$

where $\mathbf{A}$ is the 2×2 matrix defined above. Solving for m_0 yields two solutions.
When solving the above optimization problem, some of the w_i may be negative. This means that the investor will borrow a number of shares of company i, sell them at the current market price, but will be obliged to return them to the lender in the future (short selling). This move may cause unlimited loss to the investor, since the share price can (theoretically) increase without bound in the future, and the investor will be forced to buy at a very high price the shares that were borrowed and must be returned.
The possible loss of the investor can be bounded above if they also purchase a call option, for each share borrowed, to secure themselves. Call and put options mainly serve to secure the investor. However, they can also be used for speculative purposes like shares, especially when such contracts are traded on the stock exchange.
In the case where we assume that $w_i \geq 0$, this must be included in the conditions 6.1.

6.4 Three Optimization Problems for Portfolio Construction

In the following three variants of portfolio optimization, we additionally assume that $w_i \geq 0$, considering portfolios with a small number of shares so that the calculations are feasible.

6.4.1 Minimum Variance Portfolio

Assume that the minimum variance portfolio consists of three shares. The variance of the portfolio return is

$$\begin{aligned} \sigma &= f(w_1, w_2, w_3) \\ &= w_1^2\sigma_1 + w_2^2\sigma_2 + w_3^2\sigma_3 + 2w_1w_2\sigma_{12} + 2w_1w_3\sigma_{13} + 2w_2w_3\sigma_{23} \end{aligned}$$

where $\sigma_i = \sigma_{ii}$. The expected portfolio return is

$$m_1w_1 + m_2w_2 + m_3w_3 = m_0 \tag{6.4}$$

If $m_1 = m_2 = m_3$, no further calculations are required, so without loss of generality, we assume $m_1 \neq m_2$.
Clearly, the following must hold:

$$w_1 + w_2 + w_3 = 1 \tag{6.5}$$

From 6.4 and 6.5 we get

$$\begin{aligned} w_1 &= \frac{t(m_2 - m_3) + m_0 - m_2}{m_1 - m_2} \\ w_2 &= \frac{t(m_3 - m_1) + m_1 - m_0}{m_1 - m_2} \end{aligned}$$

where $t = w_3$. Substituting w_1, w_2 into $f(w_1, w_2, w_3)$ yields a function of a single variable, t. That is, the function must be minimized with respect to t over all $\mathbb{R}$ if no other condition is required for w_1, w_2, w_3.

If in addition w_1, w_2, w_3 are assumed to be positive, then t must lie in a suitable subset of $[0,1]$ where w_1, w_2 are positive, and hence the variance must be minimized in this interval. If $m_1 - m_2 > 0$, then $w_1 \geq 0$ implies $t(m_2 - m_3) + m_0 - m_2 \geq 0$ and $w_2 \geq 0$ implies $t(m_3 - m_1) + m_1 - m_0 \geq 0$. Now the interval is chosen so that t lies in a subset of $[0,1]$ while all conditions are satisfied. If this is not feasible, then the minimization problem has no solution. Similar steps are followed in the case $m_1 - m_2 < 0$.

6.4.2 Maximum Return Portfolios

Another perspective of the optimization problem is to find the vector $w = (w_1, w_2)$ such that the portfolio variance is equal to a given number σ_0 and to maximize the mean return of the portfolio. Thus the mathematical problem is defined as:

$$\begin{aligned} &\max_{(w_1,w_2)} (w_1 m_1 + w_2 m_2) \\ &\text{subject to} \\ \sigma = \sigma_1 w_1^2 + \sigma_2 w_2^2 + 2\sigma_{12} w_1 w_2 &= \sigma_0 \\ w_1 + w_2 &= 1 \\ w_1 \geq 0, \quad w_2 &\geq 0 \end{aligned}$$

To solve the above problem, set $w_2 = t$ and $w_1 = 1 - t$, and substitute into the variance equality. Then the maximization problem becomes

$$\max_{t\in[0,1]} (m_1(1-t) + m_2 t)$$
$$\sigma_1(1-t)^2 + \sigma_2 t^2 + 2\sigma_{12}(1-t)t - \sigma_0 = 0$$

The second equation may have two solutions, say t_1, t_2. We choose the one that lies in $[0,1]$ (if it exists) and at the same time maximizes the desired quantity.

6.4.3 Maximum Return-Variance Difference Portfolios

Another way to solve the problem is to maximize the difference between the expected return and the variance, that is

$$\begin{aligned} &\max_{(w_1,w_2)} \big(m_1 w_1 + m_2 w_2 - \lambda(\sigma_1 w_1^2 + \sigma_2 w_2^2 + 2\sigma_{12} w_1 w_2)\big) \\ &\text{given that } w_1 + w_2 = 1 \\ &w_1 \geq 0, \quad w_2 \geq 0 \end{aligned}$$

for a given $\lambda > 0$. To solve this problem we set again $w_2 = t$ so that $w_1 = 1 - t$. Then the original problem is reduced to

$$\max_{t\in[0,1]} (m_1(1-t) + m_2 t - \lambda(\sigma_1(1-t)^2 + \sigma_2 t^2 + 2\sigma_{12}(1-t)t))$$

6.5 Value at Risk

We assume that a portfolio is constructed where the expected value and the variance have been calculated. These values refer to a specific time period, e.g. one day, one month, etc. An interesting question is: what is the maximum loss (in euros) per day with probability a?

Let V_0 denote the present value of the portfolio and V_t its value tomorrow. Then the above question is expressed by the following mathematical problem: find $x > 0$ (amount in euros) such that

$$P(V_t - V_0 \leq -x) = 1 - a$$

Using the portfolio return, the above equation can be written as

$$P\left(\frac{V_t - V_0}{V_0} \leq -\frac{x}{V_0}\right) = 1 - a$$

where $\mu_t = \frac{V_t - V_0}{V_0}$ is the one-day portfolio return.
The mean m_0 and variance σ^2 of the return are known (more precisely, we have assumed them) and we assume they follow the normal distribution with mean m_0 and variance σ^2. This assumption allows the computation of the appropriate $x > 0$, usually denoted by $VaR_a(t)$ where t is the time horizon of interest, e.g. one day.
Since the return follows a normal distribution with mean m_0 and variance σ^2, then

$$P\left(\frac{V_t - V_0}{V_0} \leq -\frac{x}{V_0}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{-\frac{x}{V_0}} e^{-\frac{(t-m_0)^2}{2\sigma^2}} dt \tag{6.6}$$

The above can be related to the error function $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ via the relation

$$P(Y \leq x) = \frac{1}{2} + \frac{1}{2}\,\text{erf}\left(\frac{x - m_0}{\sqrt{2}\sigma}\right)$$

where Y follows the normal distribution with mean m_0 and variance σ^2.
This relation can be proved as follows

$$\begin{aligned}
P(Y \leq x) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{(t-m_0)^2}{2\sigma^2}} dt \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\frac{x-m_0}{\sigma\sqrt{2}}} e^{-y^2} \sigma\sqrt{2} dy \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x-m_0}{\sigma\sqrt{2}}} e^{-y^2} dy \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-y^2} dy + \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x-m_0}{\sigma\sqrt{2}}} e^{-y^2} dy \\
&= \frac{1}{2} + \frac{1}{2}\,\text{erf}\left(\frac{x - m_0}{\sigma\sqrt{2}}\right)
\end{aligned}$$

Returning to 6.6 we obtain

$$g(x) = P\left(\frac{V_t - V_0}{V_0} \leq -\frac{x}{V_0}\right) = \frac{1}{2} + \frac{1}{2}\,\text{erf}\left(\frac{-\frac{x}{V_0} - m_0}{\sigma\sqrt{2}}\right)$$

The expected loss given that it exceeds a given amount $p < 0$ can be computed. We have

$$\mathbb{E}(V_t - V_0 | \{V_t - V_0 < p\}) = \frac{\frac{1}{\sqrt{2\pi V^2\sigma^2}} \int_{-\infty}^{p} x e^{-\frac{(x-m_0V)^2}{2V^2\sigma^2}} dx}{\frac{1}{\sqrt{2\pi V^2\sigma^2}} \int_{-\infty}^{p} e^{-\frac{(x-m_0V)^2}{2V^2\sigma^2}} dx}$$

The above calculations are part of the so-called **risk measurement** which is associated with a portfolio. Obviously, the method of measuring risk is closely linked to the choice of the probability measure and the assumptions about the distributions and parameters, and is therefore subjective.
On the other hand, the technique of constructing a portfolio which also consists of call and put options, is part of the so-called **risk management** which is associated with a portfolio. Of course, risk management has an objective nature since it does not depend on the choice of probability measure.

Lab **42 (Construction of a Markowitz Portfolio with Python)** *With the code* `Markowitz2` *we can perform the above calculations for stocks of our choice using real historical numerical data.*

6.6 Exercises

True or False

1. **Statement:** The efficient frontier is the set of portfolios that maximize the expected return for a given variance.
   True / False: _______ **Justification:** ____

2. **Statement:** If all the off-diagonal terms of a covariance matrix are zero, then the returns are uncorrelated.
   True / False: _______ **Justification:** ____

3. **Statement:** In the Markowitz problem with the constraint $w_i \geq 0$, the minimum variance portfolio may involve short selling.
   True / False: _______ **Justification:** ____

4. **Statement:** The formula for the Value at Risk of a portfolio with normally distributed returns uses the quantile of the standard normal distribution.
   True / False: _______ **Justification:** ____

5. **Statement:** The use of Lagrange multipliers in the problem of minimizing the variance always leads to positive weights w_i.
   True / False: _______ **Justification:** ____

Multiple Choice

1. In the classical Markowitz problem without constraints, which choice yields the global minimum variance portfolio?

   (A) $\Sigma^{-1}\mathbf{1}$ (normalized so that $\sum w_i = 1$).
   (B) $\Sigma^{-1}\mathbf{m}$ (normalized so that $\sum w_i = 1$).
   (C) Vector with equal weights $w_i = 1/n$.
   (D) Different solution for each m_0.

   Answer: _______

2. The relation $\sigma^2 = \mathbf{w}^T \Sigma \mathbf{w}$ describes:

   (A) The expected return.

   (B) The variance of returns.

   (C) The probability of loss beyond a threshold.

   (D) The Sharpe ratio.

   Answer: _______

3. In the computation of VaR with μ_0, σ and confidence level a, which is the corresponding formula for the amount x?

   (A) $x = V_0\big(\mu_0 + z_{1-a}\,\sigma\big)$.

   (B) $x = V_0\big(-\mu_0 + z_a\,\sigma\big)$.

   (C) $x = V_0\big(-\mu_0 + z_{1-a}\,\sigma\big)$.

   (D) $x = V_0\big(\mu_0 - z_a\,\sigma\big)$.

   Answer: _______

4. In the Markowitz problem, what does the matrix Σ represent?

   (A) Strike prices.

   (B) Covariance matrix of returns.

   (C) Variances only.

   (D) Weight matrix.

   Answer: _______

5. Which parameter in VaR expresses the confidence level?

   (A) m_0 (expected return).

   (B) σ (standard deviation).

   (C) a (confidence level).

   (D) V_0 (initial value).

   Answer: _______

Matching

Match the description to the correct concept. Write the letter next to the number.

1.	Vector w that minimizes $\mathbf{w}^T \Sigma \mathbf{w}$ under $\sum w_i = 1$	A. Global minimum variance portfolio
2.	Graphical representation of m vs σ for optimal portfolios	B. Efficient frontier
3.	Borrowing shares to sell now and repurchase in the future	C. Short selling
4.	Maximum loss with probability $1-a$ over period t	D. Value at Risk (VaR)

Answers: 1._____ 2._____ 3._____ 4._____

Fill in the Blank

1. The portfolio weight vector is denoted by ____________________.

2. The expected return of the portfolio is given by ____________________.

3. The covariance matrix of returns is defined as ____________________.

4. In the context of VaR, the confidence level a is related to the quantile ____________________.

5. The assumption of normal distribution allows the use of the erf for computing ____________________.

Theoretical and Computational Exercises

1. **Solution with Lagrange multipliers:** Show step-by-step that the equation

$$2\Sigma\,\mathbf{w} = \lambda_1 \mathbf{1} + \lambda_2 \mathbf{m}$$

together with the conditions $\mathbf{1}^T\mathbf{w} = 1$ and $\mathbf{m}^T\mathbf{w} = m_0$ leads to the formula

$$\mathbf{w} = \Sigma^{-1}[\mathbf{m}\ \mathbf{1}]\, A^{-1} \begin{pmatrix} m_0 \\ 1 \end{pmatrix}.$$

2. **Computation of σ^2 as a function of m_0:** Prove that

$$\sigma^2(m_0) = \frac{A_{11} - 2A_{12}m_0 + A_{22}m_0^2}{|A|},$$

where $A = [\mathbf{m}\ \mathbf{1}]^T \Sigma^{-1} [\mathbf{m}\ \mathbf{1}]$.

3. **Effect of weights:** For $n = 3$, write $\sigma^2(w_1, w_2, w_3)$ and minimize it without the constraint $w_i \geq 0$ using derivatives.

4. **Computation of VaR:** Given $V_0 = 1\,000$, $m_0 = 0.001$, $\sigma = 0.02$ and $a = 0.99$, compute the one-day $\mathrm{VaR}_{0.99}$.

5. **Extension with costs:** Describe how you would modify the problem $\min \mathbf{w}^T\Sigma\mathbf{w}$ to account for transaction costs in the formula for σ^2.

Python-Based Exercises

1. **Computation of efficient frontier:** Write Python code with numpy and scipy.optimize that, for given $\mathbf{m}, \boldsymbol{\Sigma}$, computes the efficient frontier for $m_0 \in [m_{\min}, m_{\max}]$.

2. **Minimum variance with constraints:** Implement in Python the optimization problem $\min \mathbf{w}^T\Sigma\mathbf{w}$ subject to $\sum w_i = 1$, $w_i \geq 0$, using scipy.optimize.linprog.

3. **Computation of VaR:** Write a Python function that takes V_0, m_0, σ, a and returns VaR_a, using the corresponding quantile from scipy.stats.norm.

Chapter 7

Generalizing the Markowitz Theory

Note that this scenario, which originates from Markowitz's portfolio theory, resulted from a statistical study of past numerical data without taking into account recent events. This means that while the data may indicate a downward trend, the investor's forecast could be opposite, relying on recent events or other information.

In the Markowitz theory, it was not taken into account that the investor might purchase certain call and put options. Therefore, the only thing we can do is calculate the probability of bankruptcy, specifically according to the forecasts that have been made or, alternatively, with respect to the probability measure and the distribution of the chosen random variables. This implies that for the same portfolio, different investors may end up with different bankruptcy probabilities because different forecasts have been made!

In other words, if a portfolio has been constructed consisting of n stocks, then different bankruptcy probabilities will be associated with it by various financial analysts, specifically lying throughout the interval $(0, 1)$.

The above discussion essentially encourages us to equip our portfolios with appropriate call and put options, if available. This will lead, on the one hand, to a more complex mathematical problem but, on the other hand, to more efficient portfolios (from a financial point of view) and safer results.

Therefore, to construct a portfolio consisting of n stocks, the most profitable scenario should be forecast in advance. For example, the investor's forecast may be as follows:

$$(S_1^T, S_2^T, \cdots, S_n^T) \in G \subseteq \mathbb{R}_+^n$$

This implies that the forecast must be made for the stock prices at time T. Therefore, the portfolio construction problem can be transformed into an optimization problem with a solution that proposes the appropriate portfolio with the maximum profit for the specific scenario.

`Lab` **43 (Computing an Appropriate Region G)** *Using historical data and appropriate statistical analysis, we can compute a region G for which the probability that the stock value vector lies inside is the desired one. One can use the code* ***`Two-Assets-ComputingG.ipynb`****. The conclusion one reaches is the following: regardless of the method used to compute region G, if that* *method does not also use historical events to match them to the corresponding numerical data, it will always be far behind the way experienced investors make their forecasts.*

`Problem` **44** *Suppose the investor predicts that the values of two stocks will lie in a region* $G \subseteq \mathbb{R}^2_+$. *We are interested in constructing a portfolio that is profitable in this scenario and at the same time the maximum possible loss in the bad scenario does not exceed the amount* D. *Then we can solve the following linear programming problem:*

$$\begin{aligned} &\max && \sum w_{ij} D_{ij} \\ &\textit{subject to} && \Pi(x,y) \geq -D \textit{ for all } x,y \geq 0 \\ & && \Pi(U_i,V_j) \geq D_{ij} \geq 0 \\ & && ax + by + ce^{rT} + \sum \gamma_i P_i = Y \end{aligned} \tag{7.1}$$

where P_i *are the values of the available contracts.*

`Lab` **45 (Solving Problem 7.1)** *We can use the code* ***MultiAsset4.ipynb*** *to solve problem 7.1 for two stocks. Use also the code* ***Markowitz-Portfolio-Profit-Function-2Assets.ipynb*** *to see the profit function for a portfolio with two assets. The profit function shows us the scenario, that is, the set of values, under which our portfolio will be profitable. We see that, in this case, the payoff function is not as easily adaptable as when we equip the portfolio with options contracts. This is evident, since increasing the number of parameters leads to better calibration. Note that this scenario resulted from a statistical study of past numerical data without taking into account recent events. This means that while the data may indicate a downward trend, the investor's forecast could be opposite, relying on recent events or other information.*

Note that the smaller the set G, the larger the profit for the investor, in the case where the scenario is realized. If the investor chooses $G = \mathbb{R}^n_+$, then for the proposed portfolio there may exist a certain riskless profit (arbitrage), which however may be very small, even smaller than the corresponding transaction costs.
A different type of forecast is through the distribution followed by the vector $(S_1^T, S_2^T, \cdots, S_n^T)$. Based on this distribution, a portfolio will be constructed where it is possible to maximize a quantity that may contain, for example, the expected value of the profit, the variance, etc.
Thus, the investor can solve the following optimization problem:

$$\begin{aligned} &\min && \mathrm{VaR}(\Pi(S_1^T, S_2^T, \cdots, S_d^T)) \\ &\text{subject to} && \mathbb{E}(\Pi(S_1^T, S_2^T, \cdots, S_d^T)) \geq m_0 \\ & && \sum_{i=1}^{d} a_i S_i^0 + \sum \gamma_j C_j = V \\ & && \Pi(x_1, \cdots, x_d) \geq -D \ \forall (x_1, \cdots, x_d) \in \mathbb{R}^d_+ \end{aligned} \tag{7.2}$$

where C_j are all the available call and put options corresponding to the d stocks. This portfolio will, in general, have lower variance and higher mean than in the case where the available options are not taken into account, and additionally, it is certain that the loss will not exceed the amount D.
The last inequality is equivalent to a finite set of linear inequalities, exactly as in the single-stock case. All the points where the function Π changes behavior must be taken into account, as well as that the partial derivatives in each direction must be positive.
Therefore, the value at risk may prove to be entirely useless, especially when there are many available contracts. If there are no derivatives available in the market, then the above problem reduces to that of the Markowitz theory.

`Remark` **46 (If past numerical data exhibit a declining trend...)** *Solving the above optimization problem yields a portfolio that is profitable on a set* $A \subseteq \mathbb{R}^n_+$ *and, in general, incurs a loss on its complement. This is equivalent to predicting that the stock price will lie within the set* A *and constructing the corresponding portfolio. In this case, however, the distribution chosen by the investor is based on past numerical data. For instance, these data may indicate a downward trend in the stock price, but this does not necessarily imply that the same will occur in the following period. On the contrary, by evaluating recent events, the investor may predict that the stock price will follow an upward trajectory and, therefore-contrary to historical data-construct the corresponding portfolio.* □

In all the above cases, transaction costs, the bid-ask spread, dividends, and similar factors must be incorporated.

`Remark` **47 (The Markowitz theory is outdated)** *The methodology of the Markowitz framework produces profitable portfolios only under highly specific circumstances. Such a portfolio is profitable when the stock price increases; however, it would be even more profitable if certain call options were purchased. Similarly, when the stock price falls, it would be preferable to have purchased put options. More critically, the entire methodology relies exclusively on historical numerical data, ignoring all other recent and potentially relevant information.*
Conversely, the above portfolio construction approach can generate profitable portfolios for any market scenario the investor chooses to predict. This is feasible provided the portfolio includes, in addition to stocks, all derivatives that may be bought or sold. From this perspective, the proposed construction methodology generalizes the Markowitz theory. The only reason for an investor to disregard the full range of available financial instruments in solving the optimization problem is the lack of the necessary computational tools to carry out such calculations. Indeed, this is often the case, as the relevant literature is outdated, and consequently so is the corresponding freely available software. □

`Remark` **48 (Ensure adequate protection!)** *Assume you have decided to invest an amount* V *across* d *different companies, using a Markowitz-style approach to determine the portfolio composition. The calculation of the probability of default is based on the following:*

- *The assumption that the relevant random variables follow a specific distribution with specified parameters.*
- *The parameters are estimated from a given historical dataset, e.g., a past time interval* $[T_1, T_2]$.

A different choice of the historical time window will, of course, yield different parameters. These parameters are therefore functions of the historical price trajectory used. Do you truly believe that the future price path depends on the past price path-or perhaps on future events? Remember: you will be committing real capital, and any loss will be real. A deeper reflection leads to the conclusion that it is preferable to purchase some put options per stock (instead of relying solely on Value at Risk), along with some call options to maximize potential gains. The question remains: how many of each? □

`Example` **49** *Suppose we wish to invest 1.000 in shares of Tesla and Apple, and that the corresponding Markowitz portfolio consists of 300 in Apple and 700 in Tesla. Following the approach described above, we might compute the probability of a loss greater than 600 at the end of one period.*
Let us now examine the effect of purchasing some put options prior to the above calculation. Suppose that:

$$P^{Tesla}(135) = 4, \quad P^{Apple}(120) = 1.25$$
$$C^{Tesla}(300) = 34.3, \quad C^{Apple}(240) = 3$$

while the current stock prices are $S_0^{Tesla} = 269$ and $S_0^{Apple} = 198$. We may invest 600 in Tesla shares (instead of 700) and 200 in Apple shares (instead of 300), allocating the remainder to call and put options. This allocation is, of course, arbitrary and may be adjusted as desired.
The profit function is then:

$$\begin{aligned}\Pi(x,y) &= \frac{600}{269}x + \frac{200}{198}y \\ &+\frac{60}{4}(135-x)^+ + \frac{60}{1.25}(120-y)^+ \\ &+\frac{40}{34.3}(x-300)^+ + \frac{40}{3}(y-240)^+ - 1{,}000\end{aligned}$$

At maturity T, x represents the Tesla share price and y the Apple share price. Minimizing the profit function yields the maximum possible loss.
It can be verified that in the worst-case scenario, the loss does not exceed 578, while the maximum profit for $x \leq 400, y \leq 400$ is 2686. Thus, by purchasing call options we amplify potential profits, and by purchasing put options we limit the maximum loss to well below the initial 1000 investment. This suggests that the investor should first decide the maximum acceptable loss and the desired profit scenario, and then determine the optimal allocation of the 1000. This process leads to infinitely many optimization problems, depending on the investor's criteria-much like constructing a portfolio consisting solely of the shares of a single company. For portfolios involving two or more companies, the criteria become even more numerous and complex, as price correlations may also be incorporated.
In the portfolio constructed above, calculating the probability of losing more than 578 is clearly unnecessary. It may be meaningful, however, to estimate the probability of losing more than 400, etc., while also considering the opportunity cost of time. □

`Conclusion` **1** *Ultimately, the investor must forecast both the approximate future stock prices and the maximum acceptable loss in order to determine the optimal portfolio.*

`Remark` **50 (Financial and Actuarial Mathematics)** *By incorporating appropriate call and put options into the portfolio, one transfers part of the risk to other investors. In actuarial science, risk transfer is achieved through reinsurance. Just as in the Markowitz framework it is meaningless to compute the probability of default without first selecting an appropriate hedging strategy via call and put options, in insurance mathematics it is meaningless to compute similar probabilistic quantities without first determining the reinsurance strategy.*

`Remark` **51 (Markowitz conclusions evolve over time)** *Suppose we construct a portfolio using the Markowitz theory. After some time, incorporating updated data, the newly constructed portfolio will have a different composition. Obviously, the mean returns and correlations depend on the dataset, which will differ at another point in time. If they were constant, they would be global invariants. This observation implies that the Markowitz approach (and similar ones) should be applied only over short time intervals, so that the relevant quantities remain relatively stable.*

`Remark` **52 (More effective forecasting techniques)** *For short-term investments, one should consider not only historical data but also recent events and their expected impact on the asset's price. Here, machine learning can serve as a valuable tool. For long-term investments, one should assess the company's robustness, product quality, competitive landscape, and related indicators. These measures provide a forward-looking view of the company's prospects, although interim price fluctuations will still occur.*
In no case will over-analysis of past numerical data-without evaluating recent events and other indicators-provide meaningful additional insight. Such an approach is simply a waste of time and resources.

Question 3 *What other optimization problems can be formulated by considering a portfolio construction problem involving n assets together with their call and put options? How can these optimization problems be solved?*

Remark 53 **(Two types of predictions)** *In general, when constructing a portfolio composed of many stocks, two types of predictions can be made:*

- ***Distribution-based prediction:*** *Estimating the joint distribution of stock prices over a given time horizon. This allows us to solve an optimization problem that minimizes a quantity involving variance, mean, or other moments of the profit.*
- ***Set-based prediction:*** *Estimating a set* $G \subseteq \mathbb{R}^d$ *(where d is the number of companies) in which the investor predicts the stock prices will lie. Solving a corresponding optimization problem then yields the appropriate portfolio.*

Adding options for each company clearly increases portfolio diversification. If the maximum possible loss (denoted D) is not sufficiently small, we can compute the probability that the loss will exceed a specified threshold.

7.1 Separating the forecasting process from portfolio construction

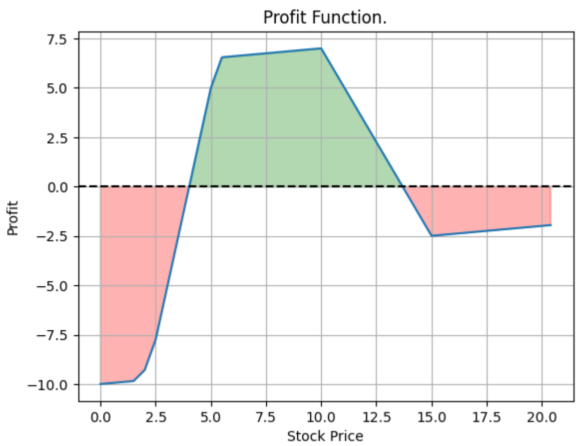
Our approach essentially separates the forecasting problem from that of constructing the optimal portfolio. The latter belongs to the domain of financial mathematics, while the former is the domain of predictive modelling techniques (e.g., machine learning). This separation allows the investor to apply advanced forecasting methods and subsequently employ the above methodology for optimal portfolio construction. As noted earlier, the Markowitz method relies on outdated forecasting techniques and, more critically, does not incorporate available options contracts in constructing the optimal portfolio.

7.2 A new type of multi - asset option: Options on Correlation

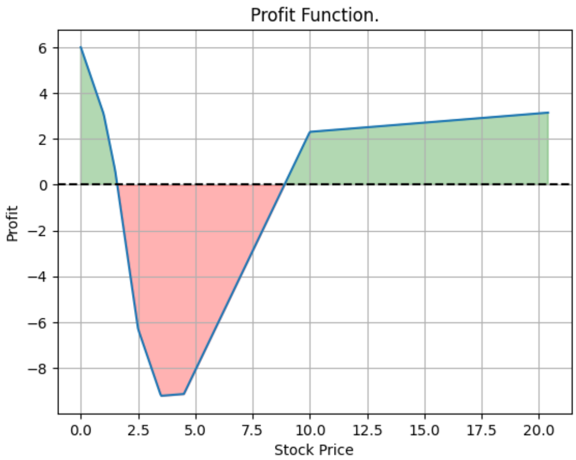
Let two assets S_1, S_2 and the corresponding call and put options with strike prices $K_1 < \cdots < K_n$ and $L_1 \cdots L_m$ respectively. As we can see in fig. 7.1a-fig. 7.1d the call and put options are enough to manipulate the profit function as we want. But in two dimensions (see fig. 7.1e) they are not. If we want to refine the profit function more efficiently, it would be easier if the following contracts (options on correlation) existed,

$$
\begin{aligned}
P_{CC}(K_i, L_j) &= \min\{(S_1^T - K_i)^+, (S_2^T - L_j)^+\}, \quad \text{for } i = 1, \cdots, n, j = 1, \cdots, m \\
P_{CP}(K_i, L_j) &= \min\{(S_1^T - K_i)^+, (L_j - S_2^T)^+\}, \quad \text{for } i = 1, \cdots, n, j = 1, \cdots, m \\
P_{PC}(K_i, L_j) &= \min\{(K_i - S_1^T)^+, (S_2^T - L_j)^+\}, \quad \text{for } i = 1, \cdots, n, j = 1, \cdots, m \\
P_{PP}(K_i, L_j) &= \min\{(K_i - S_1^T)^+, (L_j - S_2^T)^+\}, \quad \text{for } i = 1, \cdots, n, j = 1, \cdots, m
\end{aligned}
$$

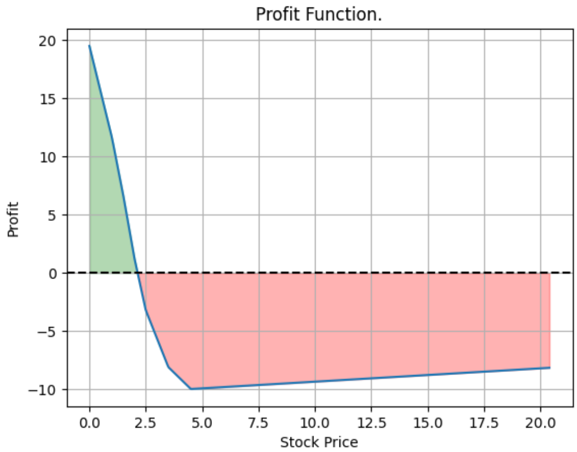
where P_{CC} etc are the payoff functions.
To be more precise, suppose the investor places a bet on the event $(S_1^T, S_2^T) \in G \subseteq \mathbb{R}_+^2$. If the investor only has access to call and put options on the underlying assets, the resulting profit function will be positive in the region G, but it may also be positive in other regions. This



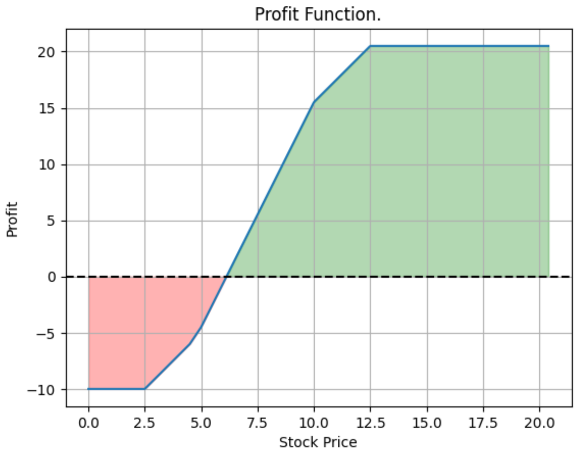
(a)



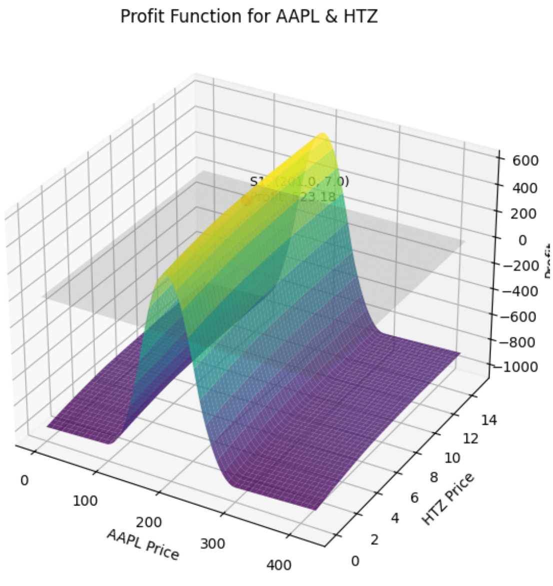
(b)



(c)



(d)



(e)

Figure 7.1

Figure 7.2: These graphs show the profit functions of various portfolios. We observe that it is possible to construct portfolios that are profitable under any chosen scenario. The selection of the preferred scenario follows from the application of an appropriate forecasting technique.

implies that the profit generated within G will be diluted compared to what could be achieved if the profit function were concentrated exclusively on G.
By utilizing options on correlation, however, we can construct a portfolio such that the profit function is strictly positive within G while maximizing the potential profit. Thus, options on correlation are not only of financial and economic interest for trading purposes but also present a mathematical challenge related to the optimal design of a portfolio.

Arbitrage Considerations and Pricing

How should such contracts be priced? Using standard no-arbitrage reasoning, we can easily deduce that, for example, the price of a contract with payoff $P_{CC}(K_i, L_j)$ must be lower than the prices of the individual call options with strike prices K_i and L_j. Otherwise, the writer of the contract could construct a portfolio that guarantees a riskless profit. For a more accurate bounds one should compute the arbitrage-free interval that we describe at 9.4.
It is important to note that if one attempts to price such an option using any of the classical option pricing models, the resulting value will, in general, lead to an arbitrage opportunity. This is because these models do not take into account the existing prices of related call and put options in the market. This limitation applies broadly to all multi-asset options.
The only method that can consistently yield arbitrage-free and fair prices is the one we propose in [2]; see also section **??** below for more details.

Motivation for Writing these Multi-Asset Contracts

Why would someone wish to write a contract with payoff $P_{CC}(K_i, L_j)$? A possible motivation is that the writer believes there is virtually no chance that the prices of the two underlying assets will increase simultaneously.
Conversely, an investor may believe that the two assets are sufficiently positively correlated to justify purchasing contracts with payoffs P_{PP} and P_{CC} for suitable strike prices, anticipating joint movements in the asset's values.
Of course, these type of options can be generalized concerning d-assets.
In this direction one should study the correlation swaps.
Further details on the portfolio construction problem can be found in [**?**], [4] and the references therein.

7.3 Exercises

True or False

1. Statement: Incorporating call and put options into portfolio optimization always reduces the variance of returns.
   True / False: _______ Justification: ______________________________________

2. Statement: The choice of region G has no impact on the maximum profit achievable by the portfolio.
   True / False: _______ Justification: ______________________________________

3. Statement: If $G = \mathbb{R}^n_+$, the portfolio may exhibit arbitrage opportunities, but these profits are often negligible after accounting for transaction costs.

True / False: _______ Justification: ____

4. Statement: The inclusion of derivatives invalidates the use of Value-at-Risk (VaR) as a risk measure.
True / False: _______ Justification: ____

5. Statement: In Problem (7.2), the expected profit constraint ensures that the portfolio achieves at least a minimum return m_0.
True / False: _______ Justification: ____

Multiple Choice

1. In the generalized Markowitz framework, which factor primarily determines the efficiency of a portfolio when options are included?

   (A) Expected return.
   (B) Variance of returns.
   (C) Forecast accuracy of stock price scenarios.
   (D) Number of available derivative contracts.

   Answer: _______

2. What is the role of the region G in portfolio construction?

   (A) It specifies the range of stock prices where the portfolio is profitable.
   (B) It determines the maximum allowable loss.
   (C) It represents the distribution of stock prices.
   (D) It defines the set of feasible portfolios.

   Answer: _______

3. Which of the following best describes the impact of including call and put options in a portfolio?

   (A) Increases both expected return and risk.
   (B) Reduces risk while maintaining or increasing expected return.
   (C) Eliminates the need for forecasting stock prices.
   (D) Simplifies the optimization problem.

   Answer: _______

4. In Problem (7.1), what does the term $\Pi(U_i, V_j) \geq D_{ij}$ represent?

   (A) A constraint ensuring non-negative profits in specific scenarios.
   (B) A condition limiting the maximum loss.
   (C) A requirement for risk-free arbitrage.
   (D) A penalty for exceeding transaction costs.

   Answer: _______

5. Which of the following is NOT a benefit of using options in portfolio optimization?
   - (A) Hedging against downside risk.
   - (B) Increasing potential upside gains.
   - (C) Guaranteeing positive returns.
   - (D) Enhancing portfolio diversification.

   Answer: _______

Matching

Match the description with the correct concept.

1. Region G	A. Specifies the range of stock prices considered in forecasts
2. Portfolio variance	B. Measure of risk in portfolio optimization
3. Call and put options	C. Tools for hedging and enhancing returns
4. Arbitrage	D. Riskless profit opportunity

Answers: 1._____ 2._____ 3._____ 4._____

Fill in the Blank

1. The parameter D in Problem (7.1) represents the ____________________.
2. The region G is computed using ____________________ and statistical analysis.
3. The inclusion of derivatives introduces ____________________ into the optimization problem.
4. The variance minimization problem balances ____________________ and expected return.
5. The function $\Pi(x_1, \ldots, x_d)$ represents the ____________________ of the portfolio.

Theoretical and Computational Exercises

1. **Impact of G:** Show mathematically how reducing the size of G increases the potential profit of the portfolio, assuming the predicted scenario materializes.
2. **Portfolio Variance:** Derive the expression for portfolio variance when call and put options are included, and compare it to the classical Markowitz case.
3. **Optimization Formulation:** Reformulate Problem (7.2) to include additional constraints, such as a limit on the number of derivative contracts.
4. **Simulation Analysis:** Simulate stock price paths under geometric Brownian motion and solve Problem (7.1) numerically for different values of G.
5. **Transaction Costs:** Extend Problem (7.1) to account for proportional transaction costs c and analyze their impact on portfolio performance.

Python-Based Exercises

1. **Computing G:** Implement the code ‘Two-Assets-ComputingG.ipynb‘ to compute the region G for two stocks using historical data.
2. **Solving Problem** (7.1)**:** Use the notebook ‘MultiAsset4.ipynb‘ to solve the linear programming problem for two stocks and analyze the results.

3. **Monte Carlo Simulation:** Write a Python script to simulate stock price paths and evaluate the performance of portfolios constructed using different regions G.

Chapter 8

Dynamic Trading

Dynamic trading is inherently risky because the future bid–ask spread is undefined and unpredictable. Even with accurate models for asset prices, the uncertainty in transaction costs can significantly affect trading outcomes.

In this section, we describe a dynamic trading strategy based on the principle sell high - buy low (and borrow high - repay low). We assume that an investor has decided to spend an amount Y to purchase a shares of a company, borrowing an amount b from their own bank account at a (risk-free) interest rate r.
We also assume that the investor has decided to rebalance their portfolio at times $0 < t_1 < t_2 < \cdots < t_N = T$, and that the current price of the asset is S_0. We construct a sequence a_k representing the number of shares held at each time t_k, with $a_0 = a$.
If $a > 0$, the investor can buy or sell at time t_k a number of shares so that

$$a_k = a_{k-1} - a_{k-1}\delta\left(\frac{S_{t_k}}{S_0}\right) + (a - a_{k-1})\mathbb{I}_{\{S_0 > S_{t_k}\}} \tag{8.1}$$

while if $a < 0$,

$$a_k = a_{k-1} - a_{k-1}\delta\left(\frac{S_0}{S_{t_k}}\right) + (a - a_{k-1})\mathbb{I}_{\{S_{t_k} > S_0\}} \tag{8.2}$$

For example, we can choose $\delta(x)$ as

$$\delta(x) = \begin{cases} \frac{z_1(x-1)^{z_2}}{z_1(x-1)^{z_2}+z_3}, & \text{when } x > 1, \\ 0, & \text{otherwise,} \end{cases}$$

for some $z_1, z_2, z_3 > 0$. There are, of course, infinitely many possible choices for the function δ, and the choice must be made by the investor. For example, one could choose

$$\delta(x) = \begin{cases} 1, & \text{for } x > M, \\ p(x), & \text{for } x \in (1, M), \\ 0, & \text{for } x \leq 1, \end{cases}$$

where $p(x) = a_n x^n + \cdots + a_0$ is such that $p(M) = 1$ and $p(1) = 0$.

`Question 4` *Does the choice of the optimal function $\delta(x)$ depend on the price of the asset, or is there a function that achieves the best performance for any such price behavior?*

If transaction costs are taken into account, we must choose a suitable $\varepsilon > 0$ and define $\delta(x)$ as follows:

$$\delta(x) = \begin{cases} \frac{z_1(x-(1+\varepsilon))^{z_2}}{z_1(x-(1+\varepsilon))^{z_2}+z_3}, & \text{when } x > 1+\varepsilon, \\ 0, & \text{otherwise,} \end{cases}$$

Assume that at time t the price S_t is below S_0, that is $S_t < S_0$. If the investor buys some shares at this time that they had previously sold (if any), they can reset S_0 to a smaller value.
We can also construct a different trading strategy by setting, when $a > 0$:

$$a_k = a_{k-1} - a_{k-1}\delta\left(\frac{S_{t_k}}{S_0}\right) \tag{8.3}$$

and when $a < 0$:

$$a_k = a_{k-1} - a_{k-1}\delta\left(\frac{S_0}{S_{t_k}}\right) \tag{8.4}$$

Finally, a similar strategy can be defined by setting, when $a > 0$:

$$a_k = a_{k-1} - a_{k-1}\delta\left(\frac{S_{t_k}}{S_0}\right) + (a - a_{k-1})\delta\left(\frac{S_0}{S_{t_k}}\right)$$

and when $a < 0$:

$$a_k = a_{k-1} - a_{k-1}\delta\left(\frac{S_0}{S_{t_k}}\right) + (a - a_{k-1})\delta\left(\frac{S_{t_k}}{S_0}\right)$$

In this trading strategy, one can choose different δ functions for each case.
Therefore, at time T, the portfolio value is given by (considering, for example, the trading strategy (8.3) and (8.4) and including transaction costs):

$$\Pi = a_N S_T + \sum_{k=1}^{N} S_{t_k}(1-\varepsilon)(a_k - a_{k-1})e^{r(T-t_k)} + be^{rT}$$

`Question 5` *For the choice of the optimal trading strategy, one can assume that the price of the asset follows a stochastic differential equation (SDE). It would be useful to apply numerical schemes that preserve positivity in this SDE. Then we must formulate a suitable optimization problem to find the best trading strategy, i.e., to determine the optimal parameters of the function δ.*
The above study can also be carried out for a portfolio of n assets, assuming that each asset price follows an SDE and taking into account possible correlations. Our goal is not only to find the best parameters of δ but also to determine how often the portfolio should be rebalanced.
An interesting mathematical problem is to consider dynamic trading in a portfolio containing n assets together with all related call and put options. Solving this problem requires tools from SDEs, stochastic analysis, and numerical analysis of SDEs.

`Lab` **54 (Parameters for Dynamic Trading)** *Using the code* ***`Pricing-and-Hedging-Lookback-Options-Sell-High-and-Binomial.ipynb`****, we can compute suitable parameters for dynamic trading. You should choose the amount Y in place of the payoff function. This code was originally designed for pricing and hedging of path-dependent options. With the above modification, it can be used for computing parameters for dynamic trading as well.*

8.1 Exercises

True or False

1. Statement: In strategy (8.3)-(8.4), the function $\delta(x)$ determines when rebalancing occurs.
   True / False: ________ Justification: ____________________

2. Statement: Choosing a larger $\varepsilon > 0$ reduces the number of trades but increases the tracking error.
   True / False: ________ Justification: ____________________

3. Statement: When $a < 0$, equations (8.2) and (8.4) are equivalent if $\delta(x)$ is symmetric.
   True / False: ________ Justification: ____________________

4. Statement: The parameter r (risk-free rate) does not affect the dynamic choice of a_k.
   True / False: ________ Justification: ____________________

5. Statement: Strategy (8.1) ensures profit if $S_{t_k} > S_0$ for all k.
   True / False: ________ Justification: ____________________

Multiple Choice

1. In the dynamic strategy, which term in $\Pi = a_N S_T + \sum_{k=1}^{N} S_{t_k}(1-\varepsilon)(a_k - a_{k-1})e^{r(T-t_k)} + be^{rT}$ represents the transaction cost?

   (A) $a_N S_T$

   (B) $\sum_{k=1}^{N} S_{t_k}(a_k - a_{k-1})e^{r(T-t_k)}$

   (C) The parameter ε inside $(1-\varepsilon)$

   (D) be^{rT}

   Answer: ________

2. Choosing $\delta(x)$ with $x = S_{t_k}/S_0$ in (8.3) determines:

   (A) How much sample data is used for backtesting.

   (B) The size of the position closed or opened at each t_k.

   (C) The interest rate r.

   (D) Only the rebalancing frequency.

   Answer: ________

3. Increasing the number of rebalances N is likely to:

   (A) Reduce S_T.

   (B) Increase transaction costs.

(C) Reduce a_0.

(D) Increase r.

Answer: ________

4. In the original strategy (8.1), the term $(a - a_{k-1})\mathbb{I}_{\{S_0 > S_{t_k}\}}$ means:

(A) Reinvesting profits when the price drops.

(B) Resetting to the initial position a when the price is lower.

(C) Selling all shares when the price rises.

(D) Buying a new option.

Answer: ________

5. What is the main difference between (8.1) and (8.3)?

(A) The first has an additional $\mathbb{I}$ term; the second does not.

(B) Only the second takes into account the interest rate r.

(C) The first uses S_0/S_{t_k}, the second S_{t_k}/S_0.

(D) The second requires $\delta(x) \equiv 1$.

Answer: ________

Matching

Match the description with the correct strategy or concept.

1. $a_k = a_{k-1} - a_{k-1}\,\delta(S_{t_k}/S_0)$	A. Simple dynamic rebalancing
2. $a_k = a_{k-1} - a_{k-1}\,\delta(S_0/S_{t_k})$	B. Short-sell adjustment
3. Where $\varepsilon > 0$ in $(1 - \varepsilon)$	C. Transaction cost buffer
4. $(a - a_{k-1})\mathbb{I}_{\{S_0 > S_{t_k}\}}$	D. Reset to initial position

Answers: 1._____ 2._____ 3._____ 4._____

Fill in the Blank

1. The parameter ε in the model is introduced to cover ____________________.

2. In strategy (8.1), the indicator $\mathbb{I}_{\{S_0 > S_{t_k}\}}$ signals ____________________.

3. The term $a_k - a_{k-1}$ in the cost term is the ____________________ of the position.

4. Choosing parameters z_1, z_2, z_3 in $\delta(x)$ determines ____________________.

5. The term be^{rT} in the payoff represents ____________________.

Theoretical and Computational Exercises

1. **Analysis of δ-forms:** Show that for $\delta(x) = \frac{z_1(x-1)^{z_2}}{z_1(x-1)^{z_2}+z_3}$, the parameter $z_2 > 1$ accelerates selling after a large price increase.

2. **Cost optimization:** Formulate an optimization problem for choosing $(\varepsilon, z_1, z_2, z_3)$ to maximize $\mathbb{E}[\Pi]$ subject to $\mathrm{Var}(\Pi) \le \sigma_0^2$.

3. **Strategy comparison:** Numerically compute (e.g., with Python) the performance of two strategies: simple rebalancing vs. full (8.1), for a sample of S_t paths.

4. **Effect of N:** Show theoretically how increasing the number of rebalances N reduces tracking error but increases total cost.

5. **Stochastic modeling:** Assume S_t follows GBM $dS_t = \mu S_t dt + \sigma S_t dW_t$. Write the recursive relation for the optimal a_k given $\{S_{t_k}\}$.

6. **Dynamic programming:** Formulate the dynamic programming problem to find the optimal $\delta(x)$ when there are N rebalances, per-trade costs $c > 0$, and discount factor $e^{-r\Delta t}$.

Python-Based Exercises

1. **Backtest engine:** Implement in Python a function that takes $\{S_{t_k}\}$ and a $\delta(x)$ parameter and returns the sequence a_k and the payoff Π.

2. δ **parameters:** Using `scipy.optimize`, find $(z_1, z_2, z_3, \varepsilon)$ that maximize the mean Π over simulated GBM paths.

3. **Frequency N:** Write a script to compare average performance for $N \in \{5, 10, 20, 50\}$ and present the results in a matplotlib plot.

Chapter 9

Option Pricing

Therefore, if you want to design a mathematical based option pricing method, i.e., to compute a fair and arbitrage-free price, do not force it to match the prices that will be observed in the market. Investors neither know nor think according to your method (or model)!

Suppose one intends to buy or sell a call option. A natural question arises: is there a fair price for this option? This question is known as the option pricing problem. In addressing it, we shall examine two classical frameworks: the binomial model and the *Black-Scholes* model. As will become evident, both approaches face significant limitations when applied in practice. We will therefore develop an alternative theory that is not only theoretically sound but also directly implementable.
It is essential to clarify that the pricing of goods is governed by the fundamental law of supply and demand. However, the presence of arbitrage opportunities also exerts a significant influence on this process. Specifically, when market prices generate such opportunities, arbitrageurs will promptly exploit them. Consequently, over time, prices tend to adjust and converge toward an equilibrium range in which arbitrage possibilities are eliminated.
A contract's seller or buyer will typically seek an approximate valuation prior to entering the bargaining stage.

`Definition` **55 (Order of Magnitude of the Price)** *Order of magnitude of the price: the interval of prices at which two expert investors would willingly act as buyer and seller.*

As we shall demonstrate, the arbitrage-free interval, when it exists, provides a natural starting point. Beyond this, we define a fair value, denoted by Y^{D^*}, which is arbitrage-free by construction, assuming such prices exist, and prove that this value is the only one that is fair and arbitrage free by construction. Hence, Y^{D^*} serves as a canonical reference point for initiating the bargaining process.

9.1 Binomial Model

The binomial model is based on constructing a hedging portfolio in discrete time. Our goal is to construct a portfolio at time zero such that at time T it has the same value as the payoff of the option. More details about the option pricing theory according to the binomial model can be found in [3] and the references therein.

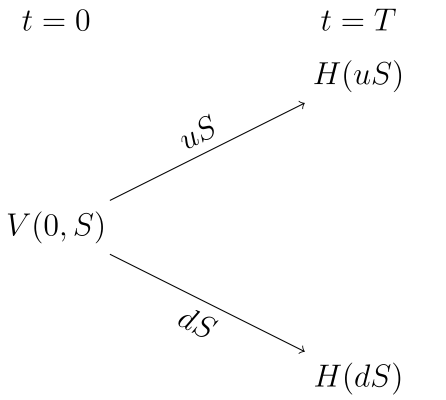


Figure 9.1: One-period binomial model.

In the binomial model, we assume that the stock price will either go up by a factor u with probability p or go down by a factor d with probability $1 - p$. The portfolio value at time 0 is $V_0 = aS + b$, where a is the number of shares and b is the amount in the risk-free investment. If the payoffs at time T are assumed to be $H(uS)$ and $H(dS)$, then

$$a = \frac{H(uS) - H(dS)}{(u-d)S} \quad \text{and} \quad b = \frac{H(dS)u - H(uS)d}{(u-d)e^{rT}}$$

where r is the risk-free interest rate. The above can be generalized to n periods.

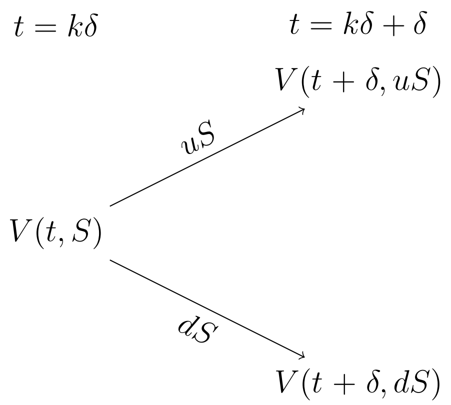


Figure 9.2: Multi-period binomial model.

If the portfolio values at time $t = k\delta + \delta$ are assumed to be $V(t + \delta, uS)$ and $V(t + \delta, dS)$, then the minimum portfolio value at $t = k\delta$ is

$$V(t, S) = e^{-r\delta} \left(q\, V(t + \delta, uS) + (1 - q)V(t + \delta, dS)\right)$$

where $q = \frac{e^{r\delta} - d}{u - d}$ when $d \leq e^{r\delta} \leq u$. Notice that $V(t, S) = aS + b$, where a is the number of shares at time t and b is the amount in the risk-free investment at time t with continuous interest rate r.

By working backwards, the initial value of the hedging portfolio and the investment strategy (a, b) can be computed. This construction is practically feasible but has one major drawback.

Remark 56 *In the binomial model, we have assumed that the next stock price will be either uS or dS if today it is S. This prediction can never be verified, so the final value of the hedging* *portfolio will certainly not match the payoff almost surely. In fact, the situation gets even worse as more periods are added to the binomial model (see Theorem 59 below). These problems stem from the assumption that the stock can take only two values in the next period.*

`Question` **6** *If the stock price rises at a rate $u^* \neq u$, what is the actual consequence for the portfolio value and profits? Similarly, what happens if it falls at a rate $d^* \neq d$?*

`Lemma` **57** *Assume the writer has applied the one-period binomial model to price a call or put option with strike K. Then they will choose rates d, u such that $uS_0 > K$ and $dS_0 < K$.*

`Proof.` For the call option, the writer will prefer d such that $dS_0 < K$ (and of course $uS_0 > K$). Indeed, if $dS_0 > K$ then it is easy to see that $a = 1$ and $be^{rT} = -K$. Thus the profit Π will be

$$\Pi = aS_T + be^{rT} - (S_T - K)^+ = S_T - K - (S_T - K)^+.$$

If $S_T > K$ then $\Pi = 0$, while if $S_T \leq K$ then $\Pi \leq 0$, meaning selling will never yield a profit. Similarly, for the put option they will choose u such that $uS_0 > K$ (and $dS_0 < K$). If $uS_0 < K$ then $a = -1$ and $be^{rT} = K$, giving

$$\Pi = aS_T + be^{rT} - (K - S_T)^+ = K - S_T - (K - S_T)^+.$$

If $S_T < K$ then $\Pi = 0$, while if $S_T \geq K$ then $\Pi \leq 0$ - no profit in either case. □

`Theorem` **58** *If the writer uses the one-period model to price a call or put option with strike K and rates d, u as above, then they will profit if $\frac{S_T}{S_0} \in (d, u)$, and incur a loss if $\frac{S_T}{S_0} \notin (d, u)$. The possible profit is bounded by $\frac{(uS_0 - K)(K - dS_0)}{(u-d)S_0}$, while the possible loss is unbounded.*

`Proof.` Consider a call with $a \in (0, 1)$, $uS_0 > K$, and $dS_0 < K$. If $S_T > K$ (so $\frac{S_T}{S_0} > d$), then

$$\Pi = a\left(\frac{S_T}{S_0} - u\right)S_0 + auS_0 + be^{rT} - (uS_0 - K) - \left(\frac{S_T}{S_0} - u\right)S_0$$

and since $auS_0 + be^{rT} - (uS_0 - K) = 0$, we have

$$\Pi = \left(\frac{S_T}{S_0} - u\right)S_0(a - 1).$$

Thus if $\frac{S_T}{S_0} \in (d, u)$, $\Pi > 0$, while if $\frac{S_T}{S_0} > u$, $\Pi < 0$ (and $\Pi \to -\infty$ as $\frac{S_T}{S_0} \to \infty$).
If $S_T < K$ (so $\frac{S_T}{S_0} < u$), then

$$\Pi = a\left(\frac{S_T}{S_0} - d\right)S_0 + adS_0 + be^{rT}$$

and since $adS_0 + be^{rT} = 0$, we have $\Pi > 0$ if $\frac{S_T}{S_0} \in (d, u)$, and $\Pi < 0$ if $\frac{S_T}{S_0} < d$. The same applies to the put option when $a \in (-1, 0)$. □

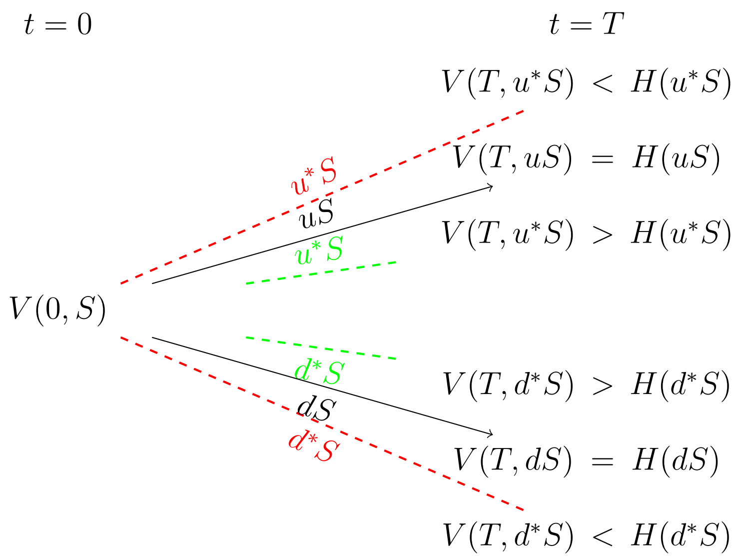


Figure 9.3: If it rises at rate u^* with $u^* \in (d,u)$ or falls at rate $d^* \in (d,u)$, then the final portfolio value will be greater than the payoffs for a call or put; otherwise it will be smaller.

Question 7 *What is the behavior of the profit when applying the binomial model for more than one period?*

A partial answer will be given in the following theorem.

Assumption 1 *If the current price of the asset is S_0, then the price at time T is such that*

$$S_T = S_0 + m\int_0^T S_r dr + \sigma \int_0^T S_r dW_r$$

where $m, \sigma \in \mathbb{R}_+$ are parameters to be determined by the writer.

Theorem 59 *Consider a call option with strike price K. We assume that the writer has priced it using the realistic n-period binomial model under Assumption 1 with*

$$u = e^{\sigma z_p \sqrt{\frac{T}{n}} + (m-\sigma^2/2)\frac{T}{n}}, \quad d = e^{-\sigma z_p \sqrt{\frac{T}{n}} + (m-\sigma^2/2)\frac{T}{n}}$$

Here z_p is such that $\frac{1}{\sqrt{2\pi\frac{T}{n}}}\int_{-z_p\frac{T}{n}}^{z_p\frac{T}{n}} e^{\frac{-t^2}{2\frac{T}{n}}} dt = p$, where p has been chosen by the writer and satisfies $u > 1$ and $d < 1$. Then the probability of profit at the last step $\mathbb{P}(\Pi \geq 0) \to 0$ as $n \to \infty$, assuming that the writer constructs the portfolio as originally planned by placing or withdrawing the corresponding amounts of money.

Proof. Suppose that the writer constructs the portfolio as originally planned, placing or withdrawing the corresponding amounts of money. This happens because the asset price will almost never take the estimated values.
The profit at time T is

$$\Pi \quad = \quad a_{n-1}US_0 + b_{n-1} - H(US_0)$$

where $H(x) = (x-K)^+$ and a_{n-1}, b_{n-1} are such that $a_{n-1}uS_{n-1} + b_{n-1} = H(uS_{n-1})$ and $a_{n-1}dS_{n-1} + b_{n-1} = H(dS_{n-1})$. Finally, $U = e^{\sigma W_T + (m-\sigma^2/2)T}$ is the actual asset price at time T after some upward and downward jumps. From now on, we denote by a, b the a_{n-1}, b_{n-1} and by S_{n-1} the estimated price after the same number of upward and downward jumps.

Suppose that $dS_{n-1} < US_0 < K$. Then the profit is

$$\begin{aligned}\Pi &= aUS_0 + b - H(US_0)\\ &= a(US_0 - dS_{n-1}) + H(dS_{n-1}) - H(US_0)\\ &= (US_0 - dS_{n-1})\left(a - \frac{H(US_0) - H(dS_{n-1})}{US_0 - dS_{n-1}}\right)\end{aligned}$$

and it follows that $\Pi \geq 0$. If $US_0 < dS_{n-1} < K$ then $\Pi \leq 0$ and if $US_0 < K < dS_{n-1}$ then $\Pi \leq 0$ because $a = 1$ and $\frac{H(US_0)-H(dS_{n-1})}{US_0-dS_{n-1}} \leq 1$. Similarly, if $K < US_0$, it follows that $\Pi \geq 0$ if $US_0 \leq uS_{n-1}$ and $\Pi \leq 0$ if $US_0 \geq uS_{n-1}$.
Therefore, the profit is non-negative when $dS_{n-1} \leq US_0 \leq uS_{n-1}$ and non-positive otherwise. The probability of profit is then

$$\mathbb{P}\left(dS_{n-1} \leq US_0 \leq uS_{n-1}\right) = \sum_{k=0}^{n-1} q^k(1-q)^{n-1-k}\mathbb{P}(u^k d^{n-1-k}(dS_0) \leq US_0 \leq u^k d^{n-1-k}(uS_0))$$

But

$$\begin{aligned}&\mathbb{P}(u^k d^{n-1-k}(dS_0) \leq US_0 \leq u^k d^{n-1-k}(uS_0))\\ =\ &\mathbb{P}\left(e^{\sigma z_p\sqrt{\frac{T}{n}}(2k-n)+(m-\sigma^2/2)T} \leq e^{\sigma W_T+(m-\sigma^2/2)T} \leq e^{\sigma z_p\sqrt{\frac{T}{n}}(2k+2-n)+(m-\sigma^2/2)T}\right)\\ =\ &\mathbb{P}\left(\sigma z_p\sqrt{\frac{T}{n}}(2k-n) \leq \sigma W_T \leq \sigma z_p\sqrt{\frac{T}{n}}(2k+2-n)\right)\\ =\ &\frac{1}{\sqrt{2\pi T}}\int_{\sigma z_p\sqrt{\frac{T}{n}}(2k-n)}^{\sigma z_p\sqrt{\frac{T}{n}}(2k+2-n)} e^{-t^2/(2T)}dt\\ \leq\ &\frac{2\sigma z_p}{\sqrt{2\pi n}}\end{aligned}$$

Therefore

$$\mathbb{P}\left(dS_{n-1} \leq US_0 \leq uS_{n-1}\right) \leq \frac{2\sigma z_p}{(1-q)\sqrt{2\pi n}}$$

Hence, the probability of profit at the last step decreases as $n \to \infty$, and of course this means that the probability of loss increases. □

`Theorem` **60 (Sell low-buy high (Binomial))** *In the Cox–Ross–Rubinstein model with* $u > 1$*,* $d \in (0,1)$*,* $d < e^{r\Delta t} < u$ *and risk-neutral* $q = \frac{e^{r\Delta t}-d}{u-d} \in (0,1)$*, let* $V_n(\cdot)$ *be the European option values for a convex payoff* F*. If at each step* n *the writer rebalances to the one-step delta*

$$\Delta_n(S) := \frac{V_{n+1}(uS) - V_{n+1}(dS)}{(u-d)S},$$

then after an up move he buys (at the higher price), and after a down move he sells (at the lower price).

`Proof.` *(1) Convexity).* Since V_{n+1} is convex and $S \mapsto V_{n+1}(uS)$, $V_{n+1}(dS)$ are convex, $V_n(S) = e^{-r\Delta t}\big(qV_{n+1}(uS) + (1-q)V_{n+1}(dS)\big)$ is convex. By induction, all V_n are convex.
(2) Monotonicity of Δ_n *in* S*).* For convex g, the secant slope $[g(b)-g(a)]/(b-a)$ is nondecreasing in a and in b. With $g = V_{n+1}$, $a = dS$, $b = uS$, we get that $S \mapsto \Delta_n(S)$ is nondecreasing; in particular $\Delta_{n+1}(uS) \geq \Delta_{n+1}(dS)$.

(3) Sandwich). For $n \le N-2$,

$$\Delta_n(S) = \frac{V_{n+1}(uS) - V_{n+1}(dS)}{(u-d)S}$$
$$= e^{-r\Delta t}(q\,u\,\Delta_{n+1}(uS) + (1-q)\,d\,\Delta_{n+1}(dS)),$$

so with $w_u = e^{-r\Delta t} qu$, $w_d = e^{-r\Delta t}(1-q)d$ and $w_u + w_d = 1$ (martingale of discounted stock),

$$\Delta_{n+1}(uS) \;\ge\; \Delta_n(S) \;\ge\; \Delta_{n+1}(dS).$$

(4) Trading). At node (n, S) the hedge holds $\Delta_n(S)$ shares. If the next move is up to uS, the target becomes $\Delta_{n+1}(uS) \ge \Delta_n(S)$, so it buys at price uS. If the move is down to dS, the target is $\Delta_{n+1}(dS) \le \Delta_n(S)$, so it sells at price dS. □

`Conclusion` **2** *Therefore, there is no reason to apply the binomial model for option pricing. This is mainly due to the fact that the choice of d, u is a personal matter with which other investors will not agree, and of course it is unrealistic to assume that the asset price will take only two values.*

However, the above model can be applied as a hedging strategy where the writer will profit if $S_T \in (dS_0, uS_0)$ and incur a loss otherwise. The disadvantage in this case is that the possible loss is unbounded, and thus it is preferable to construct a more complex hedging portfolio, such as those already described, so that the possible loss becomes bounded.
Here we describe a simple hedging strategy where not only the loss is bounded but also the profit is unbounded.
Suppose that a call option is sold at price $C(K)$. Then another call option can be purchased at price $C(K') < C(K)$ where $K' > K$. The remaining amount $Y = C(K) - C(K')$ will be used to buy $\frac{Y}{S_0}$ shares.
Therefore, the profit function of this portfolio is

$$\Pi(x) = \frac{Y}{S_0}x + (x-K')^+ - (x-K)^+$$

As already mentioned, at time T when the options expire, the stock price S_T will replace x, thus providing the valuation of the profit.

- If $S_T < K < K'$, then $\Pi(S_T) = \frac{Y}{S_0}S_T > 0$, i.e., there is profit.
- If $K < S_T < K'$, then $\Pi(S_T) = \frac{Y}{S_0}S_T - S_T + K = S_T(\frac{Y}{S_0} - 1) + K$. In this case, $\Pi(S_T) \ge K'(\frac{Y}{S_0} - 1) + K$ if $\frac{Y}{S_0} - 1 < 0$, i.e., the loss is bounded.
- If $S_T > K'$, then $\Pi(S_T) = \frac{Y}{S_0}S_T + K - K' \ge \frac{Y}{S_0}K' + K - K'$, i.e., again the loss is bounded.

Note, however, that $\Pi(S_T) \to +\infty$ as $S_T \to +\infty$, hence the profit is now unbounded.
Thus, it has been shown that by additionally purchasing another call option, we not only achieve limiting the potential loss, but we also achieve unlimited profit growth, in contrast to the binomial model where the loss is unlimited and the profit is limited.
The above example provides a rationale for why someone would want to write (sell) a call option. With this amount, they would construct a more complex portfolio that will be profitable in any scenario they believe will occur.
What happens in the case where a call option has been sold but no other call option is available on the market?
In this case, the corresponding amount can be borrowed to buy one share. Thus, the maximum possible loss will be the borrowed amount. If additional shares are to be purchased, then the potential profit will be unbounded.
Finally, if there are no other call options available and no additional shares can be purchased, then the possibility of buying other shares with similar behavior should be examined. If this is also not feasible, then the risk probability can be calculated before selling such a contract, even if this valuation does not guarantee anything.

Lab **61 (Applying the Binomial Model)** *With the code* `BinomialModel.ipynb` *we compute the fair price according to this theory and also plot the tree.*

9.1.1 Exercises

True or False

1. **Statement:** In the one-period model, the hedging portfolio $V_0 = aS + b$ is unique.
   True / False: _______ **Justification:** ______________________________

2. **Statement:** The risk-neutral coefficient $q = \frac{e^{rT} - d}{u-d}$ does not depend on the actual probability p.
   True / False: _______ **Justification:** ______________________________

3. **Statement:** If $e^{r\delta} < d$, then the binomial model violates the no-arbitrage assumption.
   True / False: _______ **Justification:** ______________________________

4. **Statement:** In the multi-period binomial model, pricing is done by backward induction.
   True / False: _______ **Justification:** ______________________________

5. **Statement:** The probability of profit for the writer in the n-period model tends to 1 as $n \to \infty$.
   True / False: _______ **Justification:** ______________________________

Multiple Choice

1. In the one-period binomial model, which formula gives a (shares) for hedging the payoff $H(uS), H(dS)$?

   (A) $a = \frac{e^{rT} H(uS) - H(dS)}{(u-d)S}$.

   (B) $a = \frac{H(uS) - H(dS)}{(u-d)S}$.

   (C) $a = \frac{H(uS) - e^{-rT} H(dS)}{(u-d)S}$.

   (D) $a = \frac{uH(uS) - dH(dS)}{(u-d)e^{rT} S}$.

   Answer: _______

2. The factor b (bond) is calculated as $b = \dfrac{uH(dS) - dH(uS)}{(u-d)e^{rT}}$. What happens if $u - d$ is very small?

   (A) b becomes very large.

   (B) b becomes very small or negative.

   (C) Hedging is undefined.

   (D) It is not affected.

   Answer: _______

3. In the multi-period model, which procedure is used for pricing?

   (A) Monte Carlo simulation.
   (B) Backward induction.
   (C) Finite difference methods.
   (D) Regression (LSM) methods.

   Answer: _______

4. The parameter $q = \frac{e^{r\delta}-d}{u-d}$ in the binomial model plays the role of:

   (A) Actual probability of up move.
   (B) Risk-neutral probability of up move.
   (C) Ratio u/d.
   (D) Volatility estimate.

   Answer: _______

5. Which term discounts the payoff back to $t = 0$?

   (A) Actual interest rate p.
   (B) $e^{-rT}\big(q\ \cdot + (1-q)\cdot\big)$.
   (C) Analytical formula $aS_0 + b$.
   (D) $\max\{u, d\}$.

   Answer: _______

Matching

Match the description with the correct concept. Write the letter next to the number.

1. $V_0 = e^{-r\delta}\big(qV_u + (1-q)V_d\big)$	A. Risk-neutral valuation
2. $q = \frac{e^{r\delta}-d}{u-d}$	B. Risk-neutral probability
3. $V_0 = aS_0 + b$	C. Replicating portfolio
4. $H(S_T) = (S_T - K)^+$	D. European call payoff

Answers: 1._____ 2._____ 3._____ 4._____

Fill in the Blank

1. In the one-period binomial model, u and d are the ____________________ rates of the stock.

2. The hedging portfolio $V_0 = aS_0 + b$ is constructed so that $V_T =$ ____________________.

3. The parameter r in the formula e^{-rT} is the ____________________.

4. In the multi-period model, pricing is done in ____________________ steps.

5. The unbounded possible loss in the binomial model arises because the ____________________ can exceed (d, u).

Theoretical and Computational Exercises

1. **Proof of uniqueness:** Show that for $d < e^{rT} < u$ the portfolio (a, b) is unique.

2. **Generalization to n periods:** Write the recursion formula $V_{k\delta} = e^{-r\delta}[qV_{(k+1)\delta} + (1 - q)V'_{(k+1)\delta}]$.

3. **Switch from p to q:** Explain why the actual p is not used for pricing.

4. **Comparison with Black-Scholes:** Explain how the binomial model converges to Black-Scholes as $n \to \infty$.

5. **Numerical demonstration:** Implement in Python a binomial tree with $n = 50$ steps and compare the call price with the analytical Black-Scholes price.

6. **Abnormal conditions:** What happens if $d \geq e^{r\delta}$? Analyze the possibility of arbitrage.

Python-Based Exercises

1. **Binomial pricing:** Write a Python function that calculates the European call price via an n-period tree.

2. **Comparison over n:** Plot the convergence of the call price for $n \in \{10, 50, 100, 500\}$ compared to Black-Scholes.

3. **Tree:** Modify the tree so that $u = e^{\sigma\sqrt{\delta}}$, $d = e^{-\sigma\sqrt{\delta}}$ and repeat the comparison.

9.2 Black-Scholes Model

Now, the theory of Black-Scholes will be presented in this topic. This theory assumes that the price of the underlying stock follows the geometric Brownian motion with parameters m, σ, that is,

$$S_t = S_0 + m \int_0^t S_r dr + \sigma \int_0^t S_r dW_r.$$

The seller's purpose, according to this theory, is to construct a portfolio at the time of selling the contract ($t = 0$) such that at time T the portfolio has the same value as the amount paid to the buyer. The portfolio consists of a number of shares and an amount in a bank account with an annual interest rate equal to r (that is, 1 euro after time t will become e^{rt}). Denoting by $a(t, S_t)$ the number of shares and by $b(t, S_t)$ the amount in the bank account in the portfolio, then the value of the portfolio will be equal to

$$V_t = a(t, S_t)S_t + b_t e^{rt}.$$

The writer's goal is to find the appropriate strategy $(a(t, S_t), b(t, S_t))$ for constructing the portfolio so that $V_T = P_T$. Liquidating the portfolio will give the money to the buyer.
This portfolio is also called a hedging strategy. It is called self-financing if the following holds:

$$V_t = V_0 + \int_0^t (ma(s, S_s)S_s + rb(s, S_s)B_s)ds + \int_0^t \sigma a(s, S_s)S_s dW_s.$$

Negative values for a, b indicate borrowing.

Theorem 62 *Let (a, b) be a self-financing Markovian strategy. Then, if there exists $u(t,x) \in C^{1,2}(D)$ (where $D = (0,T) \times (0,+\infty)$) such that $u(t, S_t) = V_t$, then $u(t,x)$ is a solution of the partial differential equation*

$$\frac{\sigma^2 x^2}{2} u_{xx}(t,x) + rxu_x(t,x) + u_t(t,x) = ru(t,x)$$

with $(t,x) \in [0,T] \times \mathbb{R}_+$. Moreover, $a(t,x) = u_x(t,x)$. The above partial differential equation is called the **Black-Scholes differential equation**.

Proof. Since (a, b) is a self-financing strategy, we have

$$V_t = V_0 + \int_0^t (ma(s, S_s)S_s + rb(s, S_s)B_s)ds + \int_0^t \sigma a(s, S_s)S_s dW_s.$$

Ito's formula is applied to $u(t, S_t)$ and we obtain

$$\begin{aligned} u(t, S_t) = \quad & u(0, S_0) \\ + & \int_0^t (u_t(s, S_s) + mS_s u_x(s, S_s) + \frac{\sigma^2 S_s^2}{2} u_{xx}(s, S_s))ds \\ + & \int_0^t \sigma S_s u_x(s, S_s) dW_s. \end{aligned}$$

Therefore V_t is expressed as an Ito process in two different ways and, using the uniqueness of the Doob-Meyer decomposition in their difference (which is zero), it follows that both integrals are zero.
Therefore, we have the identity

$$a(t, S_t) = u_x(t, S_t) \quad \text{and hence} \quad b(t, S_t)B_t = u(t, S_t) - S_t u_x(t, S_t)$$

For the ds-integral to vanish, it is necessary that $u(t,x)$ satisfies the following equation

$$\frac{\sigma^2 x^2}{2} u_{xx}(t,x) + rxu_x(t,x) + u_t(t,x) = ru(t,x)$$

after substituting appropriately $b(t, S_t)B_t$, since the integral must be zero for any path of S_t. At this point the assumption $u \in C^{1,2}(D)$ is also useful. □

It can be shown that the following partial differential equation has a unique solution:

$$\begin{aligned} \frac{\sigma^2 x^2}{2} u_{xx}(t,x) + rxu_x(t,x) + u_t(t,x) &= ru(t,x) \\ u(T,x) &= \max\{x - K, 0\}. \end{aligned}$$

Therefore the portfolio such that $V_T = P_T$ will be

$$V_t = a(t, S_t)S_t + b(t, S_t)e^{rt}$$

with a, b as chosen in the above theorem and $u(t,x)$ the solution of the Black-Scholes equation.

9.2.1 Transformation of the Black-Scholes equation into the heat equation

We now perform the appropriate changes of variables so that the equation transforms into the heat equation. Set

$$\begin{aligned} t &= T - \tfrac{2\tau}{\sigma^2}, & \tau \in \left[0, \tfrac{\sigma^2}{2}T\right] \\ S &= e^x, \quad x \in \mathbb{R} \\ V(t,S) &= v(\tau, x), \quad x \in \mathbb{R}, & \tau \in \left[0, \tfrac{\sigma^2}{2}T\right] \end{aligned}$$

We compute the partial derivatives and, noting that $V(t,S) = v\Big(\frac{\sigma^2}{2}(T-t), \ln S\Big)$, we obtain

$$\begin{aligned} V_t &= -\frac{\sigma^2}{2}\, v_\tau, \\ V_S &= \frac{1}{S}\, v_x, \\ V_{SS} &= -\frac{1}{S^2}\, v_x + \frac{1}{S^2}\, v_{xx}. \end{aligned}$$

Then the original differential equation becomes, setting $k = \dfrac{2r}{\sigma^2}$,

$$v_\tau = v_{xx} + (k-1)\, v_x - k\, v,$$

having used the fact that V satisfies the Black-Scholes equation.
Next we set

$$v(\tau, x) = e^{ax + b\tau}\, u(\tau, x).$$

Substituting v into the above equation yields

$$b\, u + u_\tau = a^2 u + 2a\, u_x + u_{xx} + (k-1)\,(au + u_x) - ku.$$

In order to simplify we choose

$$a = -\frac{1}{2}\,(k-1), \qquad b = -\frac{1}{4}\,(k+1)^2.$$

Finally, it follows that the function V is a solution of **??** `if and only if` $u(\tau, x)$ satisfies the heat equation

$$u_\tau = u_{xx}, \quad \tau \in \left[0, \frac{\sigma^2}{2}T\right], \quad x \in \mathbb{R}.$$

In the case where we have a terminal condition in the Black-Scholes equation, for example

$$V(T,S) = F(S), \quad S > 0,$$

where $F(\cdot)$ is a given function, the corresponding transformed heat equation takes the form

$$\begin{aligned} u_\tau &= u_{xx}, & \tau \in \left[0, \tfrac{\sigma^2}{2}T\right], \quad x \in \mathbb{R} \\ u(0,x) &= e^{\frac{1}{2}(k-1)x}\, F(e^x), & x \in \mathbb{R}. \end{aligned}$$

Denoting $f(x) = e^{\frac{1}{2}(k-1)x}\, F(e^x)$, we have that this problem has the solution

$$u(\tau, x) = \frac{1}{2\sqrt{\tau\pi}} \int_{-\infty}^{\infty} f(z)\, e^{-\frac{(x-z)^2}{4\tau}}\, dz, \quad \tau \in \left[0, \frac{\sigma^2}{2}T\right], \quad x \in \mathbb{R}. \tag{9.1}$$

9.2.2 Value of a European Call Option

Let us determine the value of a European call option based on the function $V(t,S)$ we have computed. In this case $F(x) = (x-K)^+$, where K is the strike price.
We will compute $u(\tau,x)$ from relation (9.1) by making the change of variable $y = \dfrac{z-x}{\sqrt{2\tau}}$. We have

$$\begin{aligned} u(\tau,x) &= \frac{1}{2\sqrt{\tau\pi}}\int_{-\infty}^{\infty} f(z)\, e^{-\frac{(x-z)^2}{4\tau}}\,dz \\ &= \frac{1}{\sqrt{2\pi}}\int_{\frac{\ln K-\ln S}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)\,(x+y\sqrt{2\tau})}\, e^{-\frac{1}{2}y^2}\,dy - \frac{K}{\sqrt{2\pi}}\int_{\frac{\ln K-\ln S}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k-1)\,(x+y\sqrt{2\tau})}\, e^{-\frac{1}{2}y^2}\,dy, \end{aligned}$$

where we have used $f(x) = e^{\frac{1}{2}(k-1)x}\max\{e^x - K, 0\}$.
The first integral above becomes

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}}\int_{\frac{\ln K-\ln S}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)\,(x+y\sqrt{2\tau})}\, e^{-\frac{1}{2}y^2}\,dy \\ &= \frac{e^{\frac{1}{2}(k+1)x}}{\sqrt{2\pi}}\int_{\frac{\ln K-\ln S}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{4}(k+1)^2\tau}\, e^{-\frac{1}{2}\left(y-\frac{1}{2}(k+1)\sqrt{2\tau}\right)^2}\,dy \\ &= \frac{e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}}{\sqrt{2\pi}}\int_{\frac{\ln K-\ln S}{\sqrt{2\tau}}-\frac{1}{2}(k+1)\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\rho^2}\,d\rho \\ &= e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}\, N(d_1), \end{aligned}$$

where

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{1}{2}u^2}\,du,$$

and

$$d_1 = \frac{\ln S - \ln K}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau} = \frac{\ln\frac{S}{K} + r(T-t) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

Similarly,

$$\frac{1}{\sqrt{2\pi}}\int_{\frac{\ln K-\ln S}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k-1)\,(x+y\sqrt{2\tau})}\, e^{-\frac{1}{2}y^2}\,dy = e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau}\, N(d_2),$$

where

$$d_2 = \frac{\ln\frac{S}{K} + r(T-t) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

Therefore,

$$V(t,S) = S\,N(d_1) - K\,e^{-r(T-t)}\,N(d_2).$$

We can relate $V(t,S)$ to the error function $\mathrm{erf}(x)$ and obtain

$$V(t,S) = S\left(\frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{d_1}{\sqrt{2}}\right)\right) - K\,e^{-r(T-t)}\left(\frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{d_2}{\sqrt{2}}\right)\right). \tag{9.2}$$

This relation follows easily as

$$\begin{aligned} N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sqrt{2}}} e^{-y^2} \sqrt{2}\, dy \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{2}}} e^{-y^2}\, dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-y^2}\, dy + \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x}{\sqrt{2}}} e^{-y^2}\, dy \\ &= \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right). \end{aligned}$$

9.2.3 Historical Volatility and Implied Volatility: Definitions and Procedures

Historical (Realized) Volatility from Prices

Let $\{P_{t_i}\}_{i=0}^{n}$ be observed prices over $[t_0, t_n]$, and define (log) returns

$$r_i = \ln\left(\frac{P_{t_i}}{P_{t_{i-1}}}\right), \qquad i = 1, \ldots, n.$$

Equal sampling step (e.g., daily). Write Δt for the time step in *years* (for daily data, $\Delta t \approx 1/252$), and let

$$\bar{r} = \frac{1}{n} \sum_{i=1}^{n} r_i.$$

Two common estimators are:

(a) Sample-variance estimator: $$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (r_i - \bar{r})^2,$$

$$\widehat{\sigma}_{\text{ann}} = \sqrt{\frac{s^2}{\Delta t}} \quad \text{(annualized volatility)}.$$

(b) Realized-variance estimator: $$RV = \sum_{i=1}^{n} r_i^2, \qquad \widehat{\sigma}_{\text{ann}} = \sqrt{\frac{RV}{n\,\Delta t}} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} r_i^2}\, \frac{1}{\sqrt{\Delta t}}.$$

In practice, $\bar{r}$ over short windows is typically small, so (a) and (b) are numerically close.

Unequal time steps. If observation gaps differ, set $T = t_n - t_0$ (in years) and use

$$RV = \sum_{i=1}^{n} r_i^2, \qquad \widehat{\sigma}_{\text{ann}} = \sqrt{\frac{RV}{T}}.$$

Implied Volatility from Option Prices (Black–Scholes)

Given a European call/put with observed mid price C_{mkt} (or P_{mkt}), spot S_0, strike K, time to maturity T (years), risk-free rate r and dividend yield q, Black–Scholes prices are

$$d_1 = \frac{\ln(S_0/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T},$$

$$C_{\text{BS}}(\sigma) = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2), \quad P_{\text{BS}}(\sigma) = K e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1),$$

where $N(\cdot)$ is the standard normal cdf. The *implied volatility* σ_{impl} is the unique $\sigma > 0$ such that

$$C_{\text{BS}}(\sigma_{\text{impl}}) = C_{\text{mkt}} \quad \text{or} \quad P_{\text{BS}}(\sigma_{\text{impl}}) = P_{\text{mkt}}.$$

9.3 Volatility over $[0, T]$, Hedging Limitations, and Model–Free Pricing

This section clarifies the distinction between the *future (realized) volatility* that matters for a contract expiring at time T, and commonly used *historical* or *implied* volatilities. It also highlights practical limitations of dynamic hedging and motivates model–free notions of fair pricing.

Remark **63 (Forward–looking volatility vs. historical and implied measures)** *Let $T > 0$ be the maturity and consider the (log-)price process $X_t = \log S_t$. The volatility relevant for the contract over $[0, T]$ depends on the* future path *realized on that interval. With discrete observations $0 = t_0 < t_1 < \cdots < t_n = T$ and returns $r_i = \Delta X_{t_i}$, a realized (pathwise) estimate is*

$$\widehat{\sigma}_{[0,T]} = \left(\frac{1}{T} \sum_{i=1}^{n} r_i^2 \right)^{1/2}.$$

Because this quantity is determined by prices within $[0, T]$, *it generally* differs *from any* historical *(backward-looking) volatility computed on a past window (e.g. $[-H, 0]$). Equality would require strong assumptions such as constant volatility or strict stationarity. Implied volatility, in turn, is extracted* today *from option prices and reflects the market's assessment of the future average volatility over $[0, T]$; it need not coincide with the future realized value above.*

Remark **64 (Instantaneous portfolio rebalancing is infeasible)** *In continuous-time hedging frameworks (e.g. Black–Scholes), the number of shares in the replicating portfolio changes with t and S_t. Exact replication would require* instantaneous *rebalancing as t advances and S_t moves. In practice, orders take time to execute, markets are discrete, and trading incurs frictions (latency, lot sizes, transaction costs), so truly instantaneous rebalancing is impossible.*

Remark **65 (On predictiveness and practical hedging strategies)** *Models such as Black–Scholes and their extensions (including binomial and related trees) are not predictive of future prices because they do not incorporate investor behavior or broader market dynamics. Although they are often presented as hedging frameworks via dynamic replication, those strategies are only approximable under real-world constraints and thus are generally not fully implementable.*

Implications for choosing σ and for hedging. If a writer could implement the hedging portfolio, they would still need to *choose* a volatility input (e.g. a range for σ) for construction. If the guess for σ matches the future average volatility over $[0, T]$, hedging profits may result; otherwise, losses may accrue. Increasing the rebalancing frequency does not guarantee proximity to the theoretical target (even before accounting for costs) and may, in adverse scenarios, amplify risk. As a dynamic hedging strategy, it suffers from the undefined future bid–ask spread and transaction costs.

Conclusion **3 (Why fair pricing should be model–free)** *Any definition of a fair price that hinges on a specific model and parameter choice is fragile in practice, since the two parties are unlikely to agree on both the model and its parameters. Therefore, fair prices should be defined in a model–free manner and justified directly in market terms observable in the real world.*

9.3.1 The Volatility Smile

Market option prices are commonly quoted via *implied volatility.* For a given maturity T and strike K, the implied volatility $\sigma_{\text{imp}}(K, T)$ is the unique σ such that the Black–Scholes price with volatility σ matches the observed market price:

$$C^{\mathrm{BS}}(S_0, K, T; r, q, \sigma_{\mathrm{imp}}) = C^{\mathrm{mkt}}(K, T) \quad \text{or} \quad P^{\mathrm{BS}}(\cdot; \sigma_{\mathrm{imp}}) = P^{\mathrm{mkt}}.$$

As a function of strike (or *moneyness*), $\sigma_{\mathrm{imp}}(K,T)$ is typically non-flat. In foreign exchange markets one often observes a (symmetric) *smile*, whereas in equity markets a *left-skew/smirk* (higher implied vol for OTM puts) is more common. The shape depends on maturity (the *term structure*) and reflects distributional features such as leverage effects, jump/crash risk, and stochastic volatility.

A practical challenge for complex payoffs is ensuring that the chosen σ (or surface $(K,T) \mapsto \sigma_{\mathrm{imp}}$) yields prices that are jointly arbitrage-free across the entire cross-section of calls and puts.

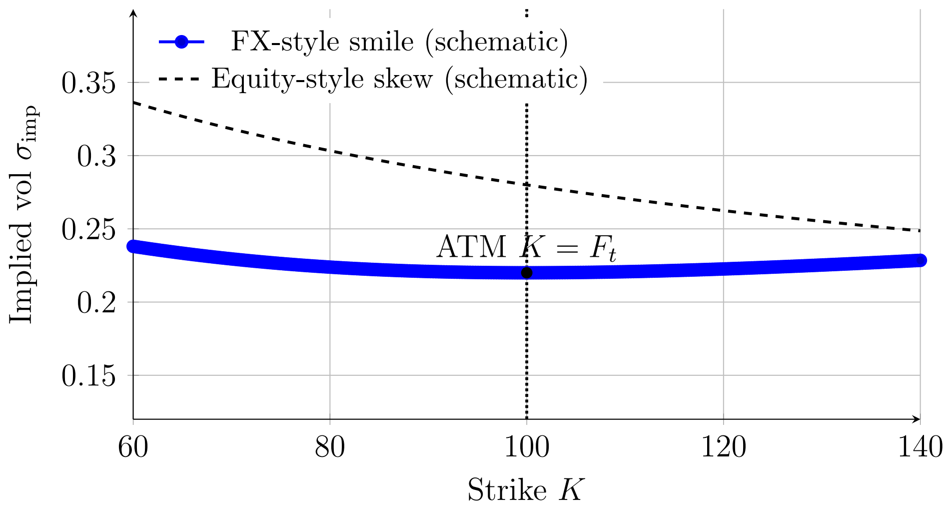


Figure 9.4: Illustrative implied volatility σ_{imp} versus strike K at a fixed maturity T. Curves are schematic (not calibrated).

Lab **66 (Applying the Black-Scholes model)** *Using the code* `Black-Scholes-Fair-Price.ipynb`*, we compute the fair price implied by this theory. It can be observed that the resulting price varies depending on the choice of historical data. As previously discussed, the proposed replicating portfolio is applicable only in discrete time. To examine its behavior, one may use the code* `DiscreteBS.ipynb`*. The output in the second cell demonstrates that, in general, the investor would be required to buy high and sell low, a strategy that directly contradicts the well, established principle of selling high and buying low. Using the code* `Volatility-Smile.ipynb` *you can see by yourself how the implied volatility related to the strike price K producing the volatility smile.*

9.3.2 Exercises

True or False

1. **Statement:** The delta-hedging strategy in the Black-Scholes model is self-financing.
   True / False: _______ **Justification:** ____

2. **Statement:** In the PDE of Black-Scholes, the term $\frac{\sigma^2 x^2}{2} u_{xx}$ comes from the Ito differential.
   True / False: _______ **Justification:** ____

3. **Statement:** The parameter r in Black-Scholes is the risk-free rate.
   True / False: _______ **Justification:** ____

4. **Statement:** The solution of the PDE with terminal condition $u(T, x) = (x - K)^+$ gives the price of a European call.
   True / False: _______ **Justification:** ____

5. **Statement:** In the Black-Scholes model, the volatility σ is constant over time.
   True / False: _______ **Justification:** ____

Multiple Choice

1. What is the formula for the delta $\Delta(t, x)$ of the hedge?

   (A) $\Delta = u(t, x) - x\, u_x(t, x)$.

   (B) $\Delta = u_x(t, x)$.

   (C) $\Delta = u_{xx}(t, x)$.

   (D) $\Delta = \frac{\partial u}{\partial t}(t, x)$.

   Answer: _______

2. Risk-neutral pricing is based on the assumption that the drift m is replaced by:

   (A) $m = 0$.

   (B) r (the risk-free rate).

   (C) $\sigma^2/2$.

   (D) $m - \sigma^2/2$.

   Answer: _______

3. Which terminal condition applies for a European put in the PDE?

   (A) $u(T, x) = \max\{x - K, 0\}$.

   (B) $u(T, x) = \max\{K - x, 0\}$.

   (C) $u(0, x) = \max\{x - K, 0\}$.

   (D) $u(0, x) = \max\{K - x, 0\}$.

   Answer: _______

4. The Black-Scholes analytical price for a European call is given by:

   (A) $S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$.

   (B) $Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$.

   (C) $S_0e^{-qT}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$.

   (D) $S_0\Phi(d_2) - Ke^{-rT}\Phi(d_1)$.

   Answer: _______

5. If the stock pays a continuous dividend at rate q, the PDE is modified with the additional term:

   (A) $-qxu_x$.

   (B) $+qu$.

   (C) $-qu$.

   (D) $+qxu_x$.

   Answer: _______

Matching

Match the description with the corresponding concept. Write the letter next to the number.

1. $u_t + \frac{\sigma^2 x^2}{2} u_{xx} + rxu_x - ru = 0$	A. Black-Scholes PDE
2. $dS_t = mS_t\,dt + \sigma S_t\,dW_t$	B. Geometric Brownian motion
3. (a,b) with $a = u_x,\ b = e^{-rt}(u - au_x)$	C. Replicating strategy
4. $u(T,x) = \max\{K - x, 0\}$	D. European put payoff

Answers: 1._____ 2._____ 3._____ 4._____

Fill in the Blank

1. The Ito process is applied to $u(t, S_t)$ to derive the Black-Scholes PDE.

2. The delta of the strategy is $\Delta(t, S_t) =$ ____________________.

3. The terminal condition for a European call is $u(T,x) =$ ____________________.

4. The decision $m \to r$ corresponds to a change to ____________________.

5. The assumption of constant σ in the model is ____________________.

Theoretical and Computational Exercises

1. **PDE Proof:** Using the Ito formula, show step-by-step that $u(t,x)$ satisfies the Black-Scholes PDE for a self-financing strategy.

2. **Closed-form solution:** Prove that the analytical solution for a European call is $C = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$ with d_i as defined.

3. **Strategy improvement:** Discuss how to modify the strategy if $\sigma = \sigma(t)$ depends on time.

4. **Comparison with binomial:** Show theoretically the convergence of the binomial model to Black-Scholes as $n \to \infty$.

5. **Error analysis:** Compute the call price difference between Black-Scholes and an n-period tree for various n.

Python-Based Exercises

1. **Monte Carlo pricing:** Implement a Geometric Brownian Motion simulation and estimate the price of a European call using Monte Carlo.

2. **PDE solver:** Code in Python a numerical finite difference solver for the Black-Scholes PDE (implicit or Crank-Nicolson).

3. **Dynamic hedging:** Simulate the hedging strategy $a(t, S_t)$ in discrete steps and compute the final payoff and tracking error.

9.4 Arbitrage-Free Price Interval

Are the prices produced by probability-based models realistic? Market reality is determined by investors; they set the prices of the options that actually trade. Such models are not realistic unless they explicitly account for investors through the observed prices of available options.

Standing assumptions. We work in a frictionless market with a single riskless bank account accruing at rate r over $[0, T]$. The variable b denotes the *time–0* cash position (positive for a deposit, negative for borrowing), which grows to $b\,e^{rT}$ at maturity. Short sales and static positions in the listed options are allowed as specified in each linear program.
Suppose that a contract is to be bought or sold with a payoff function $f(x)$. For example, for a call option we have $f(x) = (x - K)^+$, while for a put option $f(x) = (K - x)^+$. Clearly, the price of the contract is determined by the law of supply and demand. However, a decisive role will be played by the presence of an arbitrage opportunity. Therefore, in practice, to identify a no-arbitrage price interval one should at least take into account the available call and put options and solve the following linear programming problems:

$$\begin{aligned} &\text{minimize } Y \\ &\text{subject to} \quad aS_0 + b + \sum_{i=1}^{n} \big(\gamma_i C(K_i) + \delta_i P(K_i)\big) = Y, \\ &\qquad\qquad \Pi^{writer}(x) \geq 0 \quad \text{for all } x \geq 0, \end{aligned} \tag{9.3}$$

$$\begin{aligned} &\text{maximize } Y \\ &\text{subject to} \quad aS_0 + b + \sum_{i=1}^{n} \big(\gamma_i C(K_i) + \delta_i P(K_i)\big) = -Y, \\ &\qquad\qquad \Pi^{buyer}(x) \geq 0 \quad \text{for all } x \geq 0, \end{aligned} \tag{9.4}$$

where

$$\Pi^{writer}(x) = ax + be^{rT} + \sum_{i=1}^{n} \gamma_i (x - K_i)^+ + \delta_i (K_i - x)^+ - f(x),$$

$$\Pi^{buyer}(x) = ax + be^{rT} + \sum_{i=1}^{n} \gamma_i (x - K_i)^+ + \delta_i (K_i - x)^+ + f(x).$$

The solution to the first problem provides the smallest required amount to construct a portfolio (by the option writer) with no possible loss in any state. Similarly, solving the second problem we find the largest amount the buyer can pay while maintaining a portfolio with no possible loss. Let the solution to the first problem be Y^{writer}, and the solution to the second be Y^{buyer}. Then the no-arbitrage price interval is $\left(Y^{buyer}, Y^{writer}\right)$, provided $Y^{buyer} < Y^{writer}$.

`Definition` **67** *We call the interval* $\left(Y^{buyer}, Y^{writer}\right)$ *the* arbitrage-free price interval *for the option with payoff* $f(x)$.

`Conclusion` **4** *Any transaction price outside* $\left(Y^{buyer}, Y^{writer}\right)$ *creates an arbitrage for one of the parties and is therefore* ***not acceptable*** *for the side unable to exploit that arbitrage. Within the interval, no such riskless profit is available to either side.*

Model prices vs. no-arbitrage bounds

Prices output by *uncalibrated* probabilistic models (e.g., binomial or Black-Scholes with historical parameters) need not fall inside the arbitrage-free interval implied by currently traded options and may therefore be incompatible with observed markets. In practice, the no-arbitrage interval provides a robust bargaining range; any model used for valuation should respect it.

9.5 Model-Free Fair Prices of an Option

We will see below that we can define different notions of fair prices which, in general, lead to different values. This means that the buyer will choose the notion that gives the lowest price, while the writer will choose the notion that gives the highest price.

Consider an option with payoff function $f(x)$ which is piecewise linear with finitely many branches. Given an amount Y, we consider the following two linear programming problems: find the parameters $a, b, \gamma_i, \delta_i, D$ such that

$$\begin{aligned} &\text{minimize } D \\ &\text{subject to} \quad aS_0 + b + \sum_{i=1}^{n} \left(\gamma_i C(K_i) + \delta_i P(K_i)\right) = Y, \\ &\qquad\qquad \Pi^{writer}(x) \geq -D \quad \text{for all } x \geq 0, \end{aligned} \tag{9.5}$$

where $C(K_i), P(K_i)$ are the current call/put prices and

$$\Pi^{writer}(x) = ax + be^{rT} + \sum_{i=1}^{n} \gamma_i (x - K_i)^+ + \delta_i (K_i - x)^+ - f(x).$$

This is the *profit function*; substituting $x = S_T$ yields the writer's profit. Solving (9.5) finds a portfolio whose maximal loss is as small as possible.

We can include bid-ask spreads by replacing the budget constraint with

$$aS_0 + b + \sum_{i=1}^{n} \left(\gamma_i^{ask} C^{ask}(K_i) - \gamma_i^{bid} C^{bid}(K_i) + \delta_i^{ask} P^{ask}(K_i) - \delta_i^{bid} P^{bid}(K_i)\right) = Y,$$

and the profit becomes

$$\Pi^{writer}(x) = ax + be^{rT} + \sum_{i=1}^{n} \big(\gamma_i^{ask} - \gamma_i^{bid}\big)(x - K_i)^+ + \sum_{i=1}^{n} \big(\delta_i^{ask} - \delta_i^{bid}\big)(K_i - x)^+ - f(x),$$

with $\gamma_i^{bid}, \gamma_i^{ask}, \delta_i^{bid}, \delta_i^{ask} \geq 0$.
A similar construction holds for the buyer by solving

$$\begin{aligned} &\text{minimize } D \\ &\text{subject to} \quad aS_0 + b + \sum_{i=1}^{n} \big(\gamma_i C(K_i) + \delta_i P(K_i)\big) = -Y, \\ &\qquad\qquad \Pi^{buyer}(x) \geq -D \quad \text{for all } x \geq 0, \end{aligned} \tag{9.6}$$

where

$$\Pi^{buyer}(x) = ax + be^{rT} + \sum_{i=1}^{n} \gamma_i (x - K_i)^+ + \delta_i (K_i - x)^+ + f(x).$$

Define

$$\begin{aligned} D^{writer}(Y) &:= \min\Big\{ D : \exists (a,b,\gamma,\delta) \text{ s.t. } aS_0 + b + \textstyle\sum_i (\gamma_i C(K_i) + \delta_i P(K_i)) = Y, \\ &\qquad \Pi^{writer}(x) \geq -D \ \forall x \geq 0 \Big\}, \\ &= \min\Big\{ \|(-\Pi^{writer})^+\|_\infty : \ \Pi^{writer} \text{ feasible at budget } Y \Big\} \end{aligned}$$

$$\begin{aligned} D^{buyer}(Y) &:= \min\Big\{ D : \ \exists (a,b,\gamma,\delta) \text{ s.t. } aS_0 + b + \textstyle\sum_i (\gamma_i C(K_i) + \delta_i P(K_i)) = -Y, \\ &\qquad \Pi^{buyer}(x) \geq -D \ \forall x \geq 0 \Big\} \\ &= \min\Big\{ \|(-\Pi^{buyer})^+\|_\infty : \ \Pi^{buyer} \text{ feasible at budget } Y \Big\} \end{aligned}$$

It is obvious that $D^{writer}(Y)$ is the min obtained solving problem 9.5 while $D^{buyer}(Y)$ is the min obtained solving problem 9.6.

`Lemma` **68 (Monotonicity and continuity of $D^{writer}(Y)$)** *Under the standing assumptions (cash account available at rate r and free $b \in \mathbb{R}$), the minimal uniform shortfall*

$$D^{writer}(Y) := \min\Big\{ \|(-\Pi^{writer})^+\|_\infty : \ \Pi^{writer} \text{ feasible at budget } Y \Big\}$$

satisfies, for any $Y_2 > Y_1$,

$$\begin{aligned} D^{writer}(Y_2) &\leq D^{writer}(Y_1) - (Y_2 - Y_1)e^{rT} \\ \text{and} \quad D^{writer}(Y_1) &\leq D^{writer}(Y_2) + (Y_2 - Y_1)e^{rT}. \end{aligned} \tag{9.7}$$

Consequently,

$$|D^{writer}(Y_2) - D^{writer}(Y_1)| \ \leq \ e^{rT} \, |Y_2 - Y_1|,$$

so D^{writer} is Lipschitz (hence continuous) and nonincreasing in Y. Moreover, on the open interval (Y^{buyer}, Y^{writer}) one has $D^{writer}(Y) > 0$ and, for any $Y_1 < Y_2 < Y^{writer}$,

$$D^{writer}(Y_2) \ \leq \ D^{writer}(Y_1) - (Y_2 - Y_1)e^{rT} \ < \ D^{writer}(Y_1),$$

i.e. D^{writer} is strictly *decreasing on (Y^{buyer}, Y^{writer}). Finally, $D^{writer}(Y^{writer}) = 0$.*

Proof. Take an optimal portfolio at budget Y_1 and add cash $\Delta Y := Y_2 - Y_1$. The profit increases by $\Delta Y\, e^{rT}$ statewise, so the uniform shortfall decreases by at least $\Delta Y\, e^{rT}$, giving the first inequality in (9.7). Conversely, start from an optimal portfolio at Y_2 and remove ΔY of cash to meet budget Y_1; the shortfall increases by at most $\Delta Y\, e^{rT}$, giving the second inequality. The Lipschitz bound and monotonicity follow. □

Lemma 69 (Monotonicity and continuity of $D^{buyer}(Y)$) *Under the standing assumptions (cash account at rate r and free $b \in \mathbb{R}$), the minimal uniform shortfall*

$$D^{buyer}(Y) := \min\Big\{\|(-\Pi^{buyer})^{+}\|_\infty : \ \Pi^{buyer} \text{ feasible at budget } Y\Big\}$$

is nondecreasing *and* Lipschitz *in Y. Specifically, for any $Y_2 > Y_1$,*

$$\begin{aligned} D^{buyer}(Y_2) &\leq D^{buyer}(Y_1) + (Y_2 - Y_1)\, e^{rT} \\ \text{and} \quad D^{buyer}(Y_1) &\leq D^{buyer}(Y_2) - (Y_2 - Y_1)\, e^{rT}, \end{aligned} \tag{9.8}$$

hence

$$\big|D^{buyer}(Y_2) - D^{buyer}(Y_1)\big| \leq e^{rT}\, |Y_2 - Y_1|.$$

Moreover, on the open interval (Y^{buyer}, Y^{writer}) one has $D^{buyer}(Y) > 0$ and, for any $Y^{buyer} \leq Y_1 < Y_2 < Y^{writer}$,

$$D^{buyer}(Y_2) > D^{buyer}(Y_1),$$

i.e. D^{buyer} is strictly increasing *on (Y^{buyer}, Y^{writer}). Finally, $D^{buyer}(Y^{buyer}) = 0$.*

Proof. Take an optimal portfolio at budget Y_1 and *reduce* the cash position by $\Delta Y := Y_2 - Y_1$ to meet the budget $-Y_2$ in (9.6). The profit decreases statewise by $\Delta Y\, e^{rT}$, so the uniform shortfall increases by at most $\Delta Y\, e^{rT}$, giving the first inequality in (9.8). Conversely, start from an (almost) optimal portfolio at Y_2 and *add* cash ΔY to meet budget Y_1; the profit increases uniformly by $\Delta Y\, e^{rT}$, so the shortfall decreases by at least that amount, which yields the second inequality. Lipschitz continuity and monotonicity follow. Finally, by maximality of Y^{buyer} for which $D^{buyer} = 0$, one has $D^{buyer}(Y) > 0$ for all $Y > Y^{buyer}$; combining with (9.8) yields strict increase on (Y^{buyer}, Y^{writer}). □

Theorem 70 (Fair price Y^{D^*}) *Suppose that $Y^{buyer} \leq Y^{writer}$. Then there exists a unique price $Y^{D^*} \in (Y^{buyer}, Y^{writer})$ such that $D^{writer}(Y^{D^*}) = D^{buyer}(Y^{D^*})$.*

Proof. The function $G(Y) := D^{writer}(Y) - D^{buyer}(Y)$ is continuous and strictly decreasing on $[Y^{buyer}, Y^{writer}]$. Moreover, $G(Y^{writer}) = D^{writer}(Y^{writer}) - D^{buyer}(Y^{writer}) \leq 0$ and $G(Y^{buyer}) = D^{writer}(Y^{buyer}) - D^{buyer}(Y^{buyer}) \geq 0$ so there exists a unique $Y^{D^*} \in (Y^{buyer}, Y^{writer})$ with $D^{writer}(Y^{D^*}) = D^{buyer}(Y^{D^*}) =: D^* > 0$. □

As one can observe, the price Y^{D^*} is a kind of fair price because at this price both parties can construct a portfolio with at most D^* possible loss.

Definition 71 (Y^{D^*} is a fair price) *We define the price Y^{D^*} as a fair price of the contract with payoff $f(x)$.*

Theorem 72 (The fair price Y^{D^*} is arbitrage-free) *Suppose that problems (9.3) and (9.4) admit solutions Y^{writer} and Y^{buyer}, respectively, with $Y^{writer} \geq Y^{buyer}$. Then every price $p \in \big(Y^{buyer}, Y^{writer}\big)$ is arbitrage-free and therefore Y^{D^*} is also arbitrage free.*

Proof. If $p > Y^{writer}$, short one option at price p, buy the writer's superhedge costing Y^{writer}, and deposit the surplus $p - Y^{writer}$ at rate r: terminal wealth is $(p - Y^{writer})e^{rT} > 0$. If $p < Y^{buyer}$, buy the option at p and combine it with the buyer's superhedge delivering an initial inflow Y^{buyer} and deposit the surplus: terminal wealth is $(Y^{buyer} - p)e^{rT} > 0$. By theorem 70 we conclude that the fair price Y^{D^*} is also arbitrage free. □

The above can be easily extended to contracts written on many underlying assets. If the contract is American-style, then American calls and puts can also be used.

There are many fair prices There are also many other ways to define different fair prices. Given a multivariable function F, define

$$\mathcal{F}^{writer}(a,b,\gamma_i,\delta_i) = F\Big(D,\ \int_0^\infty (\Pi^{writer}(x))^-\,dx,\ \int_0^M (\Pi^{writer}(x))^+\,dx,\ \int_0^M \Pi^{writer}(x)\,dx,\ \ldots\Big),$$

for some $M > 0$ and analogously $\mathcal{F}^{buyer}$. For a given amount Y the writer solves

$$\min \mathcal{F}^{writer}(a,b,\gamma_i,\delta_i) \quad \text{s.t.} \quad aS_0 + b + \sum_{i=1}^{n}(\gamma_i C(K_i) + \delta_i P(K_i)) = Y,\ \Pi^{writer}(x) \geq -D\ \forall x \geq 0,$$

while the buyer solves

$$\min \mathcal{F}^{buyer}(a,b,\gamma_i,\delta_i) \quad \text{s.t.} \quad aS_0 + b + \sum_{i=1}^{n}(\gamma_i C(K_i) + \delta_i P(K_i)) = -Y,\ \Pi^{buyer}(x) \geq -D\ \forall x \geq 0.$$

Then we may declare $Y^{\mathcal{F}}$ a fair price if

$$\min \mathcal{F}^{writer}(a,b,\gamma_i,\delta_i) = \min \mathcal{F}^{buyer}(a,b,\gamma_i,\delta_i).$$

`Corollary` **73 (Y^{D^*} is unique in some sense)** *Suppose that $Y^{buyer} \leq Y^{writer}$. Among model-free fair values, the price Y^{D^*} defined by $D^{writer}(Y) = D^{buyer}(Y)$ is always arbitrage-free.*

This means that traders can calculate this value to obtain an order of magnitude of the derivative's value, which will be useful in the bargaining stage. The actual price will be formed by the law of supply and demand therefore it will not be necessarily fair or even arbitrage free!
Note that the "fair" values of stochastic models (e.g., Black-Scholes, Binomial) are neither fair in the real world (since the proposed replicating portfolios are not practically feasible) nor arbitrage free since they do not lie within the range of arbitrage free values.

`Lab` **74 (Python codes for the above)** *Using the code `FairValueOption1.ipynb`, we compute the fair value Y^{D^*} as well as the arbitrage free interval, if it exists.*
The price of the option will be formed by the law of supply and demand and will be an amount Y not necessarily a fair or an arbitrage free value. The writer/buyer of the option can invest this amount in various ways. For example:

- *The writer can construct a portfolio with the minimum possible loss D by solving problem 9.3. This can be done by the Python code `WriterOption.ipynb`. Similar construction can be done by the buyer using the Python code `BuyerOptions.ipynb`.*

- *The writer can make a prediction about the future price of the underlying and solve a linear programming problem like 4.13. In this case the profit function is as follows*

$$\Pi(x) = ax + be^{rT} + \sum_{i=1}^{d} \gamma_i (x - K_i)^+ + \delta_i (K_i - x)^+ - f(x)$$

  This can be done using the Python code `WriterHedgingOption1.ipynb`. Similar construction can be done by the buyer using the Python code `BuyerHedgingOption1.ipynb`.

- *The investors may employ a sell–high/buy–low trading strategy (see Section 9.6 below). However, as a form of dynamic hedging, it also suffers from the uncertainty of the future volatility, bid–ask spread and from transaction costs, just like the binomial and Black–Scholes hedging approaches.*

9.5.1 Exercises

True or False

1. **Statement:** If $Y < Y^{buyer}$, the buyer can secure a risk-free profit.
   True / False: _______ **Justification:** ____

2. **Statement:** The interval (Y^{buyer}, Y^{writer}) is empty if f is not piecewise linear with finitely many branches.
   True / False: _______ **Justification:** ____

3. **Statement:** In order to compute Y^{writer}, we only need to know the current prices of $C(K_i), P(K_i)$.
   True / False: _______ **Justification:** ____

4. **Statement:** If $Y^{buyer} < Y < Y^{writer}$, then there is no possibility of arbitrage.
   True / False: _______ **Justification:** ____

5. **Statement:** The Black-Scholes price of a European call is always arbitrage-free.
   True / False: _______ **Justification:** ____

Multiple Choice

1. In problem (9.3), the objective $\min Y$ corresponds to:
   (A) Zero maximum loss for the writer.
   (B) Zero probability of loss for the buyer.
   (C) Zero transaction cost.
   (D) Zero difference between writer and buyer prices.

   Answer: _______

2. In (9.4), the equation $aS_0 + b + \cdots = -Y$ means:
   (A) The buyer pays Y today.
   (B) The writer pays Y today.
   (C) The buyer receives Y today.
   (D) None of the above.

   Answer: _______

3. To include bid-ask spread, we introduce additional variables $\gamma^{ask}, \gamma^{bid}$ so that:
   (A) The bid and ask prices are identical.
   (B) The portfolio is always valued at the average price.
   (C) The coefficients express the number of contracts bought separately from those sold.
   (D) The spread is zeroed.

Answer: _______

4. The interval (Y^{buyer}, Y^{writer}) increases when:
   (A) The strikes K_i increase.
   (B) More available derivatives are added.
   (C) The interest rate r decreases.
   (D) The number n of available options decreases.

   Answer: _______

5. If $Y^{buyer} = Y^{writer}$, then:
   (A) There is a unique arbitrage-free price.
   (B) There is no fair price.
   (C) The price interval is unlimited.
   (D) The market is in crisis.

   Answer: _______

Matching

1. $\Pi^{writer}(x) \geq 0$ for all x	A. No-loss hedging
2. $\min Y$ in (9.3)	B. Upper arbitrage bound
3. $\max Y$ in (9.4)	C. Lower arbitrage bound
4. (Y^{buyer}, Y^{writer})	D. Arbitrage-free price interval

Answers: 1._____ 2._____ 3._____ 4._____

Fill in the Blank

1. In problem (9.3), the profit function $\Pi^{writer}(x)$ includes terms ____________________.
2. The price Y^{writer} is the lowest arbitrage-free price for the ____________________.
3. In (9.4), the condition $\Pi^{buyer}(x) \geq 0$ ensures that the buyer ____________________.
4. The range of K_i (strikes) affects the width of the ____________________.
5. The fair price Y^{D^*} is defined when $D^{writer} =$ ____________________.

Theoretical and Computational Exercises

1. **Proof of interval:** Show that $Y^{buyer} \leq Y^{writer}$ always holds.
2. **Include spread:** Formulate (9.3) with bid-ask spread for two strikes $K_1 < K_2$.
3. **Comparative analysis:** For an option with $f(x) = (x-K)^+$, compute numerically (Y^{buyer}, Y^{writer}) when only $C(K)$ and $P(K)$ are available.
4. **Extension to multiple assets:** Discuss how the interval would be generalized to two underlying assets S^1, S^2.
5. **Incorporating dividends:** Add a dividend term q to the portfolio and modify the pricing problems.

6. **Non-linear f:** If f is convex but non-linear (e.g., asymmetric payoff), discuss how to choose knot points for linearization.

Python-Based Exercises

1. **LP pricing:** Implement in Python (e.g., with `scipy.optimize.linprog`) problem (9.3) for some K_i and total $n = 3$.

2. **Interval plot:** Write a script that, for varying r and fixed $C(K_i), P(K_i)$, plots (Y^{buyer}, Y^{writer}) as a function of r.

3. **Spread sensitivity:** Add bid/ask and study how the price interval changes as the spread increases.

9.6 Path Dependent Options

An interesting problem is the hedging of path dependent options. One possible approach to this problem is to apply an appropriate sell high - buy low strategy where the parameters of this strategy will depend on the form of the payoff function.
The replicating portfolio proposed by the Black-Scholes theory can only be implemented in discrete time. However, as you will notice, this leads us to a sell low - buy high strategy! Regarding the dynamic hedging problem of an option, one can use a similar strategy and choose the parameters based on the payoff function.
This can be very useful for the pricing and hedging problems of options that depend on the path (path dependent options).

- *(Pricing Problem)* In the pricing problem, the writer, for a given $a \in (0, 1)$, can compute the minimum amount Y for which there exist parameters in the dynamic sell high - buy low trading strategy such that

  $$\mathbb{P}\big(V_n \geq f(S_1, \ldots, S_n)\big) \geq a,$$

  where $V_n = a_n S_n + b_n$ with a_n, b_n determined by the above strategy and $Y = a_0 S_0 + b_0$. By computing this amount, the writer obtains an order of magnitude for the option price, useful during negotiations.

- *(Hedging Problem)* However, the actual price will be determined by the law of supply and demand. Therefore, for the given actual price Y of the option, the writer can find the maximum $a \in (0, 1)$ for which there exist parameters in the sell high - buy low strategy such that

  $$\mathbb{P}\big(V_n \geq f(S_1, \ldots, S_n)\big) \geq a,$$

  with $Y = a_0 S_0 + b_0$. In this way, the writer obtains a practically applicable hedging strategy.

At the above minimization problem we can employ the expected shortfall as well. One can add call and put options so the minimization problem can find the best possible choice about the actual number of these options.
As can be seen, the pricing problem here is not to find a fair or arbitrage free price, but to estimate the order of magnitude of the price. Similar procedures can be followed by the buyer. Call and put options can also be added to the dynamic strategy as appropriate.

Remark **75** *The pricing and hedging strategy described above is not limited to path-dependent options; it can also be applied to simpler instruments such as call and put options. However, because it does not incorporate the market prices of existing calls and puts, it will generally fail to produce arbitrage-free prices. Finally, this approach can also be applied to options with nonlinear payoff structures.*

Lab **76 (Pricing and hedging path dependent options)** *Using the Python code* `Pricing-and-Hedging-Lookback-Options-Sell-High-and-Binomial.ipynb` *one can price and hedge a lookback option using the sell high - buy low strategy.*

9.6.1 Exercises

True or False

1. **Statement:** In the sell high - buy low strategy, the parameters a_n, b_n depend exclusively on the current price S_n.
   True / False: _______ **Justification:** ____________________

2. **Statement:** The estimate Y of the writer for the path dependent option does not depend on the tolerance level $a \in (0,1)$.
   True / False: _______ **Justification:** ____________________

3. **Statement:** The strategy yields only one portfolio $V_n = a_n S_n + b_n$, regardless of the form of $f(S_1, \ldots, S_n)$.
   True / False: _______ **Justification:** ____________________

4. **Statement:** The pricing problem is formulated as $\min Y$ with constraint $\mathbb{P}(V_n \geq f) \geq a$.
   True / False: _______ **Justification:** ____________________

5. **Statement:** The corresponding buyer's strategy requires $\max a$ for a given Y.
   True / False: _______ **Justification:** ____________________

Multiple Choice

1. In the pricing problem, $a \in (0,1)$ represents:

   (A) The probability of hedging without loss.
   (B) The percentage of capital in stocks.
   (C) The weighting of the simulation.
   (D) The confidence level in the strategy.

   Answer: _______

2. The typical form $V_n = a_n S_n + b_n$ is:

   (A) A static portfolio.
   (B) A dynamic rebalancing.
   (C) A consumption model.

(D) A benchmark model.

Answer: _______

3. If f is a lookback payoff, then $f(S_1, \ldots, S_n)$ depends on:

(A) Only S_n.
(B) All prices S_k.
(C) The final a_n, b_n.
(D) The sum $\sum S_k$.

Answer: _______

4. The sell high - buy low strategy in path dependent options involves:

(A) Appropriate choice of a_n, b_n based on f.
(B) Constant a_n.
(C) Ignoring historical data.
(D) Zero volatility.

Answer: _______

Matching

Match the description with the corresponding concept. Write the letter next to the number.

1. $\min Y$ with $\mathbb{P}(V_n \geq f) \geq a$	A. Pricing Problem
2. $\max a$ with $Y = a_0 S_0 + b_0$ given	B. Hedging Problem
3. $V_n = a_n S_n + b_n$	C. Dynamic Portfolio
4. $\{a_n, b_n\}$ depend on f	D. Path Dependence

Answers: 1._____ 2._____ 3._____ 4._____

Fill in the Blank

1. The $V_n = a_n S_n + b_n$ is called ____________________.
2. The function $f(S_1, \ldots, S_n)$ for a lookback option is ____________________.
3. In the pricing problem, $Y = a_0 S_0 + b_0$ is estimated so that ____________________.
4. In the hedging problem, $a \in (0, 1)$ is maximized for given ____________________.
5. The dynamic sell high - buy low strategy is determined by ____________________.

Theoretical and Computational Exercises

1. **LP Formulation:** Fully formulate the $\min Y$ pricing problem as an LP with variables a_n, b_n.
2. **Lookback payoff:** For $f(S_1, \ldots, S_n) = \max_{1 \leq k \leq n} S_k - K$, find criteria for choosing a_n.
3. **Monte Carlo Simulation:** Implement in Python the estimate of Y under $\mathbb{P}(V_n \geq f) \geq a$.
4. **Comparison of extreme scenarios:** Show what happens in strong upward or downward paths.
5. **Parameter adjustment:** Analyze how Y changes when the target a increases.
6. **Extension to Asian option:** Formulate corresponding pricing/hedging problems for $f = \left(\frac{1}{n} \sum S_k - K\right)^+$.

Python-Based Exercises

1. **LP solver:** Implement the pricing LP with `scipy.optimize.linprog` for an Asian option.
2. **MC calibration:** Find parameters a_n via MC optimization so that $\mathbb{P}(V_n \geq f) \approx a$.
3. **Lookback test:** Use the `Pricing-and-Hedging-Lookback-Options-Sell-High-and-Binomial.ipynb` for different a and n.

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